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BY ROBERT POTTS, M.A.,

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A FEW WORDS TO TEACHERS AND LEARNERS.

A quaint old teacher has remarked "that when a learner is at first rightly and thoroughly grounded, the rest of the work goes on with readiness, with ease, with speed, and with assurance. When he is ill-grounded, all falls out contrarywise; much labour and much patience of the master, and much diligence and industry of the scholar will hardly be able to rescue him from the mischievous consequences of previous ill-grounding. So powerful is ill habit when it hath once got hold, and so difficult to eradicate, that it is much harder to unteach the wrong than to teach the right."

Another eminent teacher of modern times has stated the cause of his success in these words: "It has ever been with me a principle that even rudimental teaching must be thorough, and must be founded upon the essential character of the mind, as accretive; and that when once it is rightly proceeded upon, the teaching of any science will not remain quiescent, but will advance to completion and success."

The present state of teaching the Elementary Principles of the Sciences of Number and Space is not satisfactory, as will appear in the following extracts from the Report of the Cambridge Board of the University Examinations, and from the twenty-second Report of the Syndicate appointed to conduct the Examination of Students not members of the University.

The Board of Examinations issued their Report of the Previous Examination of the Freshmen in March, 1880. With respect to the subjects of Elementary Arithmetic, Algebra, and Geometry (which are supposed to be acquired at School) the following remarks appear in the Report:

> In the June Examination *seven* per cent. of the candidates failed altogether and *twenty-six* per cent. only just passed in the subject of Arithmetic. Questions on Compound Interest, Discount, and Stocks, as well as those on Simplification of Decimals, were answered by very few.
>
> In the December Examination, *four* per cent. failed altogether and *ten* per cent. only just passed. Questions in Stocks and Discount were answered incorrectly by a large number of the candidates, their work shewing ignorance of the meaning of the questions.
>
> In the December Examination in Euclid, *one quarter* of the candidates did not write out more than half of the bookwork correctly. Many of the candidates had apparently prepared the subjects from text-books in which the order of Euclid is not preserved. By this means great confusion is introduced.
>
> In June, *four-ninths* of the candidates failed in Algebra; in December, *one quarter*. Questions in Ratio, Proportion, and Variation were not well done, and in many cases the candidates did not appear to have read these portions of the subject.

Of the Additional Subjects the Report states:—

> In Algebra undisguised pieces of bookwork were in general written out correctly, and an easy Problem of a common type was solved by most of the candidates. The Progressions were treated without much thought, the formulæ being often applied at random.

The Regulations for the Previous Examination revised in 1849 required the candidates to be arranged in two classes:—

> The *first class*, consisting of those candidates who have passed their examination with credit; the *second*, of those to whom the Examiners have only not refused their certificate of approval.

The facts stated, and the words employed in the Report suggest a very unsatisfactory state of the acquirements and of the intellectual habits of a large class of Freshmen in their first year of residence at Cambridge.

In the year 1879 the names of 6738 students were entered for the Cambridge Local Examinations, and forty-one per cent. of those who attended, failed to pass the examination. There were 3600 students examined in Pure Mathematics, including Elementary Arithmetic, Algebra, and Geometry.

On the results of the Examination of the Junior Students the Report of the Examiners remarks:—

In Arithmetic,

> There were fewer cases of confused and bad methods, fewer gross errors in principle; and the papers were worked in better form and more neatly. But, as compared with last year, improvement is not manifest.

In Algebra,

> Although the failures were more numerous this year than last, a fair per centage of the whole number of candidates acquitted themselves creditably.

In Euclid,

> Several lost credit by quoting axioms other than Euclid's, and by inattention to Euclid's order, for instance, proving I. 27 by I. 32. More riders than last year were sent up, but still too few of the candidates attempted them.

Of the Examination of the Senior Students the Examiners state:—

In Arithmetic,

> The work both of boys and girls shewed an improvement on that of last year, as regards both style and accuracy. There were but few candidates, however, of any very great merit. Nearly half the whole number of boys and more than half the girls were but little above the minimum standard required for passing. The girls in many cases used heavy and cumbrous modes of working.

In Algebra,

> The Algebra was far from satisfactory. More than half the whole number of candidates failed to pass, and very many could have had no reasonable expectation of passing. The bookwork questions were written out at very great length, but in most cases the important points were slurred over or altogether omitted.

In Euclid,

> The propositions of Euclid were on the whole creditably written out, but only the easier riders were solved. The per centage of failures in Euclid was higher in the case of the girls than in that of the boys.

It is more than probable that one of the causes of the unsatisfactory state of elementary knowledge of these subjects is the simple fact that too many subjects of study are prematurely forced on minds of ordinary capacity. In the case of inert and dull minds no sound and exact knowledge is acquired; and in the case of active and precocious minds a loose smattering of many subjects is crammed into the memory which have not passed through the understanding. Whatever may be forced into the memory in this way is only held in solution until it has served its purpose, and then it is precipitated.

The injudicious solicitude of some teachers to develope prematurely the mental faculties of youth will only result in furnishing illustrations of the consequences which flow from a disobedience to the laws of nature. The teacher ought not to forget that the organs of the brain equally with those of the body have their predetermined periods of growth and development. Any attempts to interfere with these cannot be made with impunity. How seldom do those who in youth have been nurtured in the forcing hot-bed exhibit in after life superior intellectual powers. They more commonly fall beneath than exceed the average in talent. But even if the mental capacity be enlarged, it is at the expense of the corporeal energies. These are not displayed in the strongly knit and active limbs, in the well-formed and robust frame. The brain may have grown, but it is almost invariably accompanied by a feeble and imperfectly developed body, which, in general, prematurely breaks up. If nature has given a superiority of mind, the less interference there is with the laws which regulate its development the more ample and gratifying will be the results. If she has withheld these conditions which are necessary for the manifestation of talent, it is not only in vain to endeavour to create what she has denied, but injudicious and fraught with danger. The vegetable kingdom illustrates the justice of these views.

In the use of ordinary language it is implied that there exists some sort of analogy between the bodily and mental faculties. As the mysterious processes of digestion and assimilation are necessary for the healthy development of the body, so also, unless the food of the mind be inwardly digested, it cannot contribute to the like development of the intellectual powers. The reception of knowledge into the mind has been described under the comparison of good seed sown in good soil or bad soil; and it is written that 'men do not gather grapes of thorns nor figs from thistles.'

As, therefore, the physical development of the organs and functions of the body take place gradually according to the order of fixed laws; it would appear that the development of the mind must, in a similar way, be subject to the same order. The mental acquirements suitable for youth cannot be put off with advantage till the approach of manhood. The consequences of neglect in early mental training will in due time become as manifest as the lack of proper food and care in the healthy growth of the body. As the body for food, so the mind for knowledge has its hungerings and thirstings. The craving appetite in both cases implies a process of assimilation. As the mental food assimilated will affect the character of the mind itself, only wholesome mental food should be supplied in the right order and quantity suitable for healthy and vigorous intellectual growth.

The study of the exact sciences is one of the most effective means of cultivating and developing the reason. Geometry is the Science of Space, and Arithmetic with Algebra in its character of Universal Arithmetic is the Science of Number. As all our knowledge of the external world must be subject to the conditions of Space and Number, the elementary portions of these sciences are from their nature better adapted than any other to form the habit of fixity of attention, of distinctness in the conception of ideas, and of precision in expression. The language, like the subjects, is fixed and definite, and does not admit of the same ambiguity and uncertainty as the language employed on other subjects. The reasonings are always conclusive and exact, expressed in terms whose meaning cannot change from the sense in which they have been defined. In one of his recent charges, the present Lord Chief Justice made a passing remark to the effect that Euclid's Elements were a mental training second to none. Teachers who need a hint or two on this subject may learn something from the experience of the Rev. Stephen Hawtrey, late Assistant Master at Eton, as stated in his "Short Introduction to the Elements of Euclid."

It may be remarked that persons of the highest acquirements in any science do not always become the best teachers. Experience has shewn that such persons may be utterly incompetent to adapt their knowledge to the capacity of minds of a lower character than their own. If a teacher has not skill to make the subject of his instruction interesting, and tact to adapt his knowledge to minds of different capacities, he does not possess one of the essential requisites of a teacher. It is a delusion to imagine that correct habits of thinking can be created or exact knowledge acquired by the mere passive attendance on lectures, however excellent. Class-teaching may be useful or useless, or even worse than useless. There are many youthful minds so constituted that they must be taught individually, if ever they are to be able to draw inductions from facts, or comprehend principles and apply them with success. In dealing with the misapprehensions and mistakes of learners, the teacher should observe how the learner was led into error, and by suitable questions lead him to perceive his mistake and to make the correction for himself. By this method the mind of the learner is brought into active exercise, and he will be less likely to repeat the mistake than if the correction were received passively from the teacher. If the learner exhibit a listless inattention or a positive dislike to the subject of study, the efforts of the most judicious teacher are in vain.

It may also be remarked that implicit obedience in the learner and the love of knowledge are also necessary conditions of improvement. A sense of duty rather than a desire of surpassing others constitutes the right motive of the learner. If a morbid appetite for praise or an eager strife for pre-eminence be encouraged, it may grow, and at length become the ruling passion, and create envy and hatred of every successful rival, and generate a feeling of discontent which may become a fatal obstruction to all mental and moral improvement.

These few words to teachers and learners may be concluded with the expressive words of the late Dr. Whewell:—

"The object of a liberal education is to develope the whole mental system of man:—to make his speculative inferences coincide with his practical convictions:—to enable him to render a reason for the belief that is in him, and not to leave him in the condition of Solomon's Sluggard, who is wiser in his own conceit than seven men that *can* render a reason."

R. P.

OF ALGEBRA.

THE science which bears this name is a generalised extension of the science of number. From this consideration Sir Isaac Newton was led to give to it the title of "Universal Arithmetic." This designation suggests a more correct idea of its nature than the inadequate name handed down by tradition. Besides, it appears calculated to lead the student to such a knowledge of the general operations of the science, as to render his comprehension of ordinary mathematical language more intellectual and less mechanical. As this science depends primarily on the same fundamental principles as Arithmetic, this necessary connection is required to be kept constantly in view. The following brief notices will be restricted to a general sketch of the origin of the science, as far as it has been ascertained, and the progressive improvements and discoveries. These will be seen to have been very slow and gradual in different countries and in different ages; and when viewed in succession, to extend over centuries.

The history of the first steps in the development of numerical systems, as of written languages, is involved in uncertainty, and it is perhaps impossible to ascertain the origin of any of the systems which existed among the people of early times. The methods of arithmetical notation employed by the Hebrews, the Phœnicians, and the Egyptians, and the still ruder systems employed by the Romans, are all of them limited and imperfect, and the remaining monuments of their literature and antiquities afford no intimation whatever that they were acquainted with any superior system. The powers of numeration at first must have been very limited before the art of writing was invented. A comparison of the written names of numbers found in the records of ancient nations may lead to some probable conjectures respecting their possible connection in more early times. However interesting it might be to trace the steps by which mankind originally applied and gradually extended their first essays of number to the various occasions of human life, there are no extant writings, known at present, which afford the needful assistance.

The art of denoting and naming numbers, in some form, became necessary among men as soon as the distinction of property was acknowledged, which appears to have been coeval with the origin of society. The earliest methods of numbering would naturally be limited to the naming of such numbers as the necessities of the people required.

The earliest extensions of the primitive mode would be made by the people whose necessities extended beyond the wants of a patriarchal and pastoral mode of life. In the early ages of which any records exist, some progress had been made in the arts of number and measure, as is clear from the narrative of the life of Abraham in the twenty-third chapter of the book of Genesis. The next improvements would be found in those nations who applied themselves to commerce and navigation. It is therefore among the earliest civilised nations which directed attention to commerce, that arithmetical calculation could have received its greatest improvement. It is recorded of the Egyptians and Phœnicians, that these two nations made the first improvements in the practice and calculation of numbers. The necessity of improved methods would be found in the management of the public revenues of states and kingdoms, and men would be naturally led to find out methods of abridging and improving the operations in which they were every day engaged. The account of the revenue in Egypt when Joseph was the chief minister, implies that their system of arithmetic was not of a very complete nature for the expression of large numbers; as it is recorded in the forty-first chapter of the book of Genesis, that Joseph left off numbering the quantity of corn laid up in the cities, "for the quantity was without number;" an expression which can only refer to the system of numbering known in Egypt at that time.

Previous to that period, their ordinary methods might have been sufficient for the necessities of the government in the regulation of the revenue. At a later period, it was in Egypt that Pythagoras learned the theories concerning the nature and properties of numbers, which he taught in the school he founded at Crotona, in Magna Græcia.

The Phœnicians, in the remotest ages, applied themselves to commerce and navigation, which required them to be conversant with numerical calculations. Ancient history[1] ascribes to them the invention of casting accounts, and keeping registers of everything that relates to the affairs of merchants. Plato[2] writes that a sophist affirms of the Lacedemonians, that they scarcely knew how to number. This censure of the ignorance of the people of Lacedæmon, seems to imply that they did not frequent the school in the grove of Academus at Athens, where that science, with others, was sedulously taught and cultivated.

By reducing the elementary processes of numbers to their first principles, some attempt may be made to discover those operations which, on account of their simplicity, might naturally present themselves to the mind. The operations of Arithmetic depend on the two simple processes of addition and subtraction. Multiplication of integers is nothing more than the addition of equal numbers. Division bears the same relation to subtraction as multiplication does to addition. It is therefore necessary that reference should be made to addition and subtraction for the origin of the methods of calculation. Addition and subtraction presuppose numeration, the source which furnishes Arithmetic with the material of all its operations. To count is nothing else than to form ideas of different sets or assemblages of counters, and to give names to each of the sets or groups. This forms the first act of the mind in learning the science of numbers. Every particular object brought before the sight suggests the idea of one thing or unity to the mind, and every assemblage of objects or units suggests the idea of numbers, or of a greater or a less assemblage of units. The idea of a simple number is an abstraction, and from the fact of men having five fingers on each hand, they would naturally be led to form ideas of as many sets of other objects as are equal to the number of fingers on one or on both hands. The fundamental notions of number most probably arose from the use of the fingers on one or both hands as an instrument of calculation. Names to express these elementary numbers would be devised, perhaps derived from the names of the instrument itself used in counting. Arithmetic most probably began with practical numeration, or determining the number of several objects, as the number of a herd of cattle or a flock of sheep. As men are provided with a kind of calculating instrument, it is highly probable that the fingers of one or both hands were the first instruments used to assist them in the counting by sets. Homer represents Proteus[1] as counting his sea calves by fives—that is, by his fingers—and employs the word *πεμπάζειν*, which literally means to count by the five fingers; so also Æschylus uses the same word[2] for counting out from the urn the black and white pebbles which expressed the verdict of the judges after the trial of an accused person, as guilty or not guilty. Plutarch and other writers have stated, that in the infancy of the Greek language, they had no other word for calculating, and it then signified what is now expressed by the word *ἀριθμεῖν*. It is another strong presumption in favour of this supposition, that all civilised nations of the Aryan and Semitic races count by tens, tens of tens or hundreds, and so on; proceeding by multiples of ten in the successive groups of numbers.

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[1] Strabo, B. xvii., p. 1136, B.

[2] Hippias Major, p. 1248, A. fol. Francofurti, 1602.

There appears no reason why the number ten should have been chosen rather than twelve or any other number for the scale of numeration, except the primitive practice of counting by the fingers of both hands. It appears highly probable that at first men counted by their fingers whatever objects did not exceed the number ten. They could also divide any multitude of objects into sets of ten in each set, and reserve what remained over when the whole multitude did not make an exact number of tens. But as the fingers could only serve to count the number of things up to ten, something else would be required to determine the number of groups of ten each; where the number was too great for the memory to retain with ease. It appears that small stones or pebbles were used to denote them, and the vestiges of the practice among the most ancient of civilised nations has been recorded by Herodotus.[1] It is therefore certain that in very early times, pebbles were employed in arithmetical calculations. The words *calculate* and *calculation*, being derived from the Latin *calculus*, a pebble, refer, in their original sense, to the use of pebbles in such operations.[2] The Greek equivalent *ψηφίζειν* is obviously derived from *ψῆφος*, a pebble.

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[1] Odyss. iv. 412.

[2] This most ancient manner of giving sentence is noted by Ovid:—

Mos erat antiquis, niveis atrisque lapillis,
His damnare reos, illis absolvere culpa.—*Ov. Met.* xv.

It is easy to see how a multitude of objects may be divided into sets or groups; first of units less than ten, next of tens less than ten tens, and so on, by the aid of the fingers and a few pebbles.[3] As soon as it was known how to number with readiness any collection of objects, it would be easy to find the number of several of these collections together, or to add them. It would be necessary first to find the sum of the groups of units, next of the groups of tens, thirdly of the groups of hundreds, and so on, and to place the pebbles in their proper places to denote the several sums of the several collections of units, tens, &c.

In proceeding to multiplication, or multiplying one number by another, it is probable that multiplication and addition at first were one and the same operation. As, for example, to multiply 12 by 4 is simply to add 12, 4 times; and in like manner to divide 48 by 12 is merely to subtract 12, 4 times from 48.

The remains of the early Greek writers on numbers are very scanty, and do not supply any adequate knowledge of the state and extent of this science in the school of Pythagoras in Magna Græcia, nor in the Academy at Athens. It appears that at a later period, B.C. 304, when Ptolemy Soter founded the great library at Alexandria, the sciences had been cultivated in that city with considerable success. Euclid promoted and advanced the sciences at Alexandria between the years B.C. 323 and 284. Some of his scientific writings have descended to modern times. His chief work, "The Elements,"[1] combines a system of geometry and a system of arithmetic in the Greek notation, and its application to geometry. The latter is comprised in the seventh, eighth, ninth, and tenth books of the Elements.

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[1] Herod. ii. 36.

[2] Πεμπάζετ' ὀρθῶς ἐκβολὰς ψήφων, ξένοι.—Æsch. Eumen. 738.

[3] See Elementary Arithmetic, section vi., pp. 1—4, where has been explained and exemplified how any given multitude of pebbles or other objects can be divided into groups, and named and recorded.

In addition to the Elements, Pappus has recorded that Euclid was the author of a work on the Division of Surfaces and on Data, both of which are extant; also other works on Porisms, Loci, and the Conic Sections, which are lost, besides treatises on Optics and on other subjects.

To Euclid has been attributed the merit of having improved and reduced into order the principles of arithmetic and geometry which had been handed down by Thales, Pythagoras, Eudoxus, and other philosophers, and with his own additions arranged the principles of arithmetic and geometry into the form of an exact science.

Proclus affirms that Euclid more correctly ordered many parts in the Elements of Eudoxus, and completed many things in those of Theætetus, and besides confirmed such propositions as before were too slightly or insufficiently established, with the most firm and convincing demonstrations.

The mathematical school of Alexandria continued to flourish for centuries, and Alexandria was regarded as the chief seat of learning and philosophy. Almost all the eminent mathematicians and philosophers, of whose writings any remains exist, had been students at the school of Alexandria.

Of those who directed their attention to numbers, Archimedes of Syracuse, who lived between B.C. 287 and 212, improved and extended the Greek numerical notation. Apollonius adopted the scheme of Archimedes, but instead of the *octads* of Archimedes he adopted *tetrads*.

The age of Diophantus has been variously stated. Bombelli has recorded that he lived in the times of the Antonines, but upon what authority it is not evident. The more commonly received opinion is the more probable, that Diophantus lived at a later period, some time in the fourth century of the Christian era. Few facts are known of his history. An epitaph on Diophantus[2] is found in the third book of the Greek Anthology, from which it appears that he died at the age of eighty-four years; but it is not certain if the Diophantus therein named is the author of the Arithmetics. Suidas, under the name "Hypatia," makes mention of Diophantus. His work is a singular production. There are not known to be extant any writings of an earlier date on the Greek arithmetic of a similar nature. It consists of a large collection of questions, some of them of considerable difficulty, with ingenious solutions. The work contains the solution both of determinate and indeterminate problems. The substance and details of the work itself appear to afford a strong presumption that, though this collection of problems was made and augmented by Diophantus himself, it is quite impossible to discriminate what he gathered from preceding writers, whose works are lost, and what additions and improvements were made by himself. The work originally consisted of thirteen books, and was dedicated to his friend Dionysius. Only the first six books are known to be extant, and an imperfect treatise on triangular and multangular numbers.

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[1] In the year 1570, Sir H. Billingsley published a translation of Euclid's Elements in folio, with a learned preface by the famous Dr. John Dee, as he states, "Written at my poore House At Mortlake, Anno. 1570. Februarij 9." The extant works of Euclid, in the original Greek, were edited by David Gregory, Savilian Professor, and published in folio at Oxford in 1703.

[2] Οὗτός τοι Διόφαντον ἔχει τάφος. ἆ μέγα θαῦμα,
Καὶ τάφος ἐκ τέχνης μέτρα βίοιο λέγει.
Ἕκτην κουρίζειν βιότου θεὸς ὤπασε μοίρην·
Δωδεκάτην δ' ἐπιθεὶς, μῆλα πόρεν χλοάειν.
Τῇ δ' ἄρ' ἐφ' ἑβδομάτῃ τὸ γαμήλιον ἥψατο φέγγος,

These writings were first translated into Latin by Xylander, a professor at Heidelberg, and published in 1575.

Both the original Greek, with a Latin version and a commentary, were published by Bachet at Paris in 1621. Another edition was edited by his son and Fermat, and published at Toulouse in 1670.[1]

These six books contain collections of problems, with their solutions, relating chiefly to properties of numbers, such as square and cube numbers, proportional numbers, &c.

'Εκ δὲ γάμων πέμπτῳ παῖδ' ἐπένευσεν ἔτει.
Αἴ αἴ τηλύγετον δειλὸν τέκος, ἥμισυ πατρὸς
Μέτρον τῷ κρυερὸς Μοῖρ' ἄφελεν βίοτου.
Πένθος δ' αὖ πισύρεσσι παρηγορέων ἐνιαυτοῖς
Τῇδὲ πόσου σοφίῃ τέρμ' ἐπέρησε βίου.

[1] Diophanti Alexandrini Rerum Arithmeticarum libri sex, quorum primi duo adjecta habent Scholia Maximi (ut conjectura est) Planudis. Item liber de numeris polygonis seu multangulis. Opus incomparabile, veræ Arithmeticæ Logisticæ perfectionem continens, paucis adhuc visum. A. Guil. Xylandro Augustano incredibili labore Latine redditum et commentariis explanatum, inque lucem editum ad Illustriss. Principem Ludovicum Vuirtembergensem. Basiliæ per Eusebium Episcopium et Nicolai Fr. hæredes, 1571. (fol.)

Diophanti Alexandrini Arithmeticorum libri sex, et de numeris multangulis liber unus. Nunc primum Græce et Latine editi, atque absolutissimis Commentariis illustrati. Auctore Claudio Gaspare Bacheto Meziriaco Sebusiano. Lutetiæ Parisiorum, sumptibus Sebastiani Cramoisy, 1621, fol.

A second edition was published, with the following title:—Diophanti Alexandrini Arithmeticorum libri sex, et de numeris multangulis liber unus. Cum Commentariis C. G. Bacheti, V.C., et Observationibus D.P. de Fermat Senatoris Tolosani. Accessit doctrinæ Analyticæ inventum novum, collectum ex variis ejusdem D. P. de Fermat Epistolis. Obloquitur numeris Septem discrimina vocum. Tolosæ, excudebat Bernardus Bosc, e regione Collegii Societatis Jesu. M. DC. LXX.

The Greek arithmetical notation is employed, but no account is given of the first principles of the science, or of the methods employed in the elementary operations of addition or subtraction, multiplication or division; nor even any explanation of these processes. In the first book, which contains forty-three problems with their solutions, are prefixed eleven definitions or explanations of the terms employed in the solution of the problems. The first three definitions explain the method of forming the successive powers of numbers, and give the initial letters of the names by which they are denoted. The second power of a number is named *δύναμις*; the third power *κύβος*. The powers above the square and cube are named from the sums, not from the products of the lower powers. Thus, the fourth power is named *δυναμοδύναμις*, which is formed by the product of two equal square numbers; the fifth power is named *δυναμόκυβος*, the product of a square by a cube number; the sixth power by *κυβόκυβος*, which last, in modern language, is more correctly designated by the square of the cube, or the cube of the square.

In the fourth book is shown how to form the third power of a binomial, and in the sixth book the fourth power.

Diophantus calls a positive quantity *ὕπαρξις* (substance) and a negative quantity *ἔλλειψις* (defect or want), and employs a decurtated *ψ* inverted and prefixed to denote minus, but employs no corresponding mark to denote plus.

The ninth definition states that minus multiplied by minus produces plus, and that minus multiplied by plus produces minus, but offers no reason nor explanation.[1]

The unknown number is named *ἀριθμός*, and is denoted by *ς*, the final letter; a lineal quantity is also designated by the same word. The monad or general unit is denoted by the letters μ^{o}.

The eleventh definition contains some general remarks for forming the equations. No general rules are laid down for the solution of them, but different assumptions are made, and ingenious artifices are employed, whereby the equations are very much simplified. In general, a direct assumption is made for the number required. Sometimes if the sum of two numbers be given in the problem, the difference is assumed; but if the difference be given, the sum is assumed for the unknown number. Sometimes when two quantities are required, their ratio is assumed, and other artifices are employed whereby almost all the solutions are effected by means of equations of the first degree.

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[1] The words of the definition are :—

Λεῖψις ἐπὶ λεῖψιν πολλαπλασιασθεῖσα, ποιεῖ ὕπαρξιν. λεῖψις δὲ ἐπὶ ὕπαρξιν, ποιεῖ λεῖψιν.

"Penuria in penuriam ducta, copiam; in copiam ducta, penuriam procreat."

The second book contains thirty-six problems with solutions; the third, twenty-four problems on squares; the fourth book, forty-six on cubes and squares; the fifth, twenty-three on square and cube numbers, and some involving numbers in geometrical progression. To these the editor has added upwards of forty questions from the Greek Anthology. The sixth book contains twenty-six problems relating to properties of right-angled triangles. The fragment on triangular and polygonal numbers contains only the ten problems that are known to be extant.

The school of Alexandria continued to flourish after the age of Diophantus. Pappus, an eminent mathematician, lived in the latter part of the fourth century. His mathematical collections consist of eight books; of these, the first book and the early part of the second are not known to be extant.

The last twelve propositions of the second book were printed by Dr. Wallis in 1688, at the end of his work entitled, "Aristarchus Samius;" from which it would appear that Arithmetic was the subject of the second book of the collections. The work of Pappus is especially valuable as containing an account of the state of the mathematical sciences at that period.

Theon was contemporary with Pappus, and became president of the school at Alexandria. As a philosopher and a mathematician, he appears from the remains of his writings to have been a student of the laws of nature, as well as of the sciences. He understood the higher use of the mathematical sciences as an intellectual discipline, as will appear from the following extract from his writings:—

"The science of number is the mover and the guide to truth. It is not to be studied with gross and vulgar views, but in such a manner as may enable the student to attain to the contemplation of the nature of numbers; not learning it merely for the purpose of selling or of dealing with merchants or retailers, but for the improvement of the mind, considering it as the path which leads to the knowledge of truth and reality. The study of this science exalts the mind, and compels it to think and reason on numbers themselves, as abstract. There are not to be admitted into its reasonings, as necessary, any objects, either visible or capable of being numbered. And besides this study renders a student competent for all branches of the mathematics, and enables the slow in intellect to become more acute."[1]

Proclus,[2] in the fifth century, was a student at Alexandria, but afterwards removed to the Platonic school at Athens. At this period, the study of the mathematical sciences at Alexandria was not pursued in the manner of Euclid and others of his school, but had degenerated into the dreams and fancies of the later Platonists. They imagined that they had discovered mysteries in numbers and in their properties. They assumed imaginary analogies as elementary truths, and constructed from them the most strange and absurd theories. It is very singular that these absurdities of the Platonists have not been without an advocate in modern times; as the author of "Theoretic Arithmetic" demands: "Whether it is possible that these philosophers could have spoken thus sublimely of number, unless they had considered it as possessing an essence separate from sensibles, and a transcendency fabricative, and at the same time paradigmatic."

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[1] Theon. Smyrn. c. i.

[2] Proclus wrote the following epitaph to the memory of his instructor Syrianus:—

Πρόκλος ἐγὼ γενόμην Λύκιος γένος, ὃν Συριανὸς
'Ενθάδ' ἀμοιβὸν ἑῆς θρέψε διδασκαλίης·
Ξυνὸς δ' ἀμφοτέρων ὅδε σώματα δέξατο τύμβος
Αἴθε δὲ καὶ ψυχὰς χῶρος ἑεὶς λελάχοι.

After the end of the fourth century, there appear no names of great eminence who advanced the knowledge of the sciences. The dreams and subtleties of the later Platonists appear to have absorbed the attention both of the philosophers and their disciples; and Alexandria continued to be the chief seat of learning until that city was besieged, in A.D. 640, and captured by the Saracens, and the great library destroyed. This magnificent library, from its foundation, rapidly increased, and became the most extensive depository of writings on science and learning in the world, at that period. Even in the time of Ptolemy Philadelphus, the son of the founder, it is reported to have contained no less than 10,000 volumes, and it continued to increase, until, at the time of its destruction, it contained between 700,000 and 800,000 volumes, among which were the original writings of Æschylus, Sophocles, and Euripides.

It has been maintained that the astronomical and other mathematical sciences of the Hindus were borrowed from the Greeks;[1] and

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[1] Professor Dugald Stewart, in his "Philosophy of the Human Mind," has adduced considerations to show that the Sanscrit language is derived from the Greek. He considers that the Sanscrit language was formed on the model of the Greek, by the Brahmins, after the invasion of India by Alexander the Great, and that it grew rapidly and attained its perfection as a new language a century before the Christian era.

The names of the countries, rivers, towns, &c., recorded in the writings of Quintius Curtius, Arrian, and others who have written on Alexander's expedition to the East, clearly evince to the reader that the Sanscrit language had an existence before the age of Alexander. And further, there are no evidences to show that the Greek language was in use in India, or that any writings in that language were extant anterior to those in Sanscrit.

Sanscrit words are found in all the various dialects spoken in India, and these dialects, so different from each other, and extending over very large tracts of country, may have been derived from a common primeval tongue, before the Sanscrit language became a language developed.

There are very striking affinities between the Sanscrit and the Greek and Latin languages, both in the words themselves and in their inflexions. All languages are further, that the Greek language, after the conquests of Alexander in the East, gave birth to the Sanscrit language. The settlement of a part of Alexander's army in Bactriana, and their government, which lasted for 130 years, might have influenced in some degree the populations of that and the neighbouring countries. It is not very likely that there was much intercourse between the rulers of Bactriana and the people of Greece. Of the numerous facts adduced in favour of the priority of Greek science, one may be here mentioned. It is true that the twelve signs of the zodiac are identical both in the astronomy of the Greeks and the Hindus.[1] But it is uncertain whether the Hindus

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subject to growth and improvement, and it cannot be conceived that all the words and constructions of the Greek, or any other language, were coeval with the earliest use of that language. This may be understood by a comparison of the English language in its present form, with the language in the time of Chaucer, and with the earlier form of the Anglo-Saxon.

The obvious affinity between Sanscrit, Greek, and Latin words, and the variety of inflected forms existing in Sanscrit, and almost the same found in Greek and Latin inflexions, afford at least a strong ground for the presumption that the tribes which founded the Greek and Latin colonies had brought with them a language, either the Sanscrit or some cognate dialect of that language. In fact, the common origin both of Sanscrit, Greek, and Latin, as well as their offshoots, must be referred to some primitive tongue. The resemblances in words, with their derivations and inflexions, notwithstanding the changes which these languages have undergone in the course of ages, supply an indication of their relation to the same family. Examples may be seen in Dr. Donaldson's New Cratylus, 3rd Ed., 1859.

From p. 477 of the third volume of E. T. Colebrooke's Essays, edited by Professor Cowell, the following note of the Professor is extracted:—"Dr. Kern gives a list of thirty-six Greek words which occur in Varahamihira's Vrihat Sanhita [Great Course of Astrology]. The signs of the zodiac (except Cancer), Kriya, Tavuri, Jituma, Leya, Pathena, Dyuka or Juka, Kaurpya, Taukshika, Akokera, Hridroga, Ittham:—Heli (ἥλιος), Himna (Ἑρμῆς), Ara (Ἄρης), Jyau (Ζεύς), Kona (Κρόνος), Asphujit (Ἀφροδίτη), hora, kendra, dreshkana or drekkana, lipta, anapha (ἀναφή), sunapha (συναφή), dumdhara (δορυφορία), kemadruma (χρηματισμός), vesi (φάσις), apoklima (ἀπόκλιμα), panaphara (ἐπαναφορά), hibuka (ὑπόγειον), jamitra (διάμετρος), meshurana (μεσουράνημα), dyunam or dyutam (δυτικόν ?), rihpha (ῥιφή), and harija (ὁρίζων)."

[1] The Hindu division of the ecliptic is the same as that of the Greeks. Their astronomical year was sidereal (being the space of time in which the sun departing from a star returns to the same), and commenced at the instant when the sun enters the first degree of the sign *Aries*, or the Hindu constellation *Mesha*. The inclination of the ecliptic to the equator is stated in the Surya Siddhanta to be twenty-four degrees.

It was the opinion of Sir William Jones that the Indian zodiac was not borrowed from the Greek; and if the solar divisions of it in India were the same as in Greece, it may reasonably be concluded, that both Greeks and Hindus had received it from an older nation which first gave names to the constellations, and from which both Greeks and Hindus, as the similarity of their language fully evinces, had a common origin. Sir William Jones received the following account from two learned Brahmins. "The Hindus divide the circle of the heavens into 360 portions, and allot 30 to each of the twelve constellations of the zodiac in this order:

received this knowledge from the Greeks, or the Greeks from the Hindus, or, what is perhaps the more probable, whether each originally received it from some independent source. There is, however, one fact which distinguishes Hindu mathematical science from that of the Greeks, and which has given both perfection and simplicity to the Hindu system of calculation. It is that which consists in "the device of place" in their arithmetical notation; a contrivance which gives to each figure employed, a power depending on the place it occupies, in addition to the value it is assumed to have by itself.

The Hindus themselves can give no historical account of the origin or discovery of the sciences of arithmetic and algebra, nor even of the inventor of the denary scale of numeral notation, one of the most simple, and at the same time the most perfect of inventions.

In the year 1817 Mr. Colebrooke published a translation of four

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Mesha, the Ram; Sinha, the Lion; Dhanus, the Bow; Vrisna, the Bull; Canya, the Virgin; Macara, the Sea Monster; Mithuna, the Pair; Tula, the Balance; Cumbha, the Ewer; Carcata, the Crab; Vrishchica, the Scorpion; Mina, the Fish." The figures of the twelve asterisms thus denominated with respect to the sun are specified in Sanscrit verses, of which the following is a verbal translation:—

"The Ram, Bull, Crab, Lion, and Scorpion have the figures of those five animals respectively: the Pair are a damsel playing on a vina, and a youth wielding a mace; the Virgin stands on a boat in the water, holding in one hand a lamp, in the other an ear of rice-corn; the Balance is held by a weigher with a weight in one hand; the Bow, by an archer, whose hinder parts are like those of a horse; the Sea Monster has the face of an antelope; the Ewer is a water-pot borne on the shoulder of a man, who empties it; the Fish are two, with their heads turned to each other's tails; and all those are supposed to be in such places as suit their several natures."

The Greek names of the constellations of the zodiac as recorded in the Phenomena of Aratus, are:—1. Κριὸς, the Ram: 2. Ταῦρος, the Bull: 3. Δίδυμοι, the Twins: 4. Καρκίνος, the Crab: 5. Λέων, the Lion: 6. Παρθένος, the Virgin: 7. Χηλαὶ, the Claws: 8. Σκορπίος, the Scorpion: 9. Τοξευτὴς, the Archer: 10. Αἰγόκερως, the Horned Goat: 11. Ὑδροχόος, the Man that holds the watering urn: 12. Ἰχθύες, the fishes. Eudoxus, from whose work Aratus composed his poem entitled "The Phenomena," made Χηλαὶ, the claws, to occupy the seventh, and the remaining portion of the Scorpion, the eighth division of the zodiac. Aratus gives to this sign the name of μεγαθήριον (line 82), the great beast. After the time of Aratus, *Libra, the Balance*, was made to occupy the eighth division of the zodiac; and Virgil (Georg. i. 32—36) suggests that the Scorpion had drawn in his claws to make room for the constellation *Libra*, in honour of Augustus. In Gen. xxxvii. 9, can there be any allusion to the eleven constellations of the zodiac?

In the Hindu system of astronomy the planets are: Surya, the Sun; Budha, Mercury; Sucra, Venus; Mangala, Mars; Vrihaspati, Jupiter; Sani, Saturn; Chandra, the Moon. These are supposed to move in their respective orbits at the same rate; the dimensions of the moon's orbit being known, those of other planets are determined, according to their periodical revolutions, by proportion. The Hindus have observed that the moon revolves once on her axis in a lunar month, and has the same side always opposed to the earth. They have also noticed the difference of her apparent magnitude on the horizon and on the meridian.

ancient treatises on Arithmetic and Algebra,[1] written in the Sanscrit language, with a learned dissertation on those subjects. Two of these treatises, one on Arithmetic and the other on Algebra, constitute the twelfth and eighteenth chapters of one of the systems of Hindu Astronomy, entitled "Brahma-Siddhanta." The fact of these treatises being found in the midst of a system of astronomy, where the principles of one science are employed to aid in the development of another, affords at least a presumption that they were not of recent invention, but had existed in a former age, and had passed through stages of improvement. This presumption is supported by the fact that these treatises are composed in Sanscrit verse.

It may also be remarked that almost all the examples in the treatise on Algebra relate to astronomy. The author has given his own date in the Brahmasphuta Siddhanta in the following form: "In the reign of Sri Vyaythramukha, of the Sri Chapa dynasty, 550 years (A.D. 628) after the Saka king (Salivahana) having passed, Brahmegupta, the son of Jishnu, at the age of thirty, composed the Brahmegupta Siddhanta, for the gratification of mathematicians and astronomers."[2]

Mr. Colebrooke considers that these treatises are not the oldest composed by Hindu writers, and remarks that Ganesa, a learned commentator, has quoted a passage from a work of Aryabhatta (an older writer than Brahmegupta), and that another commentator has named him as one of the earliest writers on these sciences. Dr. Bhau Daji, in his essay, has shown that Aryabhatta was born A.D. 476, as given by himself in the Aryashtasata. He calls himself a native of Kusumapura, or Pataliputra. His work is written with great attention to conciseness. Varaha-Mihira flourished about A.D. 505, and died A.D. 587. He has cited Aryabhatta by name in a passage given by Bhatta Utpala, and quoted in the commentary on the Varahi Sanhita. It is highly probable that the Romaka Siddhanta was composed about A.D. 505, and Varaha-Mihira founded his Pancha Siddhantika Karana on the Romaka Siddhanta, and four other works. It is therefore clear that Varaha could not have lived before A.D. 505. It appears that Aryabhatta affirmed the diurnal rotation of the earth on its axis, and calculated its circumference, which, when reduced, was equivalent to 25,080 English miles; that he possessed the true theory of lunar and solar eclipses,[3] affirming the moon and primary planets to be illuminated by the sun; that he noticed the motion of the solstitial and equinoctial points; and that he recognised a motion of the nodes and apsides of all the primary planets as well as of the moon. His work on Astronomy embraced treatises on Arithmetic, Algebra, and Geometry.

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[1] Algebra, with Arithmetic and Mensuration, from the Sanscrit of Brahmegupta and Bhascara, translated by Henry Thomas Colebrooke, Esq., F.R.S., &c. London, 1817. (Reviewed in the "Edinburgh Review" for November, 1817.)

[2] See Journal of the Asiatic Society, 1865. Brief notes on the age and authenticity of the works of Aryabhatta, Varaha-Mihira, Brahmegupta, Bhatta Utpala, and Bhascar-Acharya. By Dr. Bhau Daji.

[3] See Journal of the Asiatic Society, 1863, vol. ix. On the Surya Siddhanta,

Mr. Colebrooke was led to consider that if Brahmegupta was not the oldest writer on these sciences, he improved and extended the knowledge of them as they existed in his time. In the age of Meya, the author of the Surya Siddhanta, the moon's horizontal parallax was made to be 53′ 20″, which gave the mean distance from the earth 220,184 in English geographical miles, whereas European astronomers compute the mean distance of the moon about 240,000. The Hindus took no notice of the errors arising from refraction, and their taking the moon's motion as along the sine instead of its arc, may be noted. But they were not wholly ignorant of optics; they knew the angles of incidence and refraction to be equal, and computed the place of a star or planet as it would be seen reflected from water or a mirror.

There is also a perpetual commentary on the work of Brahmegupta, composed by Chaturveda Prithudaca Swami. He successively quotes at length every verse of the text, interprets it word by word, and subjoins elucidations and remarks. It is impossible to doubt that the knowledge of these sciences had a wide extension long before they assumed the form in which they are found exhibited.

Of the treatises of Brahmegupta forming the twelfth and eighteenth chapters of his Astronomy, the former begins with the declaration: "He who distinctly and severally knows addition and the rest of the twenty logistics, and the eight determinations, including measurement by shadow, is a mathematician."

It consists of ten sections, which are respectively entitled, Algorithm, Mixture, Progression, Plane Figure, Excavations, Stacks, Law, Mounds of Grain, Measure by Shadow, Supplement. The eighteenth chapter consists of eight sections, which are named Pulverizer, Algorithm, Simple Equation, Quadratic Equation, Equation

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and the Hindu Method of Calculating Eclipses, by William Spottiswoode, M.A., F.R.S., &c.

In the second volume of the Asiatic Researches, printed at Calcutta, is an essay on the Astronomical Computations of the Hindus. This essay was drawn up by Samuel Davis, Esq., whose curiosity had been raised to know by what means the prediction of eclipses and other celestial phenomena, published in the Hindu Patra or Almanac, had been effected. He procured a copy of the Surya Siddhanta, an ancient treatise on Astronomy, and a commentary on the text, with other Sanscrit works on the subject. The object of Mr. Davis's essay is to exhibit a computation of an eclipse of the moon which was predicted in the Hindu Patra to happen in the month of November, 1789, not merely on the principles, but strictly by the rules of the Surya Siddhanta. His essay, with the computations, extends over sixty-three pages.

of several unknown, Equation involving a Factum, Square affected by a Coefficient, Problems. The author concludes the last section in these words: "These questions are stated merely for gratification. The proficient may devise a thousand others; or may resolve by the rules taught, problems proposed by others. As the sun obscures the stars, so does the proficient eclipse the glory of other astronomers in an assembly of people, by the recital of algebraic problems, and still more by their solution."

The treatises of Brahmegupta, on Arithmetic and Algebra, are two of the most ancient known to be extant; but they are not so complete nor extensive in the subjects they embrace as the other two which form the first portion of Mr. Colebrooke's publication. These are entitled the Lilavati and the Vija Ganita, the former a treatise on Arithmetic, and the other a treatise on Algebra. They constitute the introduction to a course of Astronomy[1] entitled Siddhanta Siromani. The author,

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[1] "Bhascara argues, that it is more reasonable to suppose the earth to be self-balanced in infinite space, than that it should be supported by a series of animals, with nothing assignable for the last of them to rest upon; and Nerasinha, in his commentary, shows that by Rahu and Cetu, the head and tail of the monster, in the sense they generally bear, could only be meant the position of the moon's nodes, and the quantity of her latitude, on which eclipses do certainly depend; but he does not therefore deny the reality of Rahu and Cetu; on the contrary, he says that their actual existence and presence in eclipses ought to be believed, and may be maintained as an article of faith without any prejudice to astronomy."

Mr. Davis adds the following translation of a sentence from the Sanscrit on the controversy:—

"Fruitless are all other Sastras; in them is contention only:
Fruitful is the Jyotish Sastra, where the sun and moon are two witnesses."

"There are certain learned pundits who have truer notions of the earth and the economy of the universe than are ascribed to the Hindus in general; and that they must reject the ridiculous belief of the common Brahmins, that eclipses are occasioned by the intervention of the monster *Rahu*, with many other particulars equally unscientific and absurd. But as this belief is founded on explicit and positive declarations contained in the Vedas and Puranas, the divine authority of which writings no devout Hindu can dispute, the astronomers have, some of them, cautiously explained such passages in those writings as disagree with the principles of their own science; and, when reconciliation was impossible, have apologised, as well as they could, for propositions necessarily established in the practice of it, by observing that certain things, as stated in other *Sastras*, 'might have been so formerly, and may be so still; but for astronomical purposes, astronomical rules must be followed.' Others have, with a bolder spirit, attacked and refuted unphilosophical opinions."

Sir William Jones has given the following account of the two systems, which he received from a venerable Hindu mathematician:—

"The Pauranics will tell you that our earth is a plane figure studded with eight mountains, and surrounded by seven seas of milk, nectar, and other fluids; that the part which we inhabit is one of seven islands, to which eleven smaller isles are subordinate; that a god, riding on a huge elephant, guards each of the eight regions; and that a mountain of gold rises and gleams in the centre; but we believe the

Bhascara Acharya, thus gives the date of his work: "In the year 1036 (A.D. 1114) of the Saka king I was born; and at the age of thirty-six I composed the Siddhanta Siromani."

In the age of Brahmegupta there appear to have existed five very ancient treatises on Astronomy, which also embraced Astrology, under the names of Paulisa-Siddhanta, Romaca-Siddhanta, Vasishtha-Siddhanta, Surya-Siddhanta,[1] and Brahma-Siddhanta. It was from

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earth to be shaped like a cadamba fruit, or spheroidal, and admit only four oceans of salt water, all of which we name from the four cardinal points, and in which are many great peninsulas, with innumerable islands. They will tell you that a dragon's head swallows the moon, and thus causes an eclipse; but we know that the supposed head and tail of the dragon mean only the nodes, or points formed by intersections of the ecliptic and the moon's orbit. In short, they have imagined a system, which exists only in their fancy; but we consider nothing as true, without such evidence as cannot be questioned."

The following passage, which appeared in the *Times* of March 28, 1877, is quoted to show that faith in Rahu and Cetu, with other superstitions, such as judicial astrology, still hold a potent spell in the eastern parts of the world:—

"On the recent eclipse of the moon, the Turks at Constantinople were reported to have fired guns, according to traditional custom, in order to frighten the dragon devouring that luminary into releasing its prey. A letter from Constantinople in the *Journal de Genève* records another illustration of Oriental beliefs. It states that the opening of the new parliament was postponed from the 13th to the 19th, not, as represented, to give the deputies time to arrive, but because the astrologer of the seraglio reported against the original date, and recommended not only the day, but the precise hour of the ceremony."

[1] In the first Section of the Surya Siddhanta, it is stated that "Time of the denomination of *Murta* (sidereal time) is estimated by respirations; six respirations make a *vicala* (second); sixty *vicalas* a *danda;* sixty *dandas* a *nacshatra* (sidereal) day; and thirty nacshatra days a nacshatra month. The *Savan* (solar) month is that contained between thirty successive risings of *Surya* (the Sun), and varies in its length according to the Lagna Bhirja (right ascension). Thirty *tithis* (lunar) days compose the *chandra* (lunar) month. The *Saura* month is that in which the sun describes one sign of the zodiac, and his passage through the twelve signs is one year, and one of those years is a *Deva* day, or day of the gods."

Sir William Jones, in an essay on the Chronology of the Hindus, written in 1788, and printed in the second volume of the Asiatic Researches, has shown how far the great antiquity of the Hindus, so firmly believed by themselves, is worthy of credit. He gives a concise account of Indian chronology, extracted from Sanscrit books, or conversations with pundits, and subjoins a few remarks on their system, without attempting to decide the question which he ventured to start—"Whether it is not in fact the same with our own, but embellished by the fancy of their poets and the riddles of their astronomers."

After relating incredible stories and wonderful periods of time, he writes:—"Let us compare the two Indian accounts of the Creation and the Deluge with those delivered by Moses. It is not made a question in this tract, whether the first chapters of Genesis are to be understood in a literal or merely allegorical sense. The only points are, whether the creation described by the first *Menu,* which the Brahmins call that of the *Lotos,* be not the same with that recorded in our Scriptures? and whether the story of the seventh *Menu* be not one and the same with that of Noah? I propose the questions, but affirm nothing; leaving others to settle their these treatises that Brahmegupta compiled his treatise on Astronomy. Also from these five treatises Varaha-Mihira, a writer of an earlie

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opinions, whether Adam be derived from *adim*, which in Sanscrit means *the first*, or Menu from Nuh, the true name of the patriarch; . . . . whether the two Menus can mean any other persons than the great progenitor and the restorer of our species."

In the third volume of the Asiatic Researches, Lieutenant Wilford printed an essay on Egypt and the Nile, from the ancient books of the Hindus, with some brief remarks by Sir William Jones. In those remarks is given the following literal translation of a passage of the Padma-Puran :—"To Satyavarman, that sovereign of the whole earth, were born three sons: the eldest, Sherma; then Charma; and thirdly, Jyápeti by name. They were all men of good morals, excellent in virtue and virtuous deeds, skilled in the use of weapons to strike with or be thrown; brave men, eager for victory in battle. But Satyavarman, being continually delighted with devout meditation, and seeing his sons fit for dominion, laid upon them the burden of government, whilst he remained honouring and satisfying the gods, and priests, and kine. One day, by the art of destiny, the king having drunk mead, became senseless, and lay asleep naked; then was he seen by Charma, and by him were his two brothers called, *To whom he said*: 'What now has befallen? In what state is this our sire?' By those two was he hidden with clothes, and called to his senses again and again. Having recovered his intellect, and perfectly knowing what had passed, he cursed Charma, saying, 'Thou shalt be the servant of servants; and, since thou wast a laugher in their presence, from laughter shalt thou acquire a name.' Then he gave to Sherma the wide domain on the south of the snowy mountain, and to Jyápeti he gave all on the north of the snowy mountain; but he, by the power of religious contemplation, attained supreme bliss." On this he remarks: "Now you will probably think that even the conciseness and simplicity of this narrative are excelled by the Mosaic relation of the same adventure; but, whatever may be our opinion of the old Indian style, this extract most clearly proves that the Satyavrata, or Satyavarman of the Purans, was the same personage as the Noah of Scripture, and we consequently fix the utmost limit of Hindu chronology; nor can it be with reason inferred that the divine legislator borrowed any part of his work from the Egyptians. He was deeply versed, no doubt, in all their learning, such as it was; but he wrote what he knew to be truth itself, independently of their tales, in which truth was blended with fables; and their age was not so remote from the days of the patriarch, but that every occurrence in his life might naturally have been preserved by tradition from father to son."

If the longevity of human life during the period before the deluge be admitted, and the traditional genealogies in the book of Genesis be correct, it may be shown that the early traditions therein recorded from the time of Adam to Moses passed through seven intermediate persons. See Gray's Key to the Old Testament, pp. 80, 81.

In a supplementary note, Sir William Jones adds :—"But whatever may be the comparative antiquity of the Hindu Scriptures, we may safely conclude that the Mosaic and Indian chronologies are perfectly consistent; that Menu, son of Brahma, was the Adima, or first created mortal, and consequently our Adam; that Menu, child of the sun, was preserved with seven others in a *bahitra*, or capacious ark, from an universal deluge, and must therefore be our Noah; that Hiranyacasipu, the giant with a golden axe, and Vali cr Bali, were impious or arrogant monarchs, and most probably our Nimrod and Belus; that the three Ramas, two of whom were invincible warriors, and the third, not only valiant in war, but the patron of agriculture and wine, which derive an epithet from his name, were different repreage, is understood to have compiled his astronomical treatise named Pancha-Siddhanta.

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sentations of the Grecian Bacchus, and either the Rama of Scripture, or his colony personified, or the sun first adored by his idolatrous family; that a considerable emigration from Chaldea into Greece, Italy, and India happened about twelve centuries before the birth of our Saviour; that Sacya or Sisak, about two hundred years after Vyasa, either in person or by a colony from Egypt, imported into this country [Bengal] the mild heresy of the ancient Bauddhas; and that the dawn of true Indian history appears only three or four centuries before the Christian era, the preceding ages being clouded by allegory or fable."

Lieutenant (since Captain) Wilford employed a learned Brahmin to make extracts from Sanscrit books to aid him in a second essay he was writing on the Sacred Isles of the West. In the course of collating the Sanscrit authorities quoted or referred to, Captain Wilford discovered some discolorations in the manuscripts, which led him to suspect deception, which examination fully verified. This discovery naturally excited his apprehension that a similar imposition had been practised upon him, both with respect to the extracts for his former essay on Egypt and the Nile, and that the Purana, in which he had actually and carefully read the passage which he communicated to Sir William Jones, as an extract from it, does not contain that passage; and that it was interpolated by the dextrous introduction of a forged sheet, discoloured, and prepared for the purpose of deception, and which, having served this purpose, was afterwards withdrawn. The forgeries of the pundit (Captain Wilford observes) were of three kinds: 1. A word or two was only altered. 2. Such legends as had undergone a more material alteration. 3. All those which he had written from memory. With regard to those of the first class, when he found that I was resolved to make a collation of the manuscript, he began to adulterate and disfigure his own manuscript, mine, and the manuscripts of the college, by erasing the original name of the country, and putting that of Egypt or of Swetam in its place.

"To prevent my detecting those of the second class, which were not numerous, but of the greatest importance in their nature (and as books in India are not bound as in Europe, and every leaf is loose), he took out one or two leaves, and substituted others with an adulterous legend. In books of some antiquity, it is not uncommon to see a few new leaves inserted in the room of others that were wanting.

"To conceal the more numerous impositions of the third class, he had the patience to write two voluminous sections, supposed to belong, one to the Scanda-Purana, and the other to the Bramanda, in which he connected all the legends together, in the usual style of the Puranas. These two sections, as he wrote them, consist of no less than 12,000 slocas or lines, the title of which he borrowed.

"The first imposition is a legend of the greatest importance, and is said to be extracted from the Padma-Purana. It contains the history of Noah and his three sons, and is written in a masterly style. But unfortunately there is not a word of it to be found in that Purana."

The following is the passage referred to from Captain Wilford's Essay, pp. 312, 313:—"It is related in the Padma-Puran, that Satyavrata, whose miraculous preservation from a general deluge is told at length in the Matsya, had three sons, the eldest of whom was named Jyâpeti, or lord of the earth. The others were Charma and Sharma, which last words are, in the vulgar dialects, usually pronounced Cham and Sham; as we frequently hear Kishu for Crishna. The royal patriarch (for such is his character in the Purans) was particularly fond of Jyâpeti, to whom he gave all the regions to the north of Himalaya, or the Snowy Mountains, which extend from sea to sea, and of which Caucasus is a part. To Sharma he allotted the countries to the south of these mountains. But he cursed Charma; because, when

Aryabhatta, an astronomer of a still earlier age, is quoted by succeeding writers in support of the precession of the equinoxes. Though Brahmegupta in his treatise is silent on the subject, Bhascara and others have affirmed a periodical revolution of the places of the colures. The reason of the omission or denial of this periodical motion is attributed by Bhascara to the very small quantity of the deviation, and he states, in reference to Brahmegupta, that "in mathematical science, holy tradition is authority so far only as it agrees with demonstration," and adds: "Such motion as results from the assigned revolutions must be admitted, when the places being calculated agree with those which are observed, whether taught by a holy sage or by a temporal teacher. If, then, the same places are deducible from other revolutions, which of the assigned motions is the true one? The answer is, whichever agrees with the present observation must be admitted."

The author, Bhascara-Acharya,[1] informs his readers that his work is a compilation, and that his own corrections and improvements are not very numerous nor important. In the Vija Ganita (Section 131) he has quoted a passage from Sridhara's Algebra, and another (Section 142) from Padmanabha's.

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the old monarch was accidentally inebriated with a strong liquor made of fermented rice, Charma laughed; and it was in consequence of his father's imprecation that he became a slave to the slaves of his brothers. The children of Sherma travelled a long time, until they arrived at the bank of the Nile, or Cali; and a Brahmin informs me (but the original passage from the Puran is not yet in my possession) that their journey began after the building of the Padma-Mandira, which appears to be the Tower of Babel, on the bank of the River Cumudvati, which can be no other than the Euphrates."

[1] The following extract is taken from the translation of an inscription discovered by Dr. Bhau Daji near Chalisgam, at the foot of the hills which contain the Peetulkhora caves. The inscription contains the names of several of Bhascara-Acharya's descendants, who taught his works in a college endowed in the neighbourhood of Chalisgam. The original Sanscrit, and the translation, are printed in the Journal of the Asiatic Society for 1865:—

"Full of good fame and merit was Bhascara. The learned Bhascara's son was Lakshmidhara, the first among the learned; acquainted with the meaning of the Vedas, the first among metaphysicians, and skilful in the knowledge of sacrificial ceremonies. Jaitrapala having recognised him as well versed in the meanings of all the Sastras, took him from this town and made him the chief of pundits. His son was Changadeva, the best of the astronomers and astrologers at the court of Singhana Chakravartin. Changadeva constructs the College Matha for the spread of the treatises composed by Bhascara-Acharya. His works, the chief of which is the Siddhanta Siromani, and the works of his ancestors and descendants, ought to be duly studied in my college. Sonhadeva granted ground, with gold, &c., to the college. Others also have made some grants. Future kings ought to protect this, for the increase of merit. In the year 1128 Saka, in the year Prabhava, in the Sravana month, full moon, on the occasion of a lunar eclipse, Sonhadeva, in the presence of the people, having thrown water into the hands, granted to the college of his preceptors as follows, &c."

The following is the conclusion of the Vija Ganita, and forms the ninth chapter of Mr. Colebrooke's translation :—

"On earth was one named Maheswara, who followed the eminent path of a holy teacher among the learned. His son, Bhascara, having from him derived the bud of knowledge, has composed this brief treatise of elemental computation.

"As the treatises of Algebra by Brahmegupta, Sridhara, and Padmanabha are too diffusive, he has compressed the substance of them in a well-reasoned compendium for the gratification of learners.

"For the volume contains a thousand lines, including precept and example. Sometimes exemplified to explain the sense and bearing of a rule; sometimes to illustrate its scope and adaptation; one while to show variety of inferences; another while to manifest the principle. For there is no end of instances, and therefore a few only are exhibited. Since the wide ocean of science is difficultly traversed by men of little understanding; and on the other hand, the intelligent have no occasion for copious instruction. A particle of tuition conveys a science to a comprehensive mind; and having reached it, expands of its own impulse. As oil poured on water, as a secret entrusted to the vile, as alms bestowed upon the worthy, however little, so does science infused into a wise mind spread by intrinsic force.

"It is apparent to men of clear understanding, that the rule of three terms constitutes Arithmetic; and sagacity, Algebra. Accordingly I have said in the chapter on Spherics:

"The rule of three is Arithmetic; spotless understanding is Algebra. What is there unknown to the intelligent? Therefore, for the dull alone it is set forth.

"To augment wisdom and strengthen confidence, read, do read, mathematician, this abridgment, elegant in style, easily understood by youth, comprising the whole essence of computation, and containing the demonstration of its principles, replete with excellence and void of defect."

There are extant several commentaries on the work of Bhascara, of which the oldest appears to have been composed by Gangadhara about the year A.D. 1420.

The commentary of Ganesa (the most eminent scholiast of Bhascara) on the Lilavati bears a date which corresponds to A.D. 1545. It embraces a copious exposition of the Sanscrit text of the Lilavati, with demonstrations of the rules. Mr. Colebrooke informs his readers that he has used this commentary throughout his translation as the best interpreter of the original. Ganesa has quoted a passage of Aryabhatta, denoting the science of Algebra under the name of VIJA, and has made mention of a method for the solution of indeterminate problems.

Crishna wrote a commentary on the Vija Ganita, which bears the date of A.D. 1602. This commentary contains a clear and copious exposition of the sense of the text, with ample demonstrations of the rules, in the same manner as the commentary of Ganesa on the Lilavati.

Suryadasa was the author of a Complete Commentary on the Siddhanta Siromani. His notes on the Lilavati bear a date corresponding to A.D. 1538, and those on the Vija Ganita to A.D. 1541. He considered Aryabhatta to be at the head of the older writers on Algebra. His rule for the solution of quadratic equations by means of completing the square is given in the Section 131 of Mr. Colebrooke's translation. Ranganatha, another scholiast, appears to have flourished about the year A.D. 1621.

These are the principal of the numerous commentaries in Sanscrit on the Lilavati and the Vija Ganita, which acquired in past times a high reputation, both in Hindustan and in countries beyond its borders, and superseded the use of all the preceding treatises on the subjects of Arithmetic and Algebra.

The Vija Ganita is more complete than the treatise of Brahmegupta, and consists of nine chapters. It contains an explanation of the nature and distinction between affirmative and negative quantities known and unknown, and of the ordinary operations of the science, also the arithmetic of surd numbers. It gives methods for the solution of simple and quadratic equations, and of indeterminate equations of the first and second degree. It also explains the application of algebra to geometrical figures.

It may be added, that the preceding treatise, the Lilavati,[1]

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[1] See the author's "Elementary Arithmetic," with some notices of its history, Section 1, pp. 9—12, where some account is given of the Lilavati, which was first published in English with the following title: "Lilawati, or a Treatise on Arithmetic and Geometry by Bhascara-Acharya, translated from the original Sanscrit by John Taylor, M.D., of the Honourable East India Company's Bombay Medical Establishment. Bombay, 1816."

In the preface Dr. Taylor states that the object of his translation is to furnish an authentic document, which, by exhibiting not only the actual degree of mathematical knowledge possessed by the Hindus in the twelfth century, but also, by showing their modes and principles of operation, may lead to a fair conclusion regarding their pretensions to originality in this department of science.

Dr. Taylor has printed an Appendix to his translation of the Lilavati, giving an account of the mode of teaching arithmetic in the Hindu schools of the Mahratta country and Guzerat. The following extract is taken from this Appendix :—

"But what chiefly distinguishes the Hindu schools is the plan of instruction by the scholars themselves. When a boy joins the school, he is immediately put under the tuition and care of one who is more advanced in knowledge, and whose duty it is to give lessons to his young pupil, to assist him in learning, and to report his behaviour and progress to the master. The scholars are not classed as with us, but generally paired off, each pair consisting of an instructor and pupil. These pairs are so arranged that a boy less advanced may sit next to one who has made greater progress, and from whom he receives assistance and instruction. When, however, gives the application of arithmetic to geometry, and rules for the extraction of the square root and the cube root.

The fifth chapter of the Lilavati treats of progressions, arithmetical and geometrical. Rules are given for finding the sum, the first term, the common difference, and the number of terms of an arithmetic series.

The following rule given for finding the number of terms, requires the solution of a quadratic equation :—

"From the sum of the progression multiplied by twice the common increase, and added to the square of the difference between the first term and half that increase, the square root being extracted, this root less the first term and added to the (above-mentioned) portion of the increase, being divided by the increase, is pronounced to be the period."

One of the commentators is quoted for a method of finding the sum of an arithmetic progression, grounded on placing the numbers of the series in the reversed order under the direct one ; where it becomes obvious, that each pair of terms, multiplied by the number of terms, is twice the sum of the progression.

The following rules are also given for finding the sum of the squares, and the sum of the cubes of any series of the natural numbers.

Twice the number of terms added to one and divided by three, being multiplied by the sum of the natural numbers, is the sum of the squares.

The sum of the cubes of the numbers one, two, &c., is pronounced by the ancients equal to the square of the sum of the series, one, two, &c.

The section on geometrical progressions is restricted to progressions increasing by a multiplier, and omits altogether any notice of decreasing geometrical progressions.

The rules for variations, permutations, and combinations are given

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several of the elder boys have made considerable and nearly equal progress, they are seated together in one line, and receive their instructions directly from the master.

"The plan of getting the older boys, and those who are more advanced, to assist those who are less advanced and younger, greatly lessens the burden imposed upon the master, whose duty, according to this system, is not to furnish instruction to each individual scholar, but to superintend the whole, and see that every one does his duty. If the younger boy does not learn his lessons with sufficient promptitude and exactness, his instructor reports him to the master, who inquires into the case, orders the pupil to repeat before him what he has learned, and punishes him if he has been idle or negligent. As the master usually gives lessons to the elder scholars only, he has sufficient leisure to exercise a vigilant superintendence over the whole school, and by casting his eyes about continually, or walking up and down, and inquiring into the progress made by each pupil under his instructor, he maintains strict discipline, and keeps every one on the alert through expectation of being called upon to repeat his lesson."

in the sixth section of the fifth chapter, and in the thirteenth chapter of the Lilavati.

The following rule for combinations is taken from the Persian version, as rendered in English by Mr. Strachey :—

"First, write them all with one, in order, and below write one, the last opposite to the first in order. Then divide the first term of the first line by the number which is opposite to it in the second line. The quotient will be the number of combinations of that thing. Multiply this quotient by the second term of the first line, and divide the product by the number which is opposite to it in the second line, the quotient will be the number of combinations of that thing; and multiply this quotient by the third term, and divide by that which is below it, and add together whatever is obtained below each term, the sum will be the amount of all the combinations of these things.[1]

The first section of the first chapter of the Vija Ganita opens with the following invocation, as translated by Mr. Colebrooke :—

"I revere the unapparent primary matter, which the Sanchyas declare to be productive of the intelligent principle, being directed to that production by the sentient being; for it is the sole element of all which is apparent. I adore the ruling power, which sages conversant with the nature of soul pronounce to be the cause of knowledge, being so explained by a holy person; for it is the one element of all which is apparent. I venerate that unapparent computation, which calculators affirm to be the means of comprehension, being expounded by a fit person; for it is the single element of all which is apparent. Since the arithmetic of apparent [or known] quantity, which has been already propounded in a former treatise [Lilavati], is founded on that of unapparent [or unknown] quantity; and since questions to be solved can hardly be understood by any, and not at all by such as have dull apprehensions, without the application of unapparent quantity; therefore I now propound the operations of analysis."

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[1] The following is one of the examples given (ch. v., sect. 6) to illustrate the rule. How many are the combinations in one composition, with ingredients of six different tastes, sweet, pungent, astringent, sour, salt, and bitter, taking them one by one, by twos, by threes, &c.?

Write them thus, $\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{array}$

$\frac{6}{1}=6$ combinations when taken singly.

$\frac{6\times5}{2}=15$, when taken two and two together.

$\frac{15\times4}{3}=20$, when taken three and three.

$\frac{20\times3}{4}=15$, when taken four and four.

$\frac{15\times2}{5}=6$, when taken five and five.

$\frac{6\times1}{6}=1$, when taken all six together.

And the whole number is 63.

In these oldest treatises known to be extant of the Arithmetic and Algebra of the Hindus, their writers describe the analytic art as "a method of calculation attended with an exhibition of its principles, and aided by literal signs and other devices."

The initial syllables of words and of the names of colours, as well as single letters, were assumed to denote unknown numbers. A negative quantity was distinguished by a dot or point placed over the syllable or number, and a positive quantity was known by the absence of the dot.

There is no account given of the origin of this convention. The Sanscrit words translated *affirmative quantity* and *negative quantity*, have the literal meanings of *property or wealth*, and *debt or loss*, and imply a contrariety with respect to the possessor. Likewise two distances, one towards the right hand and the other towards the left hand, have also a contrariety of direction with respect to a person in a fixed position. If it be admitted that symbols, without any additional mark, denote affirmative quantities, then any mark, as a dot, may be employed to denote negative quantities: and this convention is sufficient, as an adjective in ordinary language, to distinguish that the symbol so marked is to be considered of a contrary character to that which is not so marked.

Crishna, a commentator on the Vija Ganita, gives the following illustrations of affirmative and negative quantities. Here negation is of three sorts, according to place, time, and things. It is, in short, contrariety. Wherefore the Lilavati, Sect. 166, expresses "the segment as negative, that is to say, is in the contrary direction." Thus of two countries, east and west; if one be taken as positive, the other is relatively negative. As an example, the situation of Patna and Allahabad relatively to Benares. Patna, on the Ganges, bears east of Benares, distant 15 yojanas, and Allahabad bears west of the same place 8 yojanas. The interval or difference between Patna and Allahabad is 23 yojanas, and is not obtained but by the addition of the numbers. Therefore, if the difference between two contrary quantities be required, their sum must be taken. Next, so when motion to the east is assumed to be positive; if a planet's motion be westward, then the number of degrees equivalent to the planet's motion is negative. In like manner, if a revolution westward be affirmative, so much as a planet moves eastward, is, in respect of a western revolution, negative. The same may be understood in regard to south and north, &c. That prior and subsequent times [reckoned from a fixed epoch] are relatively to each other as positive and negative, is familiarly understood in the reckoning of days. Thirdly. So in respect of chattels; that to which a man bears the relation of owner, is considered positive with regard to him; and the contrary [or negative quantity] is that to which another person has the relation of owner. Hence, so much as belongs to Yajnyadatta in the wealth possessed by Devadatta, is negative in respect of Devadatta.

A known quantity which is always a number, when employed as a coefficient, is written after the character which denotes the unknown quantity; but when it is not so used, but simply as a number, it is denoted by *ru*, the first syllable of *rupa*, form or species.

The first syllable of *ya* of *yavat-tavat*, the Sanscrit term for "tantum quantum," "as much as," "as many as," is used to denote an unknown quantity, when only one is employed. Thus *ya* 2 is denoted by $2x$ in modern notation.

If there be more than one unknown quantity required, the initial syllables *ca*, *ni*, *pi*, *lo*, &c., of the names of the colours, black, blue, yellow, red, &c., are used to denote the second, third, &c., unknown quantities, as *ya* $\dot{6}$ *ca* $\dot{4}$ *ni* $\dot{2}$ *ru* 2 signifies $-6x-4y-2z+2$ in modern notation.

The syllables *va*, *gha*, being the initial syllables of the Sanscrit of square and cube, are employed to denote the second and third powers, and these are combined to denote the higher powers, which are reckoned by the products, and not by the sums of the lower powers, as in the modern algebra. A surd root is in the same manner denoted by an initial syllable.

Sometimes the initial letters of the words are employed instead of the initial syllables.

The Hindus had also a figurative method of expressing numbers by the names of the objects, of which a certain number was generally known; as, for instance, the name of the sacred fire stands for 3, the arrow of Kamedeva for 5, the treasure of Kuvera for 9, the sun in reference to the month of the year for 12, the lunar mansion for 27, &c.; and larger numbers were made up by combining these names, as may be seen in the Lilavati.

They had no symbols to denote the operations of addition and subtraction. They simply declared that in the addition of two negative or two affirmative quantities, the sum must be taken; but the sum of an affirmative and negative quantity, the difference is their addition. And in subtraction, the quantity to be subtracted being affirmative, becomes negative; or being negative, becomes affirmative; and the addition of the quantities then taken, constitutes their difference.

As the Hindu algebraists employed no symbols to denote the operations of addition and subtraction, so likewise they had no marks to indicate multiplication and division. They exhibit the following directions for determining the nature of products and quotients as affirmative or negative.

The product of a negative quantity and an affirmative is negative; of two negative quantities is positive; of two affirmative quantities is affirmative.[1] The product of cipher and a negative quantity, or of cipher and an affirmative quantity, is zero.

And a positive quantity divided by a positive, or a negative quantity by a negative, is affirmative. A positive quantity divided by a negative is negative; and a negative quantity divided by an affirmative is negative.[2]

The division of one number by another was indicated by writing the dividend above the divisor without any line or mark of separation. The same notation was adopted to denote fractions by placing the denominator under the numerator. The rule for division, the reverse of multiplication, is thus stated:—"Those colours or unknown quantities and absolute numbers, by which the divisor being multiplied, the products in their several places subtracted from the dividend exactly balance it, are here the quotients in division."

The product of any quantity by cipher is cipher, and the product of cipher by any quantity is cipher. And the quotient of cipher divided by any quantity is cipher; but any quantity divided by cipher becomes a fraction, the denominator of which is cipher. On this subject one of the commentators remarks:—"If the dividend be diminished, the divisor remaining the same, the quotient is reduced, and if the dividend be reduced to cipher, the quotient becomes cipher. Again, as much as the divisor is diminished, the dividend remaining the same, so much the quotient is increased, and if the divisor be reduced to the utmost, the quotient is to the utmost increased. But if it can be specified that the amount of the quotient is so much, it has not been raised to the utmost; for a quantity greater than that can be assigned. The quotient therefore is indefinitely great, and is rightly termed infinite."

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[1] The following note is subjoined by Mr. Colebrooke:—

Multiplication, as explained by the commentators, is a sort of addition resting on repetition of the multiplicand as many times as is the number of the multiplier. Now a multiplier is of two sorts, positive or negative. If the multiplier be positive, the repetition of the multiplicand, which is affirmative or negative, will give correspondently an affirmative or negative product. The multiplication, then, of positive quantities is positive; and that of a negative multiplicand by a positive multiplier is negative. The question for disquisition concerns a negative multiplier. It has been before observed that negation is contrariety. A negative multiplier, therefore, is a contrary one; that is, it makes a contrary repetition of the multiplicand. Such being the case, if the multiplicand be positive, and the multiplier be negative, the product will be negative; if the multiplicand be negative and the multiplier be negative, the product will be affirmative. In the latter case, the multiplication of two negative quantities gives an affirmative product.

Multiplication was exhibited as in this following example:—

Statement *ya* 5 *ru* $\dot{1}$
ya 3 *ru* 2 product *ya v* 15 *ya* 7 *ru* $\dot{2}$.

In modern notation, the statement of the operation is equivalent to $(5x-1)\times(3x+2)$, and the product to $15x^2+7x-2$.

[2] Division being the reverse operation of multiplication, if the product *ya v* 15 *ya* 7 *ru* $\dot{2}$ be taken as the dividend, and the multiplier *ya* 3 *ru* 2 be taken as the divisor; division being made, the quotient is *ya* 5 *ru* $\dot{1}$, the original multiplicand.

Like quantities are denoted by the same colour, unlike by different colours. The multiplication of two like quantities produces the square, three the cube, four the biquadrate, &c. But when unlike quantities are multiplied, the result was called *bhavita* (to be) their factum, the initial syllable *bha* being annexed to the statement. Thus the product of two unknown quantities is denoted by three syllables (or three initial letters), as *ya ca bha* is equivalent to $x \times y$ or xy in modern algebraic notation. But if one or more of the quantities be of a higher power than the first, other initial syllables are employed, as *va gha* &c., for the square, cube, &c. Thus, *ya va ca gha bha* will signify the square of the first unknown quantity, multiplied by the cube of the second, $x^2 \times y^3$, or x^2y^3.

In some Sanscrit texts, and in some commentaries, where two or more factors are to be multiplied, a point or dot is found interposed between the factors, without any directions given for this notation.

The terms of compound quantities are arranged according to the descending powers of the unknown quantity, the numerical coefficients are placed after the symbols which denote the powers of the unknown quantities, and the negative mark is placed over the coefficient, and not over the power of the unknown quantity; also a known number is always placed last, thus: *ya vv* 1 *ya gh* 6 *ya v* 5 *ya* $1\dot{2}$ *ru* 4; in modern notation $x^4+6x^3+5x^2-12x+4$.

As the Hindus had no sign to denote either *equality* or relative magnitude, *greater than* and *less than*, the two sides of an equation were placed one under the other.[1] It was the general practice to arrange the terms in order according to the powers of the unknown quantity, and where a term on either side was wanting, to write the literal character with zero for its coefficient.

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[1] The following is an example of a simple equation:—

One person has 300 of known species and six horses. Another has ten horses of like price; but he owes a debt of 100 of known species. They are both equally rich. What is the price of a horse?

The statement of the equation is thus made:—

ya 6 *ru* 300
ya 10 *ru* $1\dot{0}0$

In modern notation $10x-100=6x+300$.

The following is an example of an indeterminate problem of the first degree:—

The quantity of rubies without flaw, sapphires, and pearls, belonging to one person, is 5, 8, and 7 respectively; the like gems appertaining to another is 7, 9, and 6. One has 90, the other 62, known species. They are equally rich. Tell me

If a number consist of two parts, the square of the number is twice the product of the parts added to the sum of the squares of the parts. In the same manner is stated the cube of a number consisting of two parts. The cube of the second part is to be first set down; next, the square of the second part multiplied by three times the first; thirdly, the square of the first part multiplied by three times the second; and lastly, the cube of the first part. All these added together make the cube of the number.

The same process might have been begun with the first instead of the second part of the number.

The cube of a number consisting of two parts, is also expressed in the following manner:—"Three times the proposed number multiplied by its two parts, added to the sum of the cubes of these parts gives the cube of the number."

Methods of finding the square root and the cube root are given by reversing the direct process. Also rules are given for the addition and subtraction, multiplication and division, of quadratic surd numbers, as also for their involution and evolution.

It is further noted that the square of an affirmative or of a negative quantity is affirmative; but that the square root of an affirmative quantity is sometimes affirmative, and sometimes negative, according to difference of circumstances which the process may require. As, for instance, the square of 3 affirmative or of 3 negative is 9 affirmative. Hence, the square root of 9 affirmative may be 3 affirmative or 3 negative. There is no square root of a negative quantity, for it is not a square. One of the commentators remarks:—"If it be maintained that a negative quantity may be a square, it must be shown of what it can be the square. Now, it cannot be the square of an affirmative quantity, for a square is the product of two like quantities; and if an affirmative quantity be multiplied by an affirmative one, the product is affirmative. Nor can it be the square of a negative quantity; for a negative quantity multiplied by a negative quantity is also positive. Therefore, it is clear that there is no quantity such, that its square can be negative.

One of the commentators has remarked that some have pretended to have found the root of a surd, and that this might be effected by the Cutaca Ganita; but whether or not this could have been possible, I shall relate what Bhascara and others have omitted to explain.

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quickly, then, intelligent friend, who art conversant with algebra, the prices of each sort of gem.

The statement of the problem

ya 5 *ca* 8 *ni* 7 *ru* 90
ya 7 *ca* 9 *ni* 6 *ru* 62

which expressed in modern notation is $7x+9y+6z+62=5x+8y+7z+90$, and when reduced becomes $2x+y-z=28$.

A root is of two kinds; one a line, the other a number. And the root of a square formed by a line expressing 5 may be found, though the root of 5 cannot be numerically expressed; but the numbers 1, 4, 9, &c., may be expressed both ways. The square roots of the numbers 2, 3, 5, &c., are surds, and can have their roots expressed only by lines. Thus the root of a surd quantity can be shown *geometrically* with exactness, but only approximately by *numbers*.

Quadratic equations,[1] or equations of the second degree, both pure and adfected, are discussed in the treatise of Brahmegupta, and in the Vija Ganita of Bhascara. In the solution of quadratic equations, when the square of the unknown quantity is affected with an integral coefficient different from unity, Bhascara gives this rule of Sridhara: "Multiply both sides of the equation by a number equal to four times the coefficient of the square, and add to each side a number equal to the original coefficient of the first power of the unknown quantity." He also gives an example in which fractions are avoided by the application of the rule.[2]

Brahmegupta has given this rule, and another which is thus expressed:—

"To the number multiplied by the coefficient of the square of the unknown quantity, add the square of half the coefficient of the first power of the unknown quantity; the square root of the sum less half the coefficient of the unknown quantity being divided by the coefficient of the square is the value of the unknown quantity."

The chapter on quadratic equations includes other questions involving the cube and the biquadrate,[3] besides the square and the first power

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[1] The following is their mode of writing a quadratic equation:—

$$\begin{array}{lll} ya\,v\,1 & ya\,\dot{8} & ru\,16 \\ ya\,v\,0 & ya\,0 & ru\,36 \end{array}$$

which in modern notation would be $x^2-8x+16=36$.

[2] The example is the following:—

The square root of half the number of a swarm of bees is gone to a shrub of jasmine; and so are eight-ninths of the whole swarm; a female is buzzing to one remaining male, that is humming within a lotus, in which he is confined, having been allured to it by its fragrance at night. Say, lovely woman, the number of bees.

Employing the modern notation—

Let $8x^2$ denote the number of bees; then the question reduced to an equation is $2x+\frac{64x^2}{9}+2=8x^2$: which gives the values of x, 3 and $-\frac{3}{4}$.

The positive value gives the answer to the question, and $8x^2=8\times9=72$, the number of the swarm of bees.

[3] Of the two following examples in this chapter, one leads to a cubic, and the other to a biquadratic equation.

"What is the number, learned man, which being multiplied by 12 and added to the cube of the number, is equal to six times the square added to 13?"

"If thou be conversant with the operations of algebra, tell the numbers of of the unknown quantity. With respect to such equations, the author observes, if the solutions be not accomplished by the methods given, the value of the unknown quantity must be elicited by the sagacity of he intelligent analyst. Intelligent calculators commence the work of solution, sometimes from the beginning of the conditions of the problem, sometimes from the middle, sometimes from the end, so as the solution may be best effected.

It has been supposed that the Binomial Theorem in its general form, when the index is an integer, was not unknown to the Hindus. The Lilavati and the Vija Ganita afford no direct evidence of such knowedge beyond the second and third powers of a binomial, which also ppears in the older treatises of Brahmegupta. The law for the formaion of the number of combinations of a number of different things taken, 1, 2, 3, &c., at a time was well known; but these treatises do not exhibit any connection of, or identity of, this law with the coefficients of the terms of an expanded binomial.

At the end of the second volume of the "Asiatic Researches," an Appendix has been added by Mr. Reuben Burrow, entitled, "A Proof that the Hindoos had the Binomial Theorem." He produces a question[1] and its solution in proof of his statement, and adds:—"The demonstration is evident to mathematicians; for as the second term's coefficient in a general equation shows the sum of the roots, therefore in the *n*th power of 1+1, where every root is unity, the coefficient shows the different *ones* that can be taken in *n* things; also, because the third term's coefficient is the sum of the products of all the different twos of the roots, therefore when each root is unity, the product of each two roots will be unity, and therefore the number of units, or the coefficient itself, shows the number of different *twos* that can be taken in *n* things. Again, because the fourth term is the sum of the products of the different threes that can be taken among the roots, therefore, when each root is unity, the product of each three will be unity, and herefore every unit in the fourth will show a product of three different oots, and consequently the coefficient itself shows all the different *threes* that can be taken in *n* things; and so for the rest."

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which the biquadrate, less double the sum of the square and of 200 times the simple number, is a myriad less one."

Expressed in the modern algebraic notation, the two equations are $2x+x^3=6x^2+35$, and $x^4-(2x^2+200x)=10000-1$.

[1] A rajah's palace had eight doors: now these doors may either be opened one at a time, or by two at a time, or by three at a time, and so through the whole, till at st all are opened together; it is required to tell the number of times that this can e done. The question and the solution itself appears to exhibit an example of rming the combinations of a certain number of things when taken 1, 2, 3, &c., at time, and the total number of such combinations; but no reference is made to a ries of terms arising from the expansion of the integral power of a binomial pression.

It may reasonably be doubted if this mode of demonstration which he gives, arising from the formation of an equation, was known to the Hindu mathematicians. Their books show that they understood the forms of the second and third powers of a binomial, but exhibit no instances of a higher power, or of a generalised form.

Mr. Strachey makes the following comparison of the Beej Gunnit [Vija Ganita] and the Arithmetics of Diophantus:—

"The *Beej Gunnit* will be found to differ much from Diophantus's work. It contains a great deal of knowledge which the Greeks had not, such as the use of the indefinite number of unknown quantities and the use of arbitrary marks to express them; a good arithmetic of surds; a perfect theory of indeterminate problems of the first degree; a very extensive and general knowledge of those of the second degree; a knowledge of quadratic equations, &c. The arrangement and manner of the two works will be found as essentially different as their substance. The one constitutes a body of science, the other does not. The *Beej Gunnit* is well digested and well connected, and is full of general rules, which suppose great learning; the rules are illustrated by examples, and the solutions are performed with skill. Diophantus, though not entirely without method, gives very few general propositions, and is chiefly remarkable for the ability with which he makes assumptions in view to the solution of his questions. The former teaches algebra as a science by treating it systematically; the latter sharpens the wit by solving a variety of abstruse and complicated problems in an ingenious manner. The author of the Beej Gunnit goes deeper into his subject, and treats it more abstractedly, and more methodically, though not more acutely than Diophantus. The former has every characteristic of an assiduous and learned compiler; the latter of a man of genius in the infancy of science."[1]

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[1] Asiatic Researches, vol. xii., 1816, "On the Early History of Algebra," by Edward Strachey, Esq. In this essay Mr. Strachey refers to a tract "On the Mathematical Science of the Hindoos," which he printed at Calcutta, in the year 1812, pp. 17, 4to.

ELEMENTARY ALGEBRA,

WITH BRIEF NOTICES OF ITS HISTORY.

SECTION II.

OF ALGEBRA *(continued)*.

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OF ALGEBRA.

(*Continued.*)

In the former part of the sketch, an attempt has been made to exhibit a brief account of the origin and early progress of the science of Algebra, as far as it was known and cultivated by the Greeks and the Hindus. The pre-eminence of having invented the most perfect science of Numerical and Universal Arithmetic is justly assigned to the latter; while the former stand unrivalled in their earliest achievements in the science of Pure Geometry.

In this second part of the sketch, the account is continued of the cultivation of this science by the Arabians and Persians; and very briefly, with some notices of the great improvements of the science among the nations of Western Europe.

In less than two centuries after the rapid and desolating wars of the Mohammedan power, their successors began to turn their attention to the remains of the writings on literature and science which had escaped the general destruction. It appears that the scientific writings both of the East and the West had become objects of attention in the reign of the Caliph Almansur, who in A.H. 141, A.D. 762, had transferred the seat of government of the Mohammedan empire to Bagdad. Both the Caliph Almansur and his immediate successors, Haroun Alraschid and Almamun, were munificent patrons of learning and learned men, and zealously encouraged the cultivation of the arts and sciences. The scientific writings of the Hindus and the Greeks were diligently sought, and translated into Arabic, and at present there are extant Arabic translations of works on science, the originals of which, in Greek and Sanscrit, are not known to be in existence.

The oldest treatise on Algebra in the Arabic language was composed by Abu Abdallah Mohammed Ben Musa,[1] at the command of Almamun, who succeeded to the Caliphat A.D. 813. Haji Khalfa, in his Bibliographical work, has recorded in two places that Mohammed Ben Musa was the first Mussulman who had ever written on the solution of problems by "the rules of completion and reduction." Both the title of the manuscript and the author's preface agree in stating it to be the work of Mohammed Ben Musa, and the mention of the Caliph Almamun in the preface, establishes the identity of the author. It appears also from a marginal note (written in a different hand) on the first and second pages of the manuscript, that this was the first work on Algebra composed by a Mohammedan. Arabian writers agree that Mohammed Ben Musa was the first who made known to the Arabians the Hindu method of computation by numbers, and it is highly probable that his knowledge of the science of Algebra was derived from the same source. The concurrent testimony of Arabian writers seems to be still further confirmed by the fact, that the Arabians wrote their figures from *left to right*, after the manner of the Hindus, but contrary to the order of writing their language, which is from *right to left.*

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[1] In the year 1831, the original Arabic of this treatise was edited, and translated into English by Mr. F. Rosen, and published by the Oriental Translation Fund. The manuscript which Mr. Rosen used is preserved in the Bodleian Library at Oxford, bearing the date of its transcription in the year 473 of the Hejira, which corresponds to 1342 A.D. Three other treatises on Arithmetic and Algebra are bound up with it in the same volume, which is marked "CMXVIII. Hunt, 214, fol." This copy of Ben Musa's Algebra is the only one Mr. Rosen knew to be extant. On the margin of the manuscript are written some notes in a very small character. Some of them are marked as being taken from a commentary by Al Mozaihafi, probably the same person whose Introduction to Arithmetic is one of the three treatises above mentioned.

It may further be added, there appears no evidence of facts that the Arabians had any knowledge of the Arithmetics of Diophantus anterior to the middle of the fourth century of the Hejira. It is related by Ebn Aladami in the preface to his Astronomical Tables, that Mohammed Ben Musa abridged, at the request of Almamun (but before his accession to the Caliphat), the Sindhind, or Astronomical Tables, translated by Mohammed Ben Ibrahim al Fazari, from the work of an Indian astronomer who visited the Court of Almansur at Bagdad, in the year of the Hejira 156, A.D. 777.

The Algebra of Mohammed Ben Musa is elementary, exhibiting the fundamental operations, including mensuration of triangles,[1] circles, &c., but is not extended beyond the solution of simple and quadratic equations. With respect to this treatise, Mr. Rosen, in his preface, has made the following remarks:—

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[1] The following note explains their notion of the relation of the diameter and circumference of a circle :—"In any circle, the product of its diameter multiplied by $3\frac{1}{7}$ will be equal to the periphery. This is the rule generally followed in practical life, though it is not quite exact. The geometricians have two other methods. One of them is, that you multiply the diameter by itself; then by 10, and hereafter take the root of the product; the root will be the periphery. The other method is used by the astronomers among them: it is this, that you multiply the diameter by 62832, and then divide the product by 20000; the quotient is the periphery. Both methods come nearly to the same effect." The following note appears on the margin of this passage in the Arabic text :—"This is an approximation, not the exact truth itself; nobody can ascertain the exact truth of this, and find the real circumference; for the line is not straight so that its exact length might be found. This is called an approximation, in the same manner as it is said of the square roots of irrational numbers that are an approximation, and not the exact truth. The best method here given is, that you multiply the diameter by $3\frac{1}{7}$; for it is the easiest and quickest."

The first and third rules of Ben Musa are found in the Lilavati, chap. vi., sec. 201, and the second in chap. xii., sec. 40 of the Arithmetic of Brahmegupta. The quotient of the numbers 62832 divided by 20000 in the Algebra of Ben Musa, is the same as that of 3927 divided by 1250 in the Lilavati.

"But under whatever obligation our author may be to the Hindus, as to the subject matter of his performance, he seems to have been independent of them in the manner of digesting and treating it; at least, the method which he follows in expounding his rules, as well as in showing their application, differs considerably from that of the Hindu mathematical writers. Bhascara and Brahmegupta give dogmatical precepts, unsupported by argument, which, even by the metrical form in which they are expressed, seem to address themselves rather to the memory than to the reasoning faculty of the learner. Mohammed gives his rules in simple prose, and establishes their accuracy by geometrical illustrations. The Hindus give comparatively few examples, and are fond of investing the statement of their problems in rhetorical pomp. The Arab, on the contrary, is remarkably rich in examples, but he introduces them with the same perspicuous simplicity of style which distinguishes his rules. In solving their problems, the Hindus are satisfied with pointing at the result, and at the principal intermediate steps which lead to it. The Arab shows the working of each example at full length, keeping his view constantly fixed upon the two sides of the equation, as upon the two scales of a balance, and showing how any alteration in one side is counterpoised by a corresponding change in the other."

The author's preface to his work contains the following passage:—

"The learned, in times which have passed away, and among nations which have ceased to exist, were constantly employed in writing books on the several departments of science, and on the various branches of knowledge, bearing in mind those who were to come after them, and hoping for a reward proportionate to their ability, and trusting that their endeavours would meet with acknowledgment, attention, and remembrance; content as they were, even with a small degree of praise; small, if compared with the pains which they had undergone, and the difficulties which they had encountered in revealing the secrets and obscurities of science.

"Some applied themselves to obtain information which was not known before them, and left it to posterity; others commented upon the difficulties in the works left by their predecessors, and defined the best method (of study), or rendered the access (to science) easier, or placed it more within reach; others again discovered mistakes in preceding works, and arranged that which was confused, or adjusted what was irregular, and corrected the faults of their fellow-labourers, without arrogance towards them, or taking pride in what they did themselves."

Mohammed Ben Musa, at the beginning of his treatise, gives no name or definition of the science, nor does he give a formal explanation of the terms *al jebr* and *al mokabalah*, by which he designates certain operations peculiar to the solution of equations. However, he repeatedly employs these two terms combined for the name of this entire branch of mathematical science. As the former of these terms has been introduced into the several languages of Europe, and is now used as the name of the science of universal arithmetic, some explanation appears necessary. The following particulars are chiefly taken from the notes on Mr. Rosen's translation.

The Arabic verb *jabar*, of which the substantive al jebr (algebra) is a derivative, properly and correctly means, *to restore something broken*, especially to set right a fractured bone, or a dislocated joint. As the Arabian Moors for ages maintained their conquests in Spain until the capture of Granada, in 1492, it might be expected that many words of Arabic origin would be found in the language of the Spanish people. Hence, in the Spanish and Portuguese languages, is the word *Algebrista*, unquestionably derived from the Arabic words al jebr, and employed as synonymous with the word surgeon. In its primary sense, however, it was probably restricted in its use to the person, who in England was formerly called a bone-setter.[1]

In mathematical language, the verb *jabar* means to make perfect or complete any quantity that is incomplete, or subject to diminution. As applied to equations, it denotes to add to one side of an equation as much as is removed from the other, so as to make the latter complete. As, for instance, if ten less one thing be equal to four things. Then jebr is the removal of one thing from one side of the equation, and then adding one thing to four things on the other side; so that one side is made complete (or wholly positive) by the removal of one thing; and the equality is restored by adding one thing to the four things on the other side.

The word Mokabalah is a noun of action, from a verb which means to be in front of a thing, or to compare two things with one another. In mathematical language it is employed to express a comparison between positive and negative quantities. As applied to equations, it denotes the removal of equal quantities from both sides of an equation; also the withdrawal of equal quantities affected with contrary signs from the same side of an equation; as, for example, if thing added to square be equal to square added to four, then thing is equal to four;

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[1] Mr. Rosen illustrates the meaning of the word by a passage in Arabic from Motanabbi, of which he gives the following translation in English :—"O Thou on whom I rely in whatever I hope, with whom I seek refuge from all that I dread; whose bounteous hand seems to me like the sea, as thy gifts are like its pearls: pity the youthfulness of one whose prime has been wasted by the hand of Adversity, and whose bloom has been stifled in the prison. Men will not heal a bone which thou hast broken, nor will they break one which thou hast healed."

also, if ten added to thing less ten, be equal to 4, then thing is equal to 4.

This science has therefore been called by the name of these two rules al jebr and al mokabalah; namely, the rule of jebr or restoration, and of mokabalah or reduction.[1]

Both Mohammed Ben Musa and other Arabian writers on Algebra did not adopt the Hindu notations. They have no symbols, arbitrary or abbreviated, to denote known or unknown quantities, either positive or negative. Neither have they any symbols to denote the elementary operations of the science, but express everything by words and phrases at length.[2]

The Arabian algebraists designated the unknown quantity by the word *shai, thing*. A positive quantity they characterised by the word *zaid, additional;* and a negative quantity by *nakis, deficient,* without any discriminating mark for either of them. The first power of an unknown quantity was also named *root*, in reference to other powers. The square was termed *mal, possession* or *wealth;* and the *cube, cáb;* and these two terms, *mal* and *cáb*, were combined by the sums of the indices of the powers, as compound names of higher powers of the unknown quantity. Their method of solving equations was different from that of the Hindus. The equation was prepared by two preliminary operations, by means of which all the terms on both sides of the equation were made positive, before their rules were applied to effect the solution.

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[1] Haji Khalfa, in his biographical work, gives the following explanation. *Jebr* is the adding to one side what is negative on the other side of an equation, owing to a subtraction, so as to equalise them. *Mokabalah* is the removal of what is positive from either sum, so as to make them equal.

He gives further illustration of this by an example: "For instance, if we say, 'Ten less one thing equal to four things.' Then *Jebr* is the removal of the subtraction, which is performed by adding to the minuend an amount equal to the subtrahend; hereby the ten is made complete, that which was defective in them being restored. An amount equal to the subtrahend is then added to the other side of the equation; as in the above instance, after the ten have been made complete, one thing must be added to the four things, which thus become five things. *Mokabalah* consists in withdrawing the same amount from quantities of the same kind on both sides of the equation; or, as others say, it is the balancing of certain things against others, so as to equalise them. Thus, in the above example, the ten are balanced against the five with a view to equalise them. This science has therefore been called by the name of these two rules—namely, the rule of *jebr*, or restoration, and of *mokabalah*, or reduction, on account of the frequent use that is made of them."

[2] The following is an example of their method of solving simple equations:—

"Two numbers, the difference of which is two dirhems; you divide the small one by the great one, and the quotient is equal to half a dirhem.

Then the computation is this: Multiply thing and two dirhems by the quotient, that is a half. The product is half a thing and one dirhem, equal to thing. Remove now half a dirhem on account of the half dirhem on the other side, the remainder is one dirhem equal to half a thing. Double it; then you have thing, equal to two dirhems. This is one of the two numbers, and the other is four."

In modern notation, $\frac{x}{x+2}=\frac{1}{2}$, $2x=x+2$, $x=2$ and $x+2=4$.

The numbers which are required in calculating by "completion and reduction" are described to be of three kinds; namely, roots, squares, and simple numbers relative neither to root nor square. A root is any quantity which is to be multiplied by itself. A square is the whole amount of the root multiplied by itself. A simple number is any number which may be pronounced without reference to root or square.

A number belonging to one of these three classes may be equal to a number of another class; for instance, squares are equal to roots, or squares are equal to numbers, or roots are equal to numbers. By combining these three kinds—namely, roots, squares, and numbers—there arise three forms of quadratic equations: (1) squares and roots equal to numbers; (2) squares and numbers equal to roots; (3) roots and numbers equal to squares. These, in the modern algebraic notation, are expressed by $x^2+px=q$, $x^2+q=px$, and $x^2=px+q$. The form $x^2+px=-q$ was excluded from their classification of quadratic equations. The Arabians held a less general notion of equations than the Hindus, as they only knew equals to be the equality of two or more absolute positive quantities. They had no conception of the equation $x^2+px+q=o$, the sum of three positive quantities being equal to zero, otherwise than an absurdity, or an impossibility.

In the solution of quadratic equations; in the first place, they reduced the coefficient of the square to unity, where it was necessary. The next step was to relieve the equation of negative quantities by the operation of al jebr. This name was given to the operation because it was only after negative terms had been removed that the equation itself became capable of solution by the application of their rules. After al jebr followed al mokabalah, which consisted in comparing the equal numbers on the same side, and on both sides of the equation; and as far as it could be done, to cancel them. After this operation, an equation of the second degree was reduced to one of the three specified forms, and adapted for solution.[1]

The Hindus, on account of their more complete algebraic notation, and their wider view, did not find it necessary to carry back an equation to the special form of two positive equals. On that account they had no particular names for the two preliminary operations which the Arabians had described by the names al jebr and al mokabalah.

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[1] As an example of the first form of quadratic equations, "roots and squares are equal to numbers; one square and ten roots of the same, amount to thirty-nine dirhems;" that is to say, "what must be the square, which when increased by ten of its own roots, amounts to thirty-nine?" The solution is this: you halve the number of the roots, which in the present instance yields five. You multiply by itself; the product is twenty-five. Add this to thirty-nine, the sum is sixty-four. Now take the root of this, which is eight, and subtract from it half the number of the roots,

These two operations (one of which has given the name to the science) are precisely what are enjoined, without distinctive appellations of them, in the introduction of the Arithmetics of Diophantus. He directs that if the quantities be positive on both sides, like are to be taken from like until one species be equal to one species; but if, on either side or on both, any species be negative, the negative species as positive must be added to both sides, so that all the quantities become positive on both sides of the equation; after which, like are again to be taken from like, until one species remain on each side (Def. 11).

Among the writers subsequent to Mohammed Ben Musa, may be mentioned Abufaraj, the author of a treatise on computation. He lived in the twelfth century of the Christian era, and noticed a book of numerical computation which Mohammed Ben Musa amplified, and which he declared "is a most expeditious and concise method, and testifies the ingenuity and acuteness of the Hindus."

The celebrated Abu Yusef Alkindi, contemporary with the astrologer Abu Masher, in the third century of the Hejira, was versed in the sciences of Greece, India, and Persia, and was also the author of several treatises on numbers. His pupil, Ahmed Ben Mohammed of Sarkiri in Persia, who flourished about the middle of the third century of the Hejira, was the author of a complete treatise of computation, embracing Arithmetic and Algebra. About the same time a treatise on Algebra was composed by Abu Hanifah of

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which is five; the remainder is three. This is the root of the square you sought for; the square itself is nine.

In modern notation, the equation is $x^2+10x=39$.

The following is another example from M. Rosen's translation :—

"I have divided ten into two parts; I have then multiplied each of them by itself, and when I had added the products together, the sum was fifty-eight dirhems.

Computation. Suppose one of the two parts to be thing, and the other, ten minus thing. Multiply ten minus thing by itself; it is a hundred and a square minus twenty things. Then multiply thing by thing; it is a square. Add both together. The sum is a hundred, plus two squares minus twenty things, which are equal to fifty-eight dirhems. Take now the twenty negative things from the hundred and the two squares, and add them to fifty-eight; then a hundred, plus two squares, are equal to fifty-eight dirhems and twenty things. Reduce this to one square by taking the moiety of all you have. It is then: fifty dirhems and a square, which are equal to twenty-nine dirhems and ten things. Then reduce this, by taking twenty-nine from fifty; there remains twenty-one and a square, equal to ten things. Halve the number of roots, it is five; multiply this by itself, it is twenty-five; take from this the twenty-one which are connected with the square, the remainder is four. Extract the root, it is two. Subtract this from the moiety of the roots, namely, from five, there remains three. This is one of the portions; the other is seven."

Dainawar, who lived till about the 290th year of the Hejira. At a later period flourished Abulwafa of Buzjani, a distinguished mathematician, whose death happened in the 388th year of the Hejira. He wrote several treatises on computation, and a commentary on the Algebra of Mohammed Ben Musa, also on another of less repute and of later date by Abu Yahya. He was the first Arabian who translated the Arithmetics of Diophantus from Greek into Arabic. This work was executed nearly two centuries after the Arabians had become acquainted with the sciences of the arithmetic and astronomy of the Hindus.

It was in the reign of Haroun Alraschid that the astronomy of the Greeks became known to the Arabians by a translation of Ptolemy's Almagest.

In the year 1851, Dr. Woepcke published the original Arabic of a treatise on Algebra, composed by Omar Alkhayyami, with a French translation.[1] The years of the birth and death of the author are unknown. It is known, however, that he was brought up with two youths who afterwards became distinguished. One of them was the Vizier of Malak Shah, the last of the Seljukian Sultans, who died about A.D. 1072. His work is divided into five parts. The introduction contains a preface, and the definitions of the science, &c. Next, the solution of equations of the first and second degree. Thirdly, the construction of cubic equations. Fourthly, the discussion of equations involving fractions, and having powers of the unknown quantity in the denominators. And lastly, some additions. In treating the subject of equations, he always connects the solution with some geometrical construction, or the converse. For cubic equations, however, he was obliged to confine himself to the latter course. He gives the same threefold division of quadratic equations as Mohammed Ben Musa, and exhibits a classification of cubic equations.[2] The constructions of these equations are effected in different ways, some of them requiring the properties of the conic sections, as the example (p. 62). "The equation, a cube and

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[1] The original Arabic of this work, with a translation in the French language, was printed at Paris in 1851, entitled, "L'Algèbre d'Omar Alkhayyami, publiée, traduite et accompagnée d'extraits de manuscrits inédits, par F. Woepcke, Docteur agrégé a l'Université de Bonn, Membre de la Société Asiatique de Paris."

[2] The following are examples of the classes of cubic equations, each of which consists at least of two terms on one side and one term on the other, expressed in modern notation.

$$\begin{array}{llllll} (1) & x^3+cx^2=bx: & (2) & x^3+bx=a: & (3) & x^3+cx^2=bx+a: \\ & x^3+bx=cx^2: & & x^3+a=bx: & & x^3+bx=cx^2+a: \\ & cx^2+bx=x^3: & & bx+a=x^3: & & x^3+a=cx^2+bx: \end{array}$$

$$(4)\ x^3+cx^2+bx=a: \quad cx^3+bx+a=x^3: \quad x^3+cx^2+a=bx: \quad x^3+bx+a=cx^2.$$

These different classes arise from the method adopted by the Arabians of considering an equation when reduced to consist only of positive quantities. In the text, these classes are expressed in words, as, for instance, the first example above, $x^3+cx^2=bx$, is described, "A cube and squares equal to roots."

roots are equal to squares and numbers" ($x^3+cx=bx^2+a$, in modern notation), is constructed by means of a circle and the asymptotes of an equilateral hyperbola.

The Risala Hisab is a short treatise on Arithmetic and Geometry written in the Persian language by Kazi Zadeh al Rumi, surnamed Ali Kushchi. He was one of the celebrated astronomers employed by Ulugh Begh to construct the astronomical tables which bear the name of that prince, who had caused an observatory to be erected in Samarcand for that purpose. Ulugh Begh was slain by his rebellious son, A.H. 853, or A.D. 1474.

Baha Eddin was born at Balbec A.H. 953, and died A.H. 1031, or A.D. 1652. He was the author of a work on Algebra, entitled "Kholasat al Hisab,"[1] which was considered by learned Mohammedans as the most scientific treatise on the subject. This work was translated into the Persian language about sixty years after the death of the author. Mr. Strachey has given, in the twelfth volume of the Asiatic Researches, an analysis of the contents of this work, from which the abstract in the note[2] will supply a more complete conception of the knowledge of Algebra among the Arabians.

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[1] The Arabic text of the Kholasat al Hisab, with a Persian Commentary by Roshan Ali, was printed at Calcutta in the year 1812.

[2] The work is stated by the author, Baha Eddin, in his preface, to consist of an introduction, ten books, and a conclusion.

The introduction contains the definitions of arithmetic, number, &c., and the author ascribes to the Indian sages the invention of the nine figures to express the numbers from one to nine.

Book I. comprises the arithmetic of integers, with the rules of addition, duplation, subtraction, halving, multiplication, division, and the extraction of the square root; and the method of proving the operations by casting out the nines is described under each of these rules.

Book II. contains the arithmetic of fractions.

Book III. contains the rule for finding an unknown number by four proportionals.

Book IV. gives a rule for finding an unknown number by assuming a known number, once or twice, and comparing the errors.

Book V. gives a method of finding an unknown number by reversing all the steps of the process described in the question.

Book VI. treats of mensuration in three chapters. Chapter 1 treats of the mensuration of lines and areas of rectilineal figures. Chapter 2 treats of curvilinear surfaces. Chapter 3 is on the mensuration of solids.

Book VII., on practical geometry. Chapter 1. On levelling for the purpose of making canals. In this are described the plummet level and the water level, on the same principle as the spirit level. Chapter 2. On the mensuration of heights. Chapter 3. On the measuring of the breadth of rivers and the depth of wells.

Book VIII., on finding unknown quantities by algebra. Chapter 1. Introduction. Chapter 2. On the six rules of algebra. The author remarks: "To find unknown quantities by algebra depends on acuteness and sagacity, an attentive consideration of

The Lilavati of Bhascara was translated by the celebrated Fyzee into the Persian language in the year A.D. 1587. In the preface the

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the terms of the question, and a successful application of the invention to such things as may serve to bring out what is required. Call the unknown quantity *shai*, thing, and proceed with it according to the terms of the question, as has been said, till the operation ends with an equation. Let that side where there are negative quantities be made perfect, and let the negative quantity be added to the other side; this is called *Jebr*, restoration; let those things which are of the same kind, and equal on both sides, be thrown away; this is *Mokabalah*, opposition.

Book IX. contains the following twelve rules respecting the properties of numbers :—

1. To find the sum of the products of a number multiplied into itself and into all numbers below it; add one to the number, and multiply the sum by the square of the number; half the product is the number required.

2. To add the odd numbers in their regular order; add one to the last number, and take the square of half the sum.

3. To add even numbers from two upwards; multiply half the last even number by a number greater by one than that half.

4. To add the squares of the numbers in order; add one to twice the last number, and multiply a third of the sum by the sum of the numbers.

5. To find the sum of the cubes in succession; take the square of the sum of the numbers.

6. To find the product of the roots of two numbers; multiply one by the other, and the root of the product is the answer.

7. To divide the root of one number by that of another; divide one by the other, the root of the quotient is the answer.

8. To find a perfect number—that is, a number which is equal to the sum of its aliquot parts (Euclid, Book VII., def. 22). The rule is that delivered by Euclid, Book IX., prop. 36.

9. To find a square in a given ratio to its root; divide the first number of the ratio by the second; the square of the quotient is the square required.

10. If any number is multiplied and divided by another, the product multiplied by the quotient is the square of the first number.

11. The difference of two squares is equal to the product of the sum and difference of the roots.

12. If two numbers are divided by each other, and the quotients multiplied together, the result is always one.

Book X. contains nine examples.

"Conclusion. There are many questions in this science which learned men have to this time in vain attempted to solve; and they have stated some of these questions in their writings, to prove that this science contains difficulties, to silence those who pretend they find nothing in it above their ability, to warn arithmeticians against undertaking to answer every question that may be proposed, and to excite men of genius to attempt their solution. Of these I have selected seven :—

1. To divide ten into two parts, such that when each part is added to its square root, and the sums are multiplied together, the product is equal to a supposed number.

2. What square number is that which being increased or diminished by ten, the sum and remainder are both square numbers?

3. A person said he owed Zaid ten all but the square root of what he owed

translator remarks:—"By order of the Emperor Akbar, Fyzee[1] translates into Persian from the Indian language the book Lilavati, so famous for the rare and wonderful arts of calculation and mensuration. He [Fyzee] begs leave to mention that the compiler of this book was Bhascara Acharya, whose birthplace, and the abode of his ancestors, was the city of Biddur in the country of the Deccan."

The Vija Ganita was translated into Persian by Ata Alla Rasheedee, probably at Agra or Delhi. The Persian translator has prefixed the following introduction to his work:—

"By the grace of God, in the year 1044 Hejira [A.D. 1665], being the eighth year of the king's reign, I, Ata Alla Rasheedee, son of Ahmed Nadir, have translated into the Persian language, from Indian, the book of Indian Algebra, called Beej Gunnit [Vija Ganita], which was written by Bhasker Acharij [Bhascara Acharya], the author of the Leelawuttee [Lilavati]. In the science of calculation it is a discoverer of wonderful truths and nice subtleties, and it contains useful and important problems which are not mentioned in the Leelawuttee, nor in any Arabic or Persian book. I have dedicated the work to Shah Jehan,[2] and I have arranged it according to the original in an introduction and five books."

This Persian version of the Vija Ganita was translated into English by Edward Strachey, of the East India Company's Bengal Civil Service.[3] Mr. Strachey remarks that the Persian does not afford a correct idea of the original Sanscrit, as a translation should do; and describes it as an undistinguished mixture of text and commentary. He gives a literal translation of extracts from the Persian, with re-

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Amer, and that he owed Amer five all but the square root of what he owed Zaid.

4. To divide a cube number into two cube numbers.

5. To divide ten into two parts, such that if each is divided by the other, and the two quotients are added together, the sum is equal to one of the parts.

6. There are three square numbers in continued geometrical proportion, such that the sum of the three is a square number.

7. There is a square, such that when it is increased and diminished by its root and two, the sum and the difference are squares.

"Know, reader, that in this treatise I have collected in a small space the most beautiful and best rules of this science, more than were ever collected before in one book. Do not underrate the value of this bride; hide her from the view of those who are unworthy of her, and let her go to the house of him only who aspires to wed her."

[1] Faizee's Persian translation of the Lilavati was printed in 8vo at Calcutta in 1827.

[2] In the Fitzwilliam Museum at Cambridge is preserved a splendid model in ivory of the palace tomb which Shah Jehan had erected to the memory of his wife.

[3] "Bija Ganita, or the Algebra of the Hindus," by Edward Strachey, of the East India Company's Bengal Civil Service. London, 1813. (Reviewed in the "Edinburgh Review" for July, 1813.)

marks of his own. He has translated almost all the rules, some of the examples entirely, and others partly, including whatever he thought deserving attention, as giving a distinct idea of the work. The reference to Euclid[1] in the Persian can only be regarded as an interpolation by a later hand, as no references to Euclid have been discovered in Sanscrit copies of the Vija Ganita.

Mr. Strachey has employed in his work the symbols and methods of modern algebra, and adds the following judicious remark of the Persian translator in reference to the teacher and learner. "In calculation, correctness is the chief point. A wise and considerate person will easily remove the veil from the object; but where the help of acuteness is wanting, a very clear explication is necessary."[2]

It also appears that the peculiar notation of the Hindus is entirely wanting in the Persian, where the algebraic processes are always expressed in words at length. Most of the technical terms used in the Persian translation are Arabic. The product of two quantities in general is called their rectangle.

It appears that the Persians are indebted for whatever mathematical knowledge they acquired to the Arabians, who had before derived all their knowledge from the Hindus and the Greeks.

The Persians have borrowed the word *jebr* and *mokabalah*, together with the greater part of their mathematical terminology, from the Arabs. The following extract from a short treatise on Algebra in

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[1] The reference to Euclid is contained in the solution of the question, "What right-angled triangle is that, the sum of whose three sides is 40, and the rectangle ~~of the two sides about the right angle 120?~~" In the solution occurs the expression, "For the sum of the two sides is always greater than the hypotenuse, by the asses' proposition." In a note, the English translator adds, "Meaning by the asses' proposition, the 20th of the first book of Euclid, which, we are told, was ridiculed by the Epicureans as clear even to asses." This tradition of the East, however, differs from that of the West; since it has been handed down, that the 5th, and not the 20th proposition of the first Book of Euclid, bears that designation. A Cambridge witty wag of former days has supplied some small comfort for the halting and feeble-minded who are too weak to proceed, in the following pithy couplet:—

"If this be rightly called the bridge of asses,
He's not the ass who stops, but he who passes."

[2] Mr. Strachey has printed the following extract, a translation from a Persian treatise on arithmetic, in p. 184 of vol. xii. of the Asiatic Researches. "The Indian sages, wishing to express numbers conveniently, invented the nine figures. The first figure on the right hand they made stand for units, the second for tens, the third for hundreds, the fourth for thousands. Thus, after the third rank, the next following is units of thousands, the second tens of thousands, the third hundreds of thousands, and so on. Every figure, therefore, in the first rank, is the number of units it expresses, every figure in the second, the number of tens which the figure expresses, in the third, the number of hundreds, and so on. When in any rank a figure is wanting, write a cipher like a small circle o, to preserve the rank. Thus ten is written 10, a hundred 100, five thousand and twenty-five 5025."

Persian verse, by Mohammed Nadjm-Eddin Khan, appended to the Calcutta edition of the Kholasat al Hisab, will serve as an illustration of this remark:—

"Complete the side in which the expression *illa* (less, minus) occurs, and add as much to the other side, O learned man: this is in correct language called *jebr*. In making the equation mark this: it may happen that some terms are cognate and equal on each side, without distinction; these you must on both sides remove, and this you call *mokabalah*."

In the thirteenth volume of the "Asiatic Researches" is printed an Essay by J. Tytler, Esq., on the "Binomial Theorem as known to the Arabians." The essay contains an English translation, with the Arabic original, of a passage from the Ayoun-ul-Hisab, or Rules of Arithmetic, composed by Mohammed Baquier, in the reign of Shah Abbas I., about the year A.D. 1600. In the example given, the number 12 is employed, and the mode of solution appears to be the same as that given in the Lilavati, in which the number is 6, for finding the number of combinations of 6 things taken 1, 2, 3, &c., at a time. It may reasonably be doubted whether the Hindus or the Arabians had attained to the idea that the successive coefficients of an expanded binomial are respectively equal to the numbers of combinations of a number (equal to the exponent) of things taken 1, 2, 3, &c., at a time, when the exponent itself is a positive integer.

During the middle ages the countries of Western Europe appear to have produced few students of eminence in the mathematical sciences. The scientific knowledge, such as it was in these times, was not derived directly either from Greek or Hindu sources. There were, however, individuals who, inspired with a thirst for knowledge, both in England and on the continent, travelled into foreign countries in search of knowledge. Notices of such persons are extant, as Adelard, a monk of Bath, who first brought the knowledge of Euclid's Elements into England by a translation from the Arabic; and John of Basingstoke, who introduced the knowledge of the Greek numerical characters. And others, such as Roger Bacon, a diligent student of the works of nature, whose attempts to promote the advancement of the sciences brought him under the suspicion of the clergy in those times of practising magic and of having dealings with the devil. The seventh century of the Christian era was almost barren of writers on science. The irruption and wars of the Saracens, and the increasing corruptions of the Church, mutually assisted in the destruction of all learning and science. The eighth century was nearly of the same character, with a few exceptions of great eminence. Isidorus Hispalensis flourished in the early part of the seventh century, and governed the Church of Seville as Archbishop for forty years. He was the most eminent scholar of his age. His writings were numerous; one of the most important was an Encyclopædia, which, with other subjects, treated of the liberal arts of geometry, arithmetic, music, and astronomy. Bede, styled the Venerable, was the most learned man of his time, as his extant writings clearly prove. He lived in the eighth century, and among his numerous writings is one entitled, "De Temporum Ratione." It begins with an explanation of the mode of numerical computation by means of the hands, the fingers, &c., and gives an account of the Greek and Roman notations for recording numbers. This treatise, as its name imports, contains an account of ecclesiastical chronology, and in some manuscripts it includes the tract, "De Rerum Natura."[1] The treatise "De Numerorum Divisione," attributed to Bede, clearly belongs to a later period. It will be found in pp. 159-163, book i., of the folio edition of his works printed in 1529. The treatise is one of the works of Gerbert, on the Abacus, and was dedicated by him to his friend Constantinus.

Alcuin, a disciple of the Venerable Bede, was the author of numerous works, chiefly theological. Among his miscellaneous writings will be found an elementary introduction to the different sciences, and a tract entitled, "Propositiones Arithmeticæ [LIII.] ad acuendos juvenes," which has been, without evidence, included in the works of Bede.

In the middle ages, the Councils of the Roman Church, in their decrees, directed bishops in their pastorals to recommend to their clergy the study of sacred chronology, and that in every monastery there should be one person at least who understood the order of the festivals of the Church as determined by the Calendar of the current years. In this way computation always maintained a small place in the education of the clerical order. In a manuscript of the date A.D. 944, we are told that reckoning was seldom done on paper, but more frequently on the fingers, according to the ancient custom. There is no mention even of the abacus nor of the counters.

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[1] "Bede's work, entitled, 'De Rerum Natura,' has two great merits; it assembles into one focus the wisest opinions of the ancients on the subjects he discusses, and it continually refers the phenomena of nature to natural causes. The imperfect state of knowledge prevented him from discerning the true natural cause of many things, but the principle of referring the events and appearances of nature to its own laws and agencies, displays a mind of sound philosophical tendency, which was calculated to lead his countrymen to a just mode of thinking on these subjects. Although to teach that thunder and lightning were the collisions of the clouds, and that earthquakes were the effect of winds rushing through the spongy caverns of the earth, were erroneous deductions, yet they were light itself compared with the superstitions which other nations have attached to these phenomena. Such theories directed the mind into the right path of reasoning, though the correct series of the connected events and the operating laws had not then become known. The works of Bede supply evidence that the establishment of the Teutonic nations in the Roman empire did not barbarise knowledge. He collected and taught more natural truths, with fewer errors, than any Roman book on the same subjects had accomplished. Thus his work displays an advance, not a retrogradation, of human knowledge; and from its judicious selection and concentration of the best natural philosophy of the Roman empire, it does high credit to the Anglo-Saxon good sense."—*Sharon Turner's History of the Anglo-Saxons*, vol. iii., p. 403.

In the eyes of Alcuin, and many others of the priestly order, theology gave to the science of number its real importance, in that it was supposed to throw light on the symbolical numbers found in Holy Writ. He considers the number 6 the number of God, who created all things, because it is a perfect number, or the sum of its aliquot parts; but the number 8 is an imperfect number, the sum of the aliquot parts of it, 1, 2, 4, being equal to 7, a number less than 8. From this number 8 sprung the second origin of mankind after the deluge; as we read that in Noah's ark 8 persons were saved, from whom descended the whole human race, which shows that the second origin of the race was as imperfect as the first that were made after the number 6.[1] A similar use was made of astronomy. By such speculations the learned doctor sought to make it intelligible, "How pleasant and useful was the knowledge of arithmetical discipline!"

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[1] Nichomachus of Gerasa, who lived about the time of Augustus Cæsar, the first Roman Emperor, was the author of a treatise entitled "Isagoge Arithmetica." Iamblicus wrote a commentary on this work, and surpassed, in his visionary speculations on numbers, the most extravagant of his predecessors. Porphyry also wrote on the mysteries of numbers.

The propensity for finding mysteries in numbers, and in their relations and analogies, did not end with the latest of the commentators on the Platonic philosophy, but has descended even to modern times. Several works have appeared since the introduction of the Indian science of numbers, for the express purpose of explaining their mysterious properties.

Meursius published in 1631 his work, which he called "Denarius Pythagoricus." It contains a collection of the names and properties of the first ten numbers, from the writings of the Pythagorean philosophers.

In the year 1502, a work was published at Leipsic for the use of the students of that University, under the title of "Heptalogium Virgilii Saltzburgensis," on the properties of the number seven. The book is divided into seven parts, and each part into seven chapters, and forms a collection of quaint conceits and absurdities respecting the mystical and other properties of the number seven.

But the most extraordinary of these productions is the work of Peter Bungus, published in 1618, under the title of "Numerorum Mysteria." The work is written in Latin, and contains about 700 quarto pages. It professes to illustrate all numbers and their properties, mathematical, metaphysical, and theological, embracing all the opinions of the Pythagoreans and Platonists, with the addition of speculations on the numbers which incidentally occur in the writings of the Old and New Testaments.

In England, the same propensity to treat on the mysteries of numbers and the advantages of the study, has been exhibited in a work of which the following is a copy of the title-page: "The Secrets of Numbers according to Theological, Arithmetical, Geometrical, and Harmonical Computation. Drawn, for the better part, out of those Ancients, as well as Neoteriques. Pleasing to read, profitable to understande, opening themselves to the capacities of both learned and unlearned; being no other than a key to lead men to any doctrinal knowledge whatsoever. By William Ingpen, Gent. London, 1624."

Gerbert was the most remarkable man of his age, and at an early period was devoted to the cloister, where he was brought up as a monk. He left his home to undertake a student's journey into Spain. On his return he became a pupil in the convent school of Rheims. Endowed with great abilities and with extensive knowledge, both in the mathematical sciences as well as in classical learning, and being of an ardent character, thirsting for knowledge, he soon found himself distinguished as a scholar and a teacher. In the year A.D. 980 he was appointed Abbot of the Monastery of Bobbio in Lombardy; he was promoted Bishop of Rheims, next of Ravenna, and at length closed his learned and useful life in A.D. 1003, at Rome, where for four years he filled the Papal chair, under the name of Sylvester II. Gerbert, distinguished by the title of "Reparator Studiorum," reached the highest eminence that was possible in his time. He had entirely made his own the learning of his time in its widest extent, and he knew how to employ it in a manner scarcely known in that age. Gerbert's knowledge of astronomy, his construction of the globe, and his sun-dials, appeared so wonderful to the unlearned and superstitious clergy of his time, that it was reported of him after his death that he had made a compact with the devil.[1]

Leonardo Bonacci, of Pisa, first made known to his countrymen the Hindu Arithmetic and Algebra from the Arabians. A manuscript of his treatise bearing the title of "Liber Abbaci compositus a Leonardo filio Bonacci Pisano in anno 1202," and a transcript of another treatise entitled "Leonardi Pisani de filiis, Bonacci . . . Practica Geometria, composita anno 1220," were discovered in the Magliabecchian Library at Florence by Targioni Tozzetti, about the middle of the eighteenth century. In the preface to the Liber Abbaci, Leonardo relates that he had travelled in Egypt and other countries. In his youth at Bugia in Barbary, where his father was scribe at the custom-house for the merchants of Pisa who resorted thither, he there learned the Indian Arithmetic. He describes their method to be more commodious than the methods used in other countries which he had visited. He therefore prosecuted the study, and, with some additions of his own and some things taken from Euclid's Elements, he composed the treatise which he named Liber Abbaci. In the index to the fifteenth chapter of his work he writes:—"Incipit Pars Tertia de solutione quarundam quæstionum secundum modum Algebræ et Almucabalæ, scilicet oppositionis et restaurationis." This work appears to contain the earliest notice of the introduction of the Hindu Arithmetic and Algebra into Italy. Leonardo, the earliest scholar of the Arabian algebraists, is reported by Targioni Tozzetti to have employed the small letters of the alphabet to denote quantities. But it would appear that he only did so because he represented quantities by straight lines, and designated these lines by letters in the elucidation of his Algebraic Solution of Problems. Leonardo and his disciples translated the Arabic word *shai* by the Latin *res* and the Italian *cosa*. The Arabic word *mal* was rendered by the Latin *census* and the Italian *censo*, as terms of the same import; for it was in the acceptation of amount of property or estate (census, quicquid fortunarum quis habet) that census was used by Leonardo. And it appears that the name of the science was derived from the first and second powers of the unknown quantity, "Ars Rei et Census," and sometimes "Regula Rei et Census," the rule of the thing and the product, or the rule of the root and the square, in allusion to equations of the second degree.

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[1] The student is referred for more ample information on the early history of arithmetic and algebra to the learned works which bear the following titles:—

Die Algebra der Griechen. Nach den Quellen bearbeitet von Dr. C. H. F. Nesselmann, Privat-docenten an der Universitat zu Konigsberg. Berlin, 1842.

Zur Geschichte der Mathematik, in Alterthum und Mittelalter, von Dr. Hermann Hankel, Weil. Ord. Professor der Math. an der Universitat zu Tubingen. Leipzig, 1874.

Villani, the historian of Florence, writes of Paoli di Dagomari, who died about 1350, as a great geometer and most skilful arithmetician, and who surpassed both the ancients and the moderns in the knowledge of equations, which at that time constituted the most advanced branch of the science. Also, Raffaelo Caracci, of the same century, composed a work entitled "Ragionamento di Algebra," in which he writes of Guglielmo di Lunis, who had translated a treatise on Algebra from Arabic into Italian. This was the work of Mohammed Ben Musa, and was executed not long after the introduction of that science into Italy by Leonardo of Pisa. All these combine to show that the Italians were in possession of the knowledge of the Arabic numerals and the science of Algebra long before the rest of the nations of Europe.

Paciolus, and all the Italian writers on arithmetic and algebra, declare that Leonardo of Pisa was the first who introduced the knowledge of algorithm and algebra to his countrymen. Paciolus appears to have taught these sciences at Venice about 1460, and he specially makes mention of his three predecessors, who had in succession filled the office expressly dedicated to the exposition of these sciences. Their names were Paolo della Pergola, Demetrio Bragadini, and Antonio Cornaro, the last of whom had been his fellow-disciple.

The first Italian writer on the science of Algebra whose treatise was printed in Italy was Lucas Paciolus, or Lucas de Burgo. This work was published in 1494, with the title, "L'Arte Maggiore ditta dal Volgo la Regola de la Cosa over Alghebra e Almucabala." He adopted the name of "*L'Arte Maggiore*" to distinguish it from the science of number, which was then called "*L'Arte Minore*." He generally employs both the terms *Algebra* and *Almucabala*, and from the former word devised the adjective *Algebratico*. He informs his readers that the science is named *Algebra wa Almucabala* in the Arabic language, and those words are thus explained in the Roman language: "*Algebra*, id est, *restauratio; Almucabala*, id est, *oppositio* vel *comparatio* et *solidatio*." These two words being only the names of two preliminary operations in the preparation of equations for solution.

The words Algebra and Algorism were introduced at the same time into Europe, and the latter term was always applied to treatises on Arithmetic with the Indian figures. Algorism, or Algorithm, was corrupted into Augrym, and the counters which were used in calculation were called "augrym stones," as may be read in Chaucer's Miller's Tale.

After the appearance in print of the works of Lucas de Burgo on arithmetic and algebra, the knowledge of these sciences in the following century, not only in Italy, but also in other countries of Europe, engaged the attention of learned men, who promoted the advancement of them and introduced numerous improvements.

It appears from the treatises of the first Italian writers on Algebra that they did not understand how to turn the extension of the science of arithmetic to the purposes for which it has subsequently been found most useful. The application of the different methods of reckoning in common life appeared to them, as a contrast with algebra, an entirely theoretic knowledge. They appear to have held ideas as inadequate as their teachers of the real meaning of the word "equation." Neither they nor the Arabians knew equality to be any other than two or more absolutely positive quantities, and the Italians follow the special rules of their Arabian teachers.

As it has been very aptly described—"they worked with an instrument the use of which they did not fully comprehend, and employed a language which expressed more than they were prepared to understand; a language, under the notion first of negative, and then of imaginary quantities, seemed to involve such mysteries as the accuracy of mathematical science must necessarily refuse to admit."

The early Italian writers on Algebra introduced some abbreviations of the terms they learned from their Arabian instructors. They employed the initial letters *p* and *m* of the words *plus* and *minus* to denote a positive and a negative quantity, and used C^o and q^a for the first and second unknown quantities. They also adopted the characters C^o, C^e, C^u, to denote the first, second, and third powers respectively, and denominated the biquadrate and the higher powers as relato primo, secondo, tertio, &c. They assumed *R*, the initial letter of *radix*, to denote the square root.

It appears that the Italian mathematicians adopted the same classification of cubic equations as the Arabic writers; and that in the year 1505 Scipio Ferreo, Professor of Mathematics at Bononia, discovered a solution of one of these classes, which by the Arabic writers was termed "cube and things equal to numbers," and in modern notation would be expressed by $x^3+ax=b$. The like solution was also effected by Lewis Ferrari. Scipio Ferreo made known his discovery to Florido, one of his pupils, who challenged Tartaglia to solve the question. Tartaglia effected the solution, and in turn proposed to Ferreo two other cases of cubic equations, the solutions of which he himself had previously discovered.

Jerome Cardan was a physician at Bononia, and read public lectures on the mathematical sciences. In the year 1539 he published a work on Algebra in the Latin language, entitled "Ars Magna," and in 1545, to this work he published an addition which contained the recent discoveries on the solution of cubic equations, all of which he claimed as his own, except two cases discovered by Lewis Ferrari.

Cardan's great work begins with some remarks on the name and terms employed in the science. In general he expresses the terms of the science by words at length, designating a known quantity or number by the word *numerus*, the first power of the unknown quantity by *res*, the second power by *quadratum*, the third by *cubum*, &c., employing no symbol for the unknown quantity. He uses the letters *p* and *m* for *plus* and *minus*, and *R*, the initial letter of *radix*, to indicate the square root. He sometimes uses for *res* the words *positio* and *quantitas ignota*.

In treating of squares and square roots, he shows that all square numbers have two square roots, one positive (*vera*), and the other negative (*ficta*). He adds that odd powers have only one root, *vera* or positive, but none *ficta* or negative, because a negative number raised to an odd power can never produce a positive number. In quadratic equations he admits both positive and negative (*vera et ficta*) roots, whether integral, fractional, or surd, but does not allow such quantities as involve the square root of a negative quantity to be roots, which are now called imaginary or impossible roots.

Also he considers such quadratic equations as involve the fourth and second powers of the unknown quantity, may have four, two, or no roots at all: as $x^4+12=6x^2$ has no roots, all being impossible; $x^4+3x^2=28$ has two roots, $+2$ and -2, the other two being impossible; and $x^4+12=7x^2$ has four roots, $+2$, -2, $+\surd 3$, $-\surd 3$.

Among the numerous improvements of the science contained in his works, the most celebrated is the method of solving one class of cubic equations which incorrectly bears his name. He has amply exemplified the methods of dealing with all the forms of cubic equations. He remarks that the equation $x^3+6x=20$ has only one root, $+2$, the other two being imaginary; but he considers $x^3+16=12x$ has two roots, +2 and −4; of the third root, +2, being the same as the first, he takes no account. The equation $x^3 = 2x + 21$ has only one negative root, the other two being impossible, or involving the square root of a negative quantity. In this manner he shows how a cubic equation may have one, two, or three roots, according to the form of the equation and the relation of the coefficients.

He exhibits, by numerous examples, how a cubic equation can have only one root, positive or negative. This constitutes the special case which bears the name of Cardan's rule for the solution of a cubic equation. In modern language Cardan's rule is applicable in all cases where one root is positive or negative, and the other two roots imaginary or impossible. And it has been since discovered that Cardan's rule is also applicable when two roots of a cubic equation are equal. He was not ignorant of the difficulty of that case which has been called the irreducible case of cubic equations, but failed to discover the solution.

In the twenty-ninth chapter he has given the method of Lewis Ferrari for the solution of biquadratic equations, and has illustrated the method by numerous examples. And in general, it may be added, he has dealt with the transformation of equations and the nature of their roots, whether integral, surd, or imaginary.

Nicolas Tartaglia was born at Brescia, A.D. 1500, and at the capture of that city by the French in 1512 he was wounded and left for dead. By his mother's care he recovered from his wounds, but from the effect of a wound on his lips he stammered so much that he was nicknamed *Tartaglia*, from the Italian word which signifies that fault. In 1534 he settled in Venice and became Professor of the Mathematical Sciences, an office which he retained for twenty-five years.

The first part of his great work, entitled "Trattato di Numeri e Misure," was published at Venice in 1556, and the second part in 1560. The whole work consists of three large volumes. The first volume contains a system of practical and mercantile arithmetic; the other two volumes treat of geometry, mensuration, speculative arithmetic, and algebra. This work was abridged and translated into French by Guillaume Gosselin de Caen in 1578.

Tartaglia is chiefly celebrated for the discovery he made of the solution of that class of cubic equations which is commonly, but improperly, attributed to Cardan. He communicated his discovery to Cardan under the promise of secrecy with an oath, which Cardan did not keep, but published[1] it in 1545. In 1546 Tartaglia printed at

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[1] Tartaglia wrote to Cardan in the following terms, on learning the latter was about to publish his rules: "M. Hieronime, I have received your letter, in which you write that you understand the rule for the case $x^3 = ax + b$, when $\frac{1}{4}b^2$ is greater than $\frac{1}{27}a^3$; but when $\frac{1}{27}a^3$ exceeds $\frac{1}{4}b^2$, you cannot resolve the equation, and there-

Venice "Quesiti e Invenzioni Diverse," and dedicated his work to King Henry VIII. of England. This work contains his correspondence with Cardan, and gives the history of the discovery of his rules for the solution of cubic equations.

John Muller, generally called Regiomontanus, Joannes de Regio Monte (Konigsberg), in his work "De Triangulis Omnimodis libri quinque," Norimberg, 1533, writes, in reference to Cardinal Nicolaus, of Casa, "Paucis enim admodum artem Algebræ, sive rei et census satis cognitam scio."

A treatise on Algebra was composed in Latin by a native of Spain, with the title of "Arithmeticæ Practicæ seu Algorismi Tractatus, a Petro Sanchez Teruelo, noviter compilatus explicatusque. Impressus Parisiis per Thomam Rees in Domo Rubra post Carmelitas, Anno Domini 1513."

Another treatise was published in 1536 in the Spanish language, bearing the title of "Tractado subtilissimo de Arismetica y de Geometria; compuesto y ordenado por el Reverendo Padre fray Juan de Ortega, de la Orden de los predicadores."

An original treatise on arithmetic and algebra entitled, "Arithmetica Integra," was first put forth at Nuremburg, in 1544, by Michael Stifel or Stifelius, with an Introduction[1] by Philip Melancthon. The work is

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fore you request me to send you the solution of the equation $x^3=9x+10$. To which I reply that you have not used a good method in that case, and that your whole process is entirely false. As to resolving you the equation you have sent, I must say that I am very sorry that I have already given you so much as I have done; for I have been informed by a credible person that you are about to publish another algebraical work, and that you have been boasting through Milan of having discovered some new rules in algebra. But take notice, that if you break your faith with me, I shall certainly keep my word with you; nay, I even assure you to do more than I promised."

[1] Philippus Melancthon Lectori, S.:—

"Non mihi si linguæ centum sint, oraque centum, enumerare possem, quam multis in rebus usus sit numerorum. Et ita sunt in conspectu atque obviæ utilitates non solum numerorum, sed etiam artis, quæ longas et intricatas rationes mirabili dexteritate subducit et explicat, ut neminem quamlibet hebetem esse existimem, qui non et numeros miretur et de arte ipsa præclare sentiat. Quare si prolixum encomium de his utilitatibus instituerim, perinde facerem, ut si accenderem, quemadmodum Græci dicunt, *ἐν τῇ μεσημβρίᾳ λυχνόν*. Sed hominis studiosi est intelligere, quas utilitates proprie afferat Arithmetica his, qui solidam et perfectam doctrinam in cæteris philosophiæ partibus explicant. Quod enim vulgo dicunt, principium esse dimidium totius, id vel maxime in philosophiæ partibus conspicitur. Unus est aditus ad præstantissimam philosophiæ partem de motibus coelestibus, cognitio arithmetices. Et hæc tantam vim habet, ut mediocriter exercitatus in Arithmetica, facile cætera perspiciat et assequatur. Ita plus quam dimidium totius ejus philosophiæ tenet is, qui mediocriter cognovit Arithmeticen. Hanc tantam utilitatem præditi generosis naturis diligenter considerent, ut et exuscitent animos ad amorem hujus artis, et præparent se ad percipiendas cæteras artes. Quid quod multum etiam in physicis, multum in historiis utimur hac subtiliori doctrina de

divided into three books. The first book treats of Arithmetic, the second expounds the tenth book of Euclid's Elements, and the third consists of a treatise on Algebra, which he describes by the words, "De numeris cossicis."[1] He greatly improved and extended the notations then in use, and was the first to introduce the use of the symbols + and −, *plus* and *minus*, to indicate the operations of addition and subtraction.[2] These symbols do not appear to have been so employed by any writer before him. In the multiplication and division of two quantities he shows that like signs give + for the product and the

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numeris. Denique turpe est homini versanti in literis, negligere hanc artem, quæ est fons et inchoatio ratiocinationis universæ, quæ primum discernit unum et multa, eaque distribuit, qua luce proprie homines distant a pecudibus. Ideo boni et docti viri summa vi anniti debent, ut in scholas revocent hanc doctrinam, ut postquam explosa est illa diluta et loquax Sophistice, quæ oppresserat Dialecticam et Physicen, nunc veteris et purioris philosophiæ studia iterum accendantur. Hæc diligentia multis modis profutura est Reipublicæ, nam simplici et pura doctrina adsuefactæ mentes rectius judicant, quærunt certa in doctrinis, non tenent mordicus incerta. Magnum et hoc bonum est, quod hæc ipsa consuetudo adfert amorem veritatis, qui bonos viros efficit, et diligenter adsuefacit animos ad moderationem etiam in reliqua vita. Quid est autem rebus humanis magis salutare, quam eos qui propter doctrinam reipublicæ præficiuntur, et veritatis amantes esse, et moderatos? Ita bene et præclare merebuntur scholæ de genere humano, si veram, utilem, puram doctrinam proponent juventuti, eamque simul ad veritatis amorem et ad diligentiam ac moderationem in vita instituant. Proinde laudandus est conatus multorum, qui hoc tempore aut vetera scripta edunt atque illustrant de his disciplinis quæ fontes philosophiæ continent, aut cudunt nova. Hoc consilio hoc Arithmeticum scriptum Michaelis Stifelii studiosæ juventuti commendandum esse duxi, quod cum ad exercitationem proderit, tum vero ad causas præceptionum quærendas plurimum lucis afferet. Bene vale, Vuitebergæ Calendis Januarii, 1543."

[1] It is a singular circumstance that the Italian name, "Arte della cosa," was again (not by Italian writers) turned into the Latin form of "Ars Cossica" and "Ars cossæ or cossæ," as Gemma Frisius writes "per Regulam cossæ sive Algebræ;" and even in more recent times may be read on the title-page of a work published at Frankfort in 1813, the words, "Ars cossæ promota," and on the title-page of another published at Stutgard in 1815, "de functionibus cossæ resolventibus tractatio."

[2] In the eleventh volume of the Cambridge Philosophical Transactions there is an article by the late Professor De Morgan on the early history of the signs + and −. He describes a work on Arithmetic by John Widman, of Eger, printed at Leipsic in 1489, which contains the earliest instances known of the employment of these signs. The Professor remarks: "I observe that the invention of signs of operation did not suggest itself to the arithmeticians of the time. It will be said that + and − are here to dispute my assertion. Put these out of sight for a moment; the question I shall raise being, whether + and − did not come to the arithmetician *from the warehouse*. There is another case which shows us that the arithmetician's own *directive* symbols never became signs of *operation*, though in almost universal use for a century."

And he further remarks: "We know that the inventors of our symbols attached very little importance to them; and would have stared with wonder if they had been told that these trumpery tricks of abbreviation would one day have a philosophy of their own, and would make inquirers curious about their origin."

quotient, and unlike signs −. He employs the character $\surd$ for *r*, the initial letter of *radix*, to denote the square root.

He uses no symbol to denote the equality of two expressions, but employs the word *equatus* to connect them. He assumes certain characters to denote unknown quantities, which he explains by the words, "Est enim numerus cossicus nihil aliud quam numerus signo cossico denominatus." The symbol for the first cossic number he calls radix. He also employs the letters *A*, *B*, &c., in the same manner for different unknown quantities. Several of his problems are illustrated by the properties of geometrical diagrams. He explains a method of extracting the square root of a binomial when one of its terms is a quadratic surd, and makes reference to the fourth proposition of the second book of Euclid's Elements.

In treating of quadratic equations, he points out when the index of the unknown quantity in one term is double of that in the other, the equation can be reduced by means of the rule for quadratic equations. His work does not contain any reference to cubic equations, probably on account of that subject having been treated by Cardan, and other writers of the time.

He denotes the powers of quantities by the initial letters of their names, and shows that they form a geometrical progression beginning with unity; and that the series of the natural numbers (which he names exponents), beginning with zero, will indicate in order the successive powers of quantities. He also remarks, that when the geometrical progression is continued in a decreasing series less than unity, the exponents of the powers will, after 0, become negative numbers, and be still the true exponents of the powers which are less than unity.

He exemplifies in the following two series the relation between the order of the natural numbers and the corresponding terms of any series of numbers in geometrical progression, and the consequences arising from that relation:

-3	-2	-1	0	1	2	3	4	5	6
$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32	64.

And remarks, *whatever series of numbers a geometrical progression produces* by multiplication and division, *a similar arithmetical progression makes* by addition and subtraction.

It may be noted that the arithmetical progression is formed by the addition of unity to, and subtraction from, the natural numbers. The following are his first three examples:—

As $\frac{1}{8}$ multiplied into 64 produces 8, so -3 added to 6 produces 3. But -3 is the exponent of $\frac{1}{8}$, as 6 is the exponent of 64. Then 3 is the exponent of the number 8.

Next, 64 divided by $\frac{1}{8}$ gives the quotient 512, so -3 subtracted from 6 leaves 9. Thus 9 is the exponent of the number 512.

Thirdly, $\frac{1}{8}$ divided by 64 gives $\frac{1}{512}$ for the quotient, and 6 subtracted from -3 leaves -9; then -9 is the exponent of the fraction $\frac{1}{512}$.

These relations between corresponding powers and their exponents constitute the fundamental principle of the system which in the next century perfected the powers of numerical calculation by its combination with the descending scale of the denary notation.

It may here be remarked that the conventions introduced by Stifelius[1] surpass those of his predecessors and contemporaries, and led the way to further extensions in the symbolical language of the mathematical sciences. Improvements in notation, however simple, are of the highest importance in the history of mathematical science. Only those who know its history can appreciate how much has been achieved by the improvement of symbolical language. The notation of numbers gained its great power by the simple expedient of a character which could occupy any place in the scale, but have no significant value by itself. The simple device of employing the last letters of the alphabet to denote the objects of search before they were determined, and the first letters of the alphabet to designate such as were known, effectively distinguished the data from the quæsita in every problem in which these notations are employed. And further, numbers, both abstract and concrete, known or unknown, can be denoted by symbols of general number, so as to express any definite operation performed with indefinite numbers.

Mr. Babbage[2] has judiciously remarked on this subject:—"The invention of what we may call primary or fundamental notation has been but little indebted to analogy, evidently owing to the small extent of ideas in which comparison can be made useful. But at the same time analogy should be attended to, were it for no other reason than that, by making the invention of notation an art, the exertion of individual caprice ceases to be allowable. Nothing is more easy than the invention of notation, and nothing of worse example and consequence than the confusion of mathematical expressions by unknown symbols. If new notation be advisable, either permanently or temporarily, it should carry with it some mark of distinction from that which is already in use, unless it be a demonstrable extension of the latter."

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[1] The following are the words in which Michael Stifelius concludes his excellent work on the second page of fol. 305:—

"Existimo ergo me per ista omnia satis demonstrasse, quam solida sit ratio mea, qua pro octo regulis Algebræ unam posuerim simplicem atque facilem.

"Tu antem Domine Jesu Christe, Fili Dei vivi, Deus noster, mortuorum Resuscitator, et Judex futuri sæculi, largire nobis supplicibus tuis, ut hæ spinæ non suffocent in nobis scientiam Tui, sed Te toto affectu diligamus, quod dolores nostros tuleris, et languores nostros portaveris, propter iniquitates nostras vulneratus sis, et attritus sis propter scelera nostra. Nunc disciplina pacis nostræ super te, quod livore tuo sanati sumus. Sit igitur vilis omnis scientia, huic divinæ scientiæ tuæ comparata, et favor hominum nausea sit, ad divinam beneficentiam tuam comparatus. Amen."

[2] Encyclop. Metropol., ii., p. 338.

John Scheubel, or Scheubelius, Professor of Mathematics at the University of Tubingen, printed a treatise at Paris, in 1552, entitled, "Algebræ Compendiosa facilisque descriptio, qua depromuntur magna Arithmetices miracula." He has chiefly followed Stifel in the method of his work. In his chapter on surds he observes that the different kinds of binomial surds may be referred to the irrational lines treated of in the tenth book of Euclid's Elements. He gives the same rule as Stifel for extracting the square root of a binomial surd when the process is possible, and illustrates his rule by numerous examples.

Robert Recorde, M.D., composed the first treatise on Algebra which was published in the English language.

The first part of his treatise was entitled, "The Grounde of Arts, teaching the perfect worke and practice of Arithmeticke, both in whole numbers and fractions." This work was first printed in 1552, and it passed through many editions.

The second part of his treatise was entitled, "The Whetstone of Witte,[1] containing the extraction of rootes, the Cossike practice with the rules of equation; and the workes of surde nombers."

This work is drawn up in the form of a dialogue between the teacher and the learner, and was first published at London in 1557.

It treats of the properties of numbers and the extraction of the square and cube roots of numbers, and of the roots of compound algebraical expressions. It treats of algebra, or cossic numbers, as he calls them, and employs the notations of Stifelius, and gives a full account of the arithmetic of surd numbers. The terms binomial and residual were first employed by him. His work also contains rules for the solution of simple and quadratic equations, with numerous examples. In this work Robert Recorde first introduced the sign of equality. "In the rule of equation," he quaintly remarks, "and to avoide the tediouse repetition of these wordes—'is equal to'—I will set, as I doe often in worke use, a paire of paralleles, or gemowe lines of one length, thus =, bicause noe two thinges can be more equalle." Napier adopted this sign, and defined it in these words: "Betwixt the parts of an equation that are equal to each other, a double line is interposed, which is the sign of equation." But Mr. Babbage, in one of his papers on notation, observes: "It is a curious circumstance, that the symbol which now represents equality was first used to denote subtraction, in which sense it was applied by Albert Girard, and that a word signifying equality was always used instead until the time of Harriot." This sentence overlooks the claim of Robert Recorde.

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[1] The late Professor De Morgan has remarked: "It is rarely remembered that the old name of Algebra, 'The Cossic Art' (from *cosa*, thing), gave the first English work on Algebra its punning title, 'The Whetstone of Witte,' 'Cos Ingenii,' by substituting the Latin word *cos* for the Italian word *cosa*."

In the year 1558 was published at Paris a work by Jacques Pelletier, entitled, "De Occulta parte numerorum quam Algebram vocant, libri duo." This book contains several useful improvements, among which may be noted, that he showed that any rational root of an equation is one of the factors of the last term, when the equation is arranged according to the descending powers of the unknown quantity. He explained the method of reducing binomial and trinomial surds to their most simple forms, and exhibited some curious properties of square and cube numbers. He has remarked, "that the sum of any number of the cubes taken from the beginning, always makes a square number, the root of which is the sum of the roots of the cubes."

Oronteus Fineus was the predecessor of Peter Ramus as Professor of Mathematics at Paris. He composed a work, "De Arithmetica Practica," in four books; a second edition of which was published in 1555. The geometry was translated by William Bedwell into English, and published in a quarto volume at London, 1636.

Peter Ramus was born in the year 1515, and became one of the most distinguished scholars of his age. He was brought up a Roman Catholic, but the idolatry and superstition of the Romish Church induced him to unite himself with the Reformed Church in France, and he became afterwards a constant object of persecution, especially by the doctors of the Sorbonne. He was obliged to take refuge in Germany during the persecution of the Protestants in France, and on his return to Paris, being a Protestant, he ended his life as one of the victims at the massacre of St. Bartholomew in 1572. Ramus is reported to have concealed himself in a granary; but when discovered, he was dragged out by some doctors, who took his money from him with the pretence of saving his life. Having secured this, they gave him over to the assassins, who, after cutting his throat and mangling his body, flung it out of a window, and his bowels gushed out in the fall. Some scholars, encouraged by their masters, dragged his mutilated corpse in a most ignominious manner through the streets, and threw it into the Seine.

Peter Ramus was a man of extensive learning and an able orator. He wrote on grammar, rhetoric, and logic; but his chief works are his writings on mathematics. The first edition of his Arithmetic appeared as a quarto volume, with the title, "P. Rami Eloquentiæ et Philosophiæ Professoris Regii, Arithmeticæ Libri tres. Parisiis Anno Salutis, 1555. Cum privilegio Regis."[1]

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[1] In p. 107 of the first edition of his Arithmetic is given the following example of the cube root:—"Si propositus numerus cubus non est, nullum est verum latus:

In the year 1569 appeared "P. Rami Scholarum Mathematicarum, libri unus et triginta," and in the same year his "Arithmeticæ libri duo: Geometriæ septem et viginti," and another edition was published in the year 1580. In his method of extracting the square root and the cube root of numbers which are not perfect square and cubic numbers, he directed to annex periods of ciphers to the given number whose root was required, two in each period for the square root, and three in each for the cube root. The extraction can then be continued and an approximation to the true value of the root determined to any degree of accuracy required.

Pedro Nunez, or Nonius, was an eminent physician, and Professor of Mathematics in the University of Coimbra, in Portugal. His chief work was a treatise on Algebra, which he composed in Portuguese, but afterwards translated into the Castilian language. It was published at Lisbon in 1564, with the title, "Libro de Algebra y Arithmetica y Geometrica."

A treatise in Italian on Arithmetic was published under the title of "Le Pratische delle due Prime Mathematiche di Pietro Cataneo Sienese. In Venetia, 1567."

Raphael Bombelli, of Bologna, published in the Italian language a treatise on Algebra in 1579, the dedication of which, however, bears the date of 1572. His work contains some notices of the history of the science, and he follows the methods of preceding writers. In his

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cubi tamen in eo maximi verum latus erui potest: ut in 17616, cubus continetur 17576, et latus ejus est 26, et supersunt 40.

"Si voles tamen totius numeri 17616 exquirere latus vero propinquum, poteris per datas partes, ut in quadratis: prima multiplicatio sit nominis et cubica: secunda sit producti per propositum: nam latus secundi producti, numerator, etiam erit datarum partium, ut numerus 17616, præter cubum 17576, habet 40, quæ latere 26 retexto relinquuntur. Si quæras igitur de toto numero 17616, quot centesimas habeat ejus latus præter 26 integra, multiplica 100 in se cubice, fient 1000000: quæ multiplicata 17616, fient 17616000000, quorum latus cubicum est 2601 pro numeratore datarum centesimarum, sic $\frac{2601}{100}$, et supersunt 19712199, quæ ne centesimam quidem integri faciunt, ideoque despiciantur."

This passage of Ramus is found in the work of L. Schonerus, "De Numeris Geometricis," which was translated into English by T. Bedwell in 1614. In p. 70 it may be read in these words:—"As, for example, the same number 17616, reduced unto one hundred cubicall parts, that is unto 1,000,000, do make 17,616,000,000, for the numerator. The parts then are thus $\frac{17,616,000,000}{1,000,000}$. Now the side of the numerator is 2,601, for the numerator of the one hundred parts given (for the former denominator being made by the multiplication of 100, by itself cubically; the roote or side of it must needs be 100, for the denominator of the parts sought). Therefore the parts desired are $\frac{2601}{100}$, that is, by reduction, $26\frac{1}{100}$, and besides that, there do remaine 19,712,199, which cannot adde so much as $\frac{1}{100}$ part unto the side found; because that the difference of this cubicke from the next greater above it, is greater than this remaine: and therefore that remaine is neglected as not of any moment."

treatment of quadratic equations he admits only positive roots, and on equations of the form $ax^2+c=bx$, he remarks that the nature of the question will determine which of the two roots is the root required.

In cubic equations he follows the methods of Cardan. One of the most important improvements consists in his method of finding the cube roots of binomial surds which involve imaginary quantities. He shows that the cube root of $2+\surd(-121)$ is $2+\surd(-1)$. He also shows that an angle can be trisected by means of the solution of a cubic equation. In the discussion of equations of the fourth degree he follows the methods of Lewis Ferrari.

Gosselin's work on this science, published in 1577, bears the title of "De Arte Magna seu de Occulta parte numerorum quæ et Algebra et Almucabala vulgo dicitur." This work will be found among the latest in which the word Almucabala is employed in the name of the science.

Francis Vieta was born in the year 1540 at Fontenai, and died in 1603. He became an eminent mathematician, and his writings exhibit numerous improvements, especially in Algebra, which he designated by the title of "Arithmetica Speciosa," as distinguished from ordinary arithmetic, which he named "Arithmetica Numerosa." He published his treatise on Trigonometry at Paris in 1579, in which he showed that if the diameter of a circle be 10000 . . . . . , the perimeters of the inscribed and circumscribed polygons of 393216 sides are 31415926535 . . . . and 31415926537 . . . . . respectively, and that therefore the measure of the circumference of the circles lies between these two numbers.

The mathematical works of Vieta were collected by Francis Schooten, and published in a folio volume at Leyden in 1646. His works on different branches of the mathematical sciences are divided into fifteen parts, of which the first five are devoted to subjects of Algebra. He introduced various improvements into the science, both in its language and notation. He first used the words *affirmative* and *negative*, *pure* and *adfected*, and the term *coefficient*. Letters had been used by other writers before the time of Vieta to denote unknown quantities, but he employed capital letters, the *vowels* to denote *unknown*, and the *consonants known* quantities. He connected by the symbols + or − the quantities which were to be added or subtracted, always placing the greater before the less; and when it was not known which was the greater, he placed the symbol = between them to denote their difference. He had no symbol to denote multiplication, but expressed the operation in words; and the successive powers of a binomial he expressed partly by symbols, and partly by words at length.[1] He denoted division by placing the dividend above the divisor, like the numerator and denominator of a fraction. He used no sign for equality. The third part of his work consists of five books of questions, chiefly taken from the Arithmetics of Diophantus. In them he gives rules for the solution of quadratic and cubic equations. And here for the first time is employed a line drawn over two or more letters connected by the symbols + or —, to indicate that the aggregate of the terms is to be considered as one quantity with respect to any operations to which it may be subject with other quantities.

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[1] The following exhibits his method of forming the second, third, fourth, and fifth powers of a binomial:—

Vieta also made various other improvements in equations. He showed how to construct cubic and biquadratic equations from quadratics. He pointed out the relation between the roots and the coefficients of the terms of a complete equation, when the terms were alternately positive and negative, or all the roots positive; and showed how to increase or diminish the roots of given equations without knowing the values of the roots themselves. He also exhibited the roots of equations by means of angular sections. In his various improvements he does not appear to have appreciated the ideas of Cardan respecting negative roots, nor the important principle of exponents by which integers were employed to denote the powers, and fractions the roots of quantities.

Si fuerint duo latera, quadratum lateris primi, plus plano a duplo latere primo in latus secundi, plus quadrato lateris secundi, æquatur quadrato adgregati laterum.

Sit latus unum A, alterum B; Dico A quadratum $+A$ in B 2, $+B$ quadrato, æquari $A+B$ quadrato. Ex opere multiplicationis $A+B$ by $A+B$.

Si fuerint duo latera: cubus lateris primi, plus solido a quadrato lateris primi in latus secundum triplum, plus solido a latere primo in lateris secundi quadratum triplum, plus cubo lateris secundi, æquatur cubo adgregati laterum.

Sit latus unum A, alterum B; Dico A cubum, $+A$ quadrato in B 3, $+A$ in B quadratum 3, $+B$ cubo, æquari $A+B$ cubo. Ex opere multiplicationis A quadrati $+A$ in B2, $+B$ quadrato, per $A+B$.

Si fuerint duo latera: quadrato-quadratum lateris primi, plus cubo lateris primi in latus secundum quadruplum, plus quadrato lateris primi in lateris secundi quadratum sextuplum, plus latere primo in lateris secundi cubum quadruplum, plus quadrato-quadrato lateris secundi, æquatur quadrato quadrato adgregati laterum.

Sit latus unum A, alterum B: Dico A quad.-quadratum, $+A$ cubo in B 4, $+A$ quadrato in B quadratum 6, $+A$ in B cubum 4, $+B$ quad.-quadrato, æquari $A+B$ quad.-quadrato. Ex opere multiplicationis A cubi, $+A$ quadrato in B 3, $+A$ in B quadratum 3, $+B$ cubo, per $A+B$.

Si fuerint duo latero: Quadrato-cubus lateris primi, plus quadrato-quadrato lateris primi in latus secundum quintuplum, plus cubo lateris primi in lateris secundi quadratum decuplum, plus quadrato lateris primi in lateris secundi cubum decuplum, plus latere primo in lateris secundi quadrato-quadratum quintuplum, plus lateris secundi quadrato-cubo, æquatur quadrato-cubo adgregati laterum.

Sit latus unum A, alterum B: Dico A quadrato-cubum, $+A$ quad.-quadrato in B 5, $+A$ cubo in B quadratum 10, $+A$ quadrato in B cubum 10, $+A$ in B quad.-quadratum 5, $+B$ quadrato-cubo, æquari $A+B$ quadrato cubo. Ex opere multiplicationis A quadrato-quadrati, $+A$ cubo in B 4, $+A$ quad. in B quadratum 6, $+A$ in B cubum 4, $+B$ quadrato-quadrato, per $A+B$.

Christopher Clavius was the first German who composed a treatise on numbers. His Epitome Arithmeticæ Practicæ was written in Latin, and published at Rome in 1583.

Simon Stevin, of Bruges, first published his work on Arithmetic and Algebra in Flemish. It was afterwards translated into French and published in 1584. The whole works of Stevin were some time after collected and edited by Albert Girard, and were published in French at Leyden in 1634, the year after his death. They constitute a large folio volume, and include a translation in French of the Arithmetics of Diophantus, with the solutions exhibited in the new notation Stevin had devised. Several improvements of importance are exhibited in this work. He employed the signs + and − as his predecessors, but used a colon to denote equality, and placed the sign × between two fractions when the first was to be divided by the second, as $\frac{2}{3} \times \frac{5}{7} : \frac{15}{14}$, denoted that $\frac{2}{3}$ divided by $\frac{5}{7}$ is equal to $\frac{15}{14}$ (p. 23, fol. ed.).

He extended the meaning of coefficients to include fractions, surd numbers, and imaginary quantities. In approximating to the square root and the cube root of numbers which are not complete powers, he either continued the extraction by annexing periods of ciphers for obtaining any required number of decimals; or he employed a method of approximation in which he used the nearest exact root of the given number whose root was required. He gave new names to the successive powers of quantities, not using the terms borrowed from geometry, *square* and *cube*, and expressions compounded of these words, but simply using the natural numbers in order, 1, 2, 3, &c., and calling them the first, second, third, &c., powers.

He devised a new notation both for the unknown quantity and its successive powers, by writing the numbers 1, 2, 3, &c., in small circles to denote the first, second, third, &c., powers of the unknown quantity in equations: and unity he denoted by zero in the small circle. He also extended the use of exponents, and employed the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, &c., placed in small circles, to denote the square root, the cube root, the biquadrate root, &c., of the unknown quantity.

This extension of exponents to denote successive roots, is perfectly consistent with the assumption of the integers 1, 2, 3, 4, &c., to denote successive powers, as it is obvious that whatever operation might be assumed to be denoted by 2 or $\frac{2}{1}$, the inverse operation could be, by a consistent analogy, properly denoted by $\frac{1}{2}$.

He also showed how to form the successive terms of the powers of an expanded binomial by arranging in order under each other the coefficients (omitting the first and last, which are always unity) of the second, third, fourth, &c., powers.

In the year 1629, Albert Girard, an eminent Flemish mathematician, printed at Amsterdam a treatise with the title of "Invention Nouvelle en l'Algebre, tant pour la solution des equations, que pour recognoistre le nombre des solutions qu'elles reçoivent, avec plusieurs choses qui sont necessaires a la perfection de ceste divine science." In this treatise he followed, in general, the notations and methods of his predecessors, and employed several new symbols, most of which have not been admitted into the science. He first employed a parenthesis to include two or more quantities connected by the signs $+$ and $-$, when considered as one quantity; as, for instance, $(a-b)-(c+d)$ denotes that the sum of c and d is to be subtracted from the difference of a and b. A straight-line placed over the quantities had before been employed by Vieta in the same sense, as $\overline{a-b}-\overline{c+d}$. He described negative quantities as less than zero or nothing—an inaccurate expression—as every finite quantity retains its value assumed or determined under given circumstances; and if it be negative, it implies that it is of a contrary nature in some way to what it would have been if positive. He made several useful improvements and discoveries, the principal of which related to equations. He appears to have been the first who clearly understood the relations between the roots and coefficients of the terms of an equation, and that every equation had as many roots, real or imaginary, as the number of units in the index of the highest power of the unknown quantity. He also showed how to find the sums of the first, second, third, &c., powers of the roots of an equation in terms of the coefficients. To him also belongs the merit of having first solved the irreducible case of cubic equations by means of a table of sines. He also explained the geometrical meaning of the negative roots of an equation, by showing that they represented lines of the same magnitude, but drawn in a contrary direction to those which represented the positive roots.

The valuable papers on Algebraic Equations, left by Thomas Harriot, were not published until the year 1631, ten years after his death. They were edited by his friend Walter Warner, and printed in a folio volume, with the title, "Ars Analyticæ Praxis, ad Equationes Algebraicas nova, expedita, et generali methodo, resolvendas: Tractatus e posthumis Thomæ Harrioti Philosophi ac Mathematici celeberrimi Schediasmatis summa fide et diligentia descriptus." Harriot adopted the sign $=$ from Robert Recorde to denote equality, and first devised and used the sign $>$ to denote "is greater than," and $<$ "is less than," as is explained in the tenth page of his work. He also first introduced the use of the small letters of the alphabet, employing the vowels to denote *unknown*, and the consonants *known* quantities. He represented a product of two or more quantities by writing the letters together as a word, prefixing the coefficient before them separated by a point.

Harriot was the first who thought of arranging all the terms of an equation on one side, making the aggregate equal to zero, and showed how equations of all degrees with given roots may be formed by the multiplication of simple equations, or equations of the first degree. If the values of the unknown quantity a be b, c, d, f, &c., so that $a=b$, $a=c$, $a=d$, $a=f$, &c.; whence it follows that $a-b=o$, $a-c=o$, $a-d=o$, $a-f=o$, &c.; and the product of two of these simple equations gives a quadratic equation, three a cubic, four a biquadratic, &c., with all the terms on one side made equal to zero. From this it is obvious, on inspection, what relations subsist between the roots and the coefficients of any equation.

Since in every equation so arranged, the coefficient of the second term is always equal to the sum of the roots with contrary signs; this property can be employed for increasing or diminishing the roots by any given quantity, which may be so assumed that the second term shall disappear from the equation.

He also shows that a cubic equation may be formed with a simple and a quadratic equation; and that a biquadratic may be formed in the same manner with two quadratic equations.

He next shows how to transform and reduce equations to others which can be more readily solved, and illustrates his methods by numerous examples.

A brief notice[1] has been given of William Oughtred, a mathematician of considerable talent, whose writings greatly promoted the advancement of the sciences of calculation. In the year 1628 he was appointed tutor to the son of the Earl of Arundel, and he drew up for the use of his pupil a treatise which he called "Arithmeticæ in numeris et speciebus Institutio," which he designed as a key to the mathematics. In the year 1631 he published his treatise with the title of "Clavis Mathematicæ." In the year 1647 appeared in English, "The Key of the Mathematics new forged and filed: together with a treatise of the resolution of all kinde of affected equations in numbers. With the rule of compound usury; and the demonstration of the rule of false position. And a most easy art of delineating all manner of plaine sun-dyalls. Geometrically taught by Will. Oughtred."

The third edition in Latin, greatly improved, was put forth in 1652, with the title, "Gulielmi Oughtred Ætonensis, quondam Collegio

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[1] In Section I., p. 25, of "Elementary Arithmetic, with brief notices of its history."

David Lloyd, in his Memoires (fol. 1668, pp. 608, 609), relates that "he was as facetious in Greek and Latin, as solid in arithmetic, geometry, and the sphere of all measures, music, &c.; exact in his style as in his judgment; handling his cube and other instruments at eighty, as steadily as others did at thirty; owing his, he said, to temperance and archery; principling his people with plain and solid truths, as he did the world with great and useful arts; advancing new inventions in all things but religion; which in its old order and decency he maintained, secure in his privacy, prudence, meekness, simplicity, resolution, patience, and contentment."

Regalis in Cantabrigia Socii, Clavis Mathematicæ denuo limata, sive potius fabricata." In the preface he states that Mr. Seth Ward, a Fellow of Sidney College, was the first person who explained the use of his Clavis at Cambridge, where it was long used as a text-book. A fourth edition, after his death, was reprinted in 1667, and another in 1694, with the recommendation of Edmund Halley, F.R.S. In the details of his Clavis, which is an elementary work on Arithmetic and Algebra, he chiefly follows the notations and methods of Vieta. He devised several improvements in notation. He was the first who employed the sign × to denote multiplication;[1] and the colon and double colon to denote a ratio and a proportion. The letters q, c, qq he wrote after numbers and symbols of quantity to denote the second, third, and fourth powers;[2] and $\surd q$, $\surd c$, $\surd qq$, placed before a quantity, to denote its square root, cube root, and biquadrate root. In his Clavis appear the earliest essays of Algebra applied to Geometry for the investigation of unknown geometrical properties.

In the year 1644 was published at Paris a Course of Mathematics composed by Pierre Herigone, in six octavo volumes. This work, the first of its kind, was written in Latin and French, and printed in parallel columns on the same pages. In the treatise on Algebra, he employed the small letters of the alphabet to denote both known and unknown quantities, and placed the coefficients on the left side. He introduced several new characters, some of which have become established in ordinary symbolical language. In this work appear for the first time the exponents[3] of powers placed on the right of the letters; and of the same size; thus, the first, second, third, fourth, &c., powers of a are denoted by a, $a2$, $a3$, $a4$, &c., and named, latus seu radix, quadratum, cubus, quadrato-quadratum, &c.

The square root he denoted by $\surd$ or $\surd 2$, placed before the quantity whose square root was to be extracted. The cube root by $\surd c$ or $\surd 3$, the fourth root by $\surd\surd$ or $\surd 4$, and so on for the higher roots.

A considerable number of his examples relate to the application of Algebra to geometrical magnitudes.

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[1] "Multiplicatio Speciosa joineth together both the magnitudes proposed, with the note, *in*, or *in*, or ×, or for the most part without any note at all, if the magnitudes be set down only with one letter."

[2] Oughtred wrote the second, third, fourth powers, &c., of $A+E$ in the following forms in the fourth edition of the Clavis :—

$Aq+2AE+Eq$. Quadratum.

$Ac+3AqE+3AEq+Eq$. Cubus.

$Aqq+4AcE+6AqEq+4AEc+Eqq$. Quadrato-quadratum.

[3] His words are : "Characteres Cossici varie a variis auctoribus designantur, nos solemus uti iis in quibus exponentes numeris sunt expressi, nimirum his, a, $a2$, $a3$, $a4$, &c." He assumed the symbols 2|2, 3|2, 2|3, for the words, equal, major, minor, respectively, and employed the sign = to denote "parallel to," which had before been adopted by Robert Recorde to denote equality.

René Descartes, born in the year 1596, was of noble descent, being a younger son of a counsellor of France. The clergy of the Romish Church rose against the heresy of his philosophy, as they had ever done in all previous ages against Galileo and such men as honestly sought out the truth in the works of nature. He accepted the invitation of Christina, the Queen of Sweden, who generously offered him an asylum in her kingdom, and secure protection from the intolerance and persecution of the papists.

In the year 1637, at Leyden, was published in French, without the name of the author, the work of Descartes under the following title:—"Discours de la Methode[1] pour bien conduire sa raison, et chercher la Verité dans les Sciences. Plus la Dioptrique,[2] les meteores et la Geometrie: qui sont des Essais de cette Methode."

This work was translated from French into Latin by Francis Schooten, Professor of Mathematics at the University of Leyden, and published in 1649 with a commentary written by the translator, and some additional notes by M. de Beaune; and in 1659 a new edition was published, with further additions on the nature of equations by M. de Beaune; also another edition appeared in 1683.

The treatise entitled "Geometry" embraces the application of

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[1] The following is the first paragraph of his "Discourse de la Methode:"—"Le Bon sens est la chose du monde la mieux partagée: car chacun pense en être si bien pourvû, que ceux même qui sont les plus difficiles à contenter en toute autre chose, n'ont point coutume d'en désirer plus qu'ils en ont. En quoi il n'est pas vrai semblable que tous se trompent. Mais plutôt cela témoigne que la puissance de bien juger, et distinguer le vrai d'avec le faux, qui est proprement ce qu'on nomme le bon sens, ou la raison, est naturellement égale en tous les hommes. Et ainsi que la diversité de nos opinions ne vient pas de ce que les uns sont plus raisonnables que les autres, mais seulement de ce que nous conduisons nos pensées par diverses voies, et ne considerons pas les mêmes choses. Car ce n'est pas assez d'avoir l'esprit bon, mais le principal est de l'appliquer bien. Les plus grandes âmes sont capables des plus grands vices, aussi bien que des plus grandes vertus. Et ceux qui ne marchent que fort lentement peuvent avancer beaucoup d'avantage, s'ils suivent toujours le droit chemin, que ne font ceux qui courent, et qui s'en éloignent."

[2] In his treatise on light, entitled "La Dioptrique," Descartes claimed as his own the discovery of the law of refraction of light. This law had been previously discovered by Willebrod Snell in 1618. Snell found, after numerous experiments, that by prolonging the incident and refracted rays on each side of the point where the refraction takes place, and drawing any vertical line, the parts of the two rays comprised between the above point and the vertical line, always preserve the same constant ratio to each other, whatever may be the obliquity of the incident ray. This is the same as saying that when a ray of light passes out of one medium into another, the sines of the angles of incidence and of refraction always preserve a constant ratio for every angle of incidence in the same media.

Snell did not perceive the identity of these two propositions, but Descartes did, and published the discovery as his own in 1637. Huygens, who knew that Descartes had seen Snell's manuscripts in Holland, disclosed the truth, and exposed the improper claim of Descartes to the discovery.

Algebra not only to straight lines, but also to curve lines. The simple conception of expressing curve lines by means of equations in which the unknown quantities denote the co-ordinates of any point whatever in the curves, constituted a new method of extending the powers of the Greek geometry.

He supposed every curve to be traced out by the motion of a point in a straight line which moves along a fixed straight line called the *axis of abscissas*, and parallel to another fixed straight line called the *axis of ordinates;* the point in the movable line also moving according to some certain law. The nature of the curve is expressed by the relation which exists between the ordinate and abscissa, or the co-ordinates, in the equation to the curve. The equation to a curve can be found if one or more of its essential properties be given; and if the equation to a curve be given, its properties and peculiarities can be ascertained, and the curve can be described.

He also arranged curves in different orders, according to the highest powers of the co-ordinates in their equations. Those which involve only the first power of the co-ordinates belong to the first order, and can include only straight lines. The curves of the second order, which involve the second powers of the co-ordinates, include the circle, the ellipse, the hyperbola, and the parabola. The curves of the third order involve the third powers of the co-ordinates. These have been enumerated (with some few additions since his time) by Sir Isaac Newton, and were first printed in 1704, at the end of his Treatise on Optics, under the title of "Tractatus duo de Speciebus et Magnitudine Figurarum Curvilineorum." The curves of the fourth order, which are numerous, have not been fully investigated and classified.

In treating of Algebra, he introduced numerous improvements in the rules and methods of his predecessors. He adopted the small letters of the alphabet, as others had done before him, but employed the first letters, a, b, c, &c., to denote known quantities, and the last letters, z, y, x, &c., to denote unknown quantities. He improved the notation of powers devised by Herigone. Instead of writing the index or exponent of the power on the right of the letter, he wrote it in a smaller character, and placed it at the upper part, on the right of the letter, as, for instance, he denoted the third power of x by x^3 instead of $x3$. Sometimes, however, he wrote xx and xxx for the second and third powers, instead of x^2 and x^3. He employed the mark ∝ to denote equality,[1] and wrote (p. 372)

$$x^3 - 9xx + 26x - 24 \propto 0, \text{ and } x^4 - 4x^3 - 19xx + 106x - 220 \propto 0.$$

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[1] The character ∝ assumed by Descartes to denote equality, in several places where it is used by him appears very much like æ or œ inverted: if this be so, it was probably adopted by him from the two initial letters of the word æqualis.

In the transformation and solution of equations he introduced numerous rules and important improvements. Of these may be noticed, especially, that he pointed out and proved that the negative and imaginary roots, so-called, of equations, are as real and proper roots in the solution of equations as positive roots. He gave a rule for discriminating in any equation which contains only real roots, how many are positive and how many are negative, by the changes or the permanencies of the signs of the terms when taken in order two and two, the first with the second, the second with the third, the third with the fourth, &c. He also devised a new solution of biquadratic equations by the aid of a cubic and two quadratics.

The formation of the successive coefficients of any power of an expanded binomial by multiplication, was perfectly understood before the time of Newton, and tables of the coefficients of the second, third, fourth, &c., powers had been found and arranged by several writers. There is one of them which, on account of its various applications, can scarcely fail of being interesting to the student.

Henry Briggs left at his death, which happened in 1630, a very important work consisting of Trigonometrical Tables. These tables were printed at Gouda, under the direction of Henry Gellibrand, Professor of Astronomy in Gresham College, and published in 1633. The introduction contains the author's account of the construction of the tables, and an account of their uses was added by the editor. Briggs devised a table which he named Abacus Πάγχρηστος, on account of its frequent and excellent use. It is printed in the eighth chapter of the introduction. A portion of the table is here given, because it exhibits the numerical coefficients in order of the terms of an expanded binomial formed independently of each other.[1]

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[1] The following is a portion of the Abacus, including the first decade of the natural numbers :—

K.	*I.*	*H.*	*G.*	*F.*	*E.*	*D.*	*C.*	*B.*	*A.*
+(10)	+(9)	−(8)	−(7)	+(6)	+(5)	−(4)	−(3)	+(2)	(1)
1	1	1	1	1	1	1	1	1	1
11	10	9	8	7	6	5	4	3	2
66	55	45	36	28	21	15	10	6	3
286	220	165	120	84	56	35	20	10	4
1001	715	495	330	210	126	70	35	15	5
3003	2002	1287	792	462	252	126	56	21	6
8008	5005	3003	1716	924	462	210	84	28	7
19448	11440	6435	3432	1716	792	330	120	36	8
43758	24310	12870	6435	3003	1287	495	165	45	9
92378	48620	24310	11440	5005	2002	715	220	55	10

The first two lines contain, one the letters *A*, *B*, *C*, *D*, &c., written from right to left to denote the order of the vertical columns of the Abacus, and the line of figures beneath contains the characteristics of the figurate numbers.

Pierre de Fermat, a celebrated mathematician, was born in the year 1590, and by his talents and acquirements became Counsellor in the Parliament of Toulouse. His works were collected and published in a folio volume in 1679, sixteen years after his death. They bear the title of "Opera Varia Mathematica," and contain several original treatises. He is deservedly celebrated for his discoveries respecting the quadratures of parabolas, his treatise on Maxima and Minima, and his method of tangents.

Bonaventura Cavalieri, or Cavallerius, as he was commonly called, was a disciple of Galileo, and became an eminent mathematician. In the year 1629 he made known his invention of the method of Indivisibles.[1] This work was entitled "Geometria Indivisibilibus

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The table can be extended from unity to any number whatever, and the series of numbers reckoned from right to left can also be read in vertical columns, or in diagonal columns towards the right or the left.

It begins with the natural numbers written in order under *A* in a vertical column, and a series of units is written under *B*, *C*, *D*, &c., continued from 1 in the first vertical column.

The second series of the table is formed from right to left by addition in this manner:—2 in the first column on the right is added to 1 in the second column, and the sum 3 is written under 1 in column B. 3 in the second column is added to 1 in the third, and the sum 4 is written under column *C*.

In the same way, 4 is added to 1, 5 to 1, 6 to 1, &c., and the sums 5, 6, 7, &c., are written under the columns *D*, *E*, *F*, &c., and so on.

The third series is formed exactly in the same manner:

3 is added to 3, and the sum 6 is written under *B*;
6 is added to 4, and the sum 10 is written under *C*;
10 is added to 5, and the sum 15 is written under *D*;

and so for the rest of the series.

And exactly in the same manner are formed all the succeeding series of numbers, each beginning with the natural number in the vertical column on the right.

It is therefore obvious that all the numbers of the table are found by the successive additions of two diagonal numbers from right to left, and these form the successive horizontal series.

And upon examination it will be found that this table exhibits in the diagonal lines, beginning with the number on the right-hand column *A*, the numerical coefficients of the second, third, fourth, fifth, &c., powers of an expanded binomial. These are derived in order by the successive addition of two diagonal numbers, the numbers in the first column being the coefficients of the second terms respectively of the expansion of each power.

Thus	2, 1	are the coefficients for the	second power.
	3, 3, 1	,,	third power.
	4, 6, 4, 1	,,	fourth power.
	5, 10, 10, 5, 1	,,	fifth power, and so on.

[1] Cavallerius proposed a new hypothesis in the method of reasoning on Geometrical Magnitudes. He considered the three kinds of geometrical magnitude to consist of indivisible elements. The line he supposed to be composed of an infinite number of points; the surface or plane of an infinite number of lines; and

continuorum nova quadam ratione promota," and published at Bologna in 1635, and a second edition appeared in 1653. This is an original work, and has been considered as exhibiting the foundation of the method of fluxions or the infinitesimal calculus. In 1647 he published his "Exercitationes Geometriæ Sex," a work in which he

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the volume of an infinite number of planes or other surfaces. Or, in other words, the point was considered the indivisible element of the line; the line of indivisible breadth, that of the surface; and the plane of indivisible thickness, that of the volume. The method of Indivisibles may be divided into two parts. The object of the first part is to compare figures with each other by means of the equality or the constant ratio which exists between their elements, and which may be reduced to the following proposition:—"All figures whose elements increase or decrease in a similar manner from the base to the vertex, are to the uniform figure of the same base and altitude in the same ratio." The second part consists in showing the method of finding the ratio of an infinite number of lines or planes increasing or decreasing according to a certain law with the same number of elements homogeneous with the first, and all equal to each other. Thus the cone is composed of circles decreasing from the base to the vertex, while the cylinder of the same base and altitude is composed of an infinite number of circles all equal to each other. In the cone these circles decrease from the base to the vertex as the squares of the terms of an arithmetical progression. If, therefore, the ratio can be found between an infinite number of terms decreasing according to that law, and an infinite number of terms each equal to the first term of the first progression, the ratio between a cone and a cylinder of the same base and altitude is determined. Cavallerius showed that the one is a third of the other, and hence it follows that the cone is one third part of the cylinder. To the objections raised against his method, he replied that his hypothesis was no essential part of his theory, but that it was in fact identical to the method of Exhaustions, and only a more brief method of expressing the same ideas in other words.

The method of Exhaustions of the ancient Greek geometers is founded on the principle of exhausting a magnitude by continually taking away a part of it, as it is explained in the tenth book of Euclid's Elements, where it is stated that two quantities are equal whose difference is less than any quantity that can be assigned.

By means of this principle Euclid has proved that the areas of circles are proportional to the squares of their diameters (Euc. xii. 2): and that the volumes of spheres are as the cubes of their diameters (Euc. xii. 18).

Archimedes also has shown by the same principle that the area of a circle is equal to the area of a right-angled triangle, whose sides containing the right angle are equal, one to the radius, and the other to the perimeter of the circle (Dim. Circ.); and that the approximate ratio of the circumference to the diameter of a circle is as 22 to 7. He has also shown that the volume of a sphere is equal to two-thirds of its circumscribing cylinder, and that the surface of a sphere is the same as the curvilinear surface of the cylinder, and that each is equal to the areas of four great circles of the sphere. (De Sph. et Cylind.)

It may be remarked that Kepler, in his Nova Stereometria, published in 1615, had before suggested the hypothesis of a circle being composed of an infinite number of small triangles having their vertices at the centre and their bases on the circumference: and the volume of a sphere in like manner he supposed to consist of an infinite number of small pyramids, with their vertices at the centre and their bases on the surface of the sphere.

answered the objections of Guldinus,[1] and explained the use of Indivisibles, and also gave a series of exercises on the subject.

At Oxford, in 1656, was published an original work by Dr. John Wallis, then Savilian Professor of Geometry. It bears the title of "Arithmetica Infinitorum." This treatise exhibits a new method of reasoning on magnitudes, and on the ratio of magnitudes by means of series. He was led to consider that if the terms of a series of quantities successively decrease; then the series may be made continually to approach to some constant quantity, and when the series is indefinitely continued, to become equal to that quantity. He states that his method of the Arithmetic of Infinites is a further improvement on the method of Indivisibles, introduced by Cavallerius, and that his theory is only a shorter way of employing the method of exhaustions.[2]

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[1] The most important of Guldin's mathematical works was a treatise on the centre of gravity, in which he gave a method which consisted in the application of the centre of gravity to find the content of solids of revolution. His discovery is contained in the proposition : "Every solid formed by the revolution of any plane figure round an immovable axis, is equal to the product of the generating quantity multiplied by the length of the course of its centre of gravity." As for example, The right cone is generated by the revolution of a right-angled triangle round one of the sides which contain the right angle. The centre of gravity of this triangle is distant one-third of the base from the axis. And the solid content of the cone by Guldin's rule is equal to the area of the generating triangle multiplied by one-third of the circumference of the base of the cone, and hence is equal to one-third of the content of a cylinder of the same base and altitude as the cone. Guldin did not succeed in demonstrating the truth of the general proposition. This was shown to him by Cavallerius during their dispute, who stated that it had been done many years before by one of his pupils.

[2] In the year 1657 Dr. Wallis collected and printed his mathematical works at Oxford in two parts.

The first part consists of his inaugural lecture and two treatises :

1. Oratio Inauguralis, a Johanne Wallis, Geometriæ Professore Saviliano, in Auditorio Geometrico, Oxoniæ, habita, ultimo die Mensis Octobris Anno 1649; quum publicam geometriæ professionem auspicatus est.

2. Mathesis Universalis; sive Arithmeticum Opus integrum, tum philologice tum mathematice traditum; Arithmeticam tum Numerosam, tum speciosam sive symbolicam complectens, sive calculum geometricum; tum etiam Rationum Proportionumve traditionem; Logarithmorum item doctrinam; aliaque quæ capitum syllabus indicabit.

3. Adversus Marci Meibomii De Proportionibus Dialogum, tractatus elencticus.

The second part consists of four treatises :

1. De Angulo Contactus et Semicirculi, Disquisitio Geometrica. Oxonii, 1656.

2. De Sectionibus Conicis, Nova Methodo, Expositis, tractatus. Oxonii, 1655.

3. Arithmetica Infinitorum, sive Nova Methodus inquirendi in curvilineorum quadraturam, aliaque difficiliora Matheseos problemata. Oxonii, 1656.

4. Eclipsis solaris Oxonii visæ Anno Æræ Christianæ, 1654. 2 die Mensis Augusti, Stilo veteri, Observatio. Oxonii, 1655.

Commercium Epistolicum de Quæstionibus quibusdam Mathematicis nuper habitum. Edidit J. Wallis, S.Th.D. Oxonii, 1658. This is a collection of forty-

Dr. Wallis found his method for the sums of series to fail in the case of the progressions whose terms were the roots of the sums or differences of simple terms such as $\surd(r^2-oa^2)$, $\surd(r^2-1a^2)$, $\surd(r^2-4a^2)$, $\surd(r^2-9a^2)$, &c., to $\surd(r^2-r^2)$, which he names a series of terms in the subduplicate ratio of a series of equals diminished by a series of squares. Where, if r be the radius of a circle, and a denote any of the indefinitely small and equal distances of the ordinates in the quadrant of the circle, beginning from the centre, such quadrant is equal to the sum of the terms of this progression, as he has shown in Proposition 121 of his Arithmetic of Infinites. They being the right sines of which the quadrant is composed, each of which is known to be a mean proportional between $(r+oa)$ and $(r-oa)$; and between $(r+1a)$ and $(r-1a)$; and between $(r+2a)$ and $(r-2a)$, and so on. The same series, with the signs changed, of the second term under the vinculum, that is $\surd(r^2+0a^2)$, $\surd(r^2+1a^2)$, $\surd(r^2+4a^2)$, $\surd(r^2+9a^2)$, &c., to $\surd(r^2+r^2)$, being summed would give the quadrature of the equilateral hyperbola, supposing r to denote the semi-transverse diameter, and a any of the indefinitely small and equal distances on the asymptote, from the centre of the transverse diameter, then the ordinates composing the hyperbolic space will be expressed by the last progression. He also shows in Prop. 165, by finding the sums of the terms of two other series, the quadrature of the circle and hyperbola could be found, but involving the same difficulty, which he could not directly overcome. While he was engaged in his inquiries about the quadrature of the circle, he thought of another method, which he named the interpolation of series, by which he means, of discovering certain intermediate terms of a progression, by considering the properties of the progression and the relation of the terms to each other. He gives several examples of this method for finding the area of a circle, of which this is one. In the progression 1, $\frac{3}{2}$, $\frac{15}{8}$,

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four letters, of which eleven were written by Lord Brounker to Dr. Wallis, and seven from Dr. Wallis to Lord Brounker.

A Treatise of Algebra, both Historical and Practical, shewing the Original, progress and advancement thereof, from time to time; and by what steps it hath attained to the height at which now it is. With additional treatises, I. Of the Cono-cuneus; being a body representing in part a conus, in part a cuneus. II. Of Angular Sections; and other things relating thereto, and to Trigonometry. III. Of the Angle of Contact; with other things appertaining to the composition of magnitudes, the inceptives of magnitudes, and the composition of motions, with the results thereof. IV. Of Combinations, Alternations, and Aliquot parts. By John Wallis, D.D., Professor of Geometry in the University of Oxford, and a member of the Royal Society, London. Folio, London, M. DC. LXXXV.

Besides these works there will be found numerous papers of Dr. Wallis in the early volumes of the Philosophical Transactions of the Royal Society. In the year 1697 the whole of the mathematical works of Dr. Wallis were collected and printed in three folio volumes at the Oxford University Press, and published in 1699.

$\frac{105}{48}$, &c., whose terms are formed by taking 1 for the first term, and the products of the first two, three, four, &c., terms, of 1, $\frac{3}{2}$, $\frac{5}{4}$, $\frac{7}{8}$, &c., to find the intermediate term between 1 and $\frac{3}{2}$; which he found to be a surd, which cannot be exactly expressed by any definite number, but which may be expressed approximately.

Notwithstanding his failure, Dr. Wallis disclosed new methods of proceeding, among which may be noted that he first reduced the fraction $\frac{a}{1-r}$ by division into the infinite series;

$$a+ar+ar^2+ar^3+ar^4, + \text{\&c., ad infinitum.}$$

The converse, however, involved some difficulties, which may be worthy of some consideration.

It may be reasonably objected that since all the terms of an infinite series are unassignable, it is impossible to determine their sum.

In order to obviate this objection, let it be considered that *a number actually infinite* is a plain contradiction to all our ideas about numbers. For every number of which the human mind can actually form any suitable idea, is, and always must be, finite; so that a number greater than that can be assigned, and a greater than this last number, and so on without any possibility of ever coming to the end of the increase of numbers that can be assigned. This endless progression of numbers is all that can be understood by the infinity of number; and to assert that the number of any things is infinite, is not to assert that we comprehend their number, but the contrary. The only positive idea being simply this—that the number of these things is greater than any number which we can conceive and assign.

We easily conceive that a finite magnitude may become greater and greater without end, or that no limit can be assigned to the increase which it may admit; but we do not, therefore, clearly conceive magnitude increased an infinite number of times.

Mr. Locke, in his essay, admits that we easily form *an idea of the infinity of number*, but he distinguishes between this and *the idea of an infinite number;* and adds, that however clear our idea of the infinity of number may be, there is nothing more evident than the absurdity of the actual idea of an infinite number. And he further remarks, that while men talk and dispute of magnitudes infinitely great and infinitely small, as if they had as clear and positive ideas of these as they have of the names of finite quantity, it is not surprising if the incomprehensible nature of the thing they reason about leads them into perplexities and contradictions. Mathematicians abridge their computations by the supposition of infinites, but when they treat them in the same way as they reason on finite quantities, they are sometimes led into such conclusions as verify the observation of Mr. Locke.

We cannot apply to an infinite series the common notion of a sum, or an aggregate of several numbers that are connected and added together, one after another, for this supposes that these numbers are all known and determined. But the terms of an infinite series cannot be all separately assigned, there being no end in the number of its terms, and therefore it can have no sum in that sense. Again, the idea of an infinite series consists of two parts, one, the idea of something positive and determined, in so far as we conceive the series to be actually carried on; and the other, the idea of an inexhaustible remainder or an endless addition of terms to be made to it, one after another. Hence we may conceive it as a whole of its own kind; and therefore the infinite series may be said to have a real value determinable or not. Now, in some infinite series, which are convergent, the value is finite or limited; or in other words, a number can be assigned which the sum of no assignable number of terms of the series can ever exceed, or in fact can ever be equal to it, yet may approach to it nearer than by any difference that can be assigned. And this we may call the sum of the series; not as being a number found by the common method of addition, but as being such a limitation of the value of the series taken in its infinite extent, that if it were possible to add the terms of the series all together, the sum would be equal to that number.

Again, in other infinite series, such as are divergent, the value of their sums has no limitation. And this we may express by saying that the sum of the series is infinitely great; which signifies no more than that the series has no determinate and assignable value; and that the sum of the series when the terms are indefinitely continued, shall be greater than any number that can be assigned. And therefore on the whole it may be concluded there is a sum in the former case, yet not such a sum in its ordinary sense; and in the latter case, there is no determinate sum in any sense.

Dr. Wallis, in his methods of induction, considered the progressions as consisting of an infinite number of terms formed according to certain laws; and not at that time having read the methods of Archimedes, he adopted the method of indivisibles, of which elements geometrical figures were supposed to be formed. And these elements were naturally represented by the terms of these progressions, and his object was to discover how the sum of these terms could be found.

He effected his purpose by the following important proposition, which embodies the substance of his work, namely, "That the sum of all the terms of the same powers of a series of quantities in arithmetical progression beginning with 0, is equal to the product of the last term multiplied by the number of significant terms, divided by the index increased by unity." For example, suppose 0, 1, 2, 3 .... x be an arithmetical progression, consisting of an indefinite number of times, in the natural order of numbers, having the last term x, and let 0^m, 1^m, 2^m, 3^m .... x^m, be a progression of terms which are of any, the same power or root; then the sum of this last series will be equal to $\frac{x^{m+1}}{m+1}$. This proposition Wallis proved by induction from several particular cases, but Newton reduced all the cases into one, and gave a general demonstration by means of an indefinite index.

Dr. Wallis was the first who developed the principles of circulating decimals. The subject is treated in the eighty-ninth chapter of his "Algebra both Historical and Practical," and in the appendix he added some posthumous papers of Mr. Horrocks, which he digested into order at the desire of the Royal Society, and published. The subject of circulating or repeating decimals is more curious than useful. No distinction is made between repeating and non-repeating decimals in the computation of logarithmic and trigonometrical tables. These tables are calculated to seven or more places of decimals, and can be used to obtain approximate results of greater or less accuracy, according to the number of decimal places to which the tables are extended. By tables computed to seven places of decimals, approximate results can be found which shall differ from the exact result by an error less than a millionth part of the unit employed.

Dr. Wallis wrote and published works on theology and other subjects besides those on mathematics. In the year 1676 he published an edition of "Archimedis Arenarius et Dimensio Circuli." He also edited the Greek text of the Opus Harmonicum of Ptolemy, and published it with a Latin translation and notes in 1682.

After a long life of useful work he died on the 28th day of October, 1702, in the 88th year of his age, having steadily pursued one great object, which he has described in the following extract from a letter to his friend:—"It hath been my endeavour all along to act by moderate principles, being willing, whatever side was uppermost, to promote any good design for the true interest of religion, of learning, and of the public good."

Lord William Brounker was one of the leading men who originated the Royal Society, of which he was appointed the first President in the year 1662. He was an able mathematician, as is shown by several papers which appear in the Philosophical Transactions. He was the first person who considered the subject of Continued Fractions and applied them to the quadrature of curves and to other investigations.

He devised a method to express by a continued fraction the ratio of the area of the circumscribed square to the area of the circle, and found that the ratio of the square of the diameter to the area of a circle was as 1 to the continued fraction $1 + \frac{1}{2} + \frac{9}{2} + \frac{25}{2} + \frac{49}{2} +$ &c., and showed that the successive approximate values of the continued fraction were approximations alternately less and greater, to the true ratio of the square to the circle.

And Wallis, in his "Arithmetic of Infinites," proved that the area of a circle is to the square of its diameter, as 1 to $\frac{9}{8} \times \frac{25}{24} \times \frac{49}{48} \times \frac{81}{80} \times$ &c.,

and showed this expression to be identical with Lord Brounker's continued fraction (Arith. Inf. pp. 181, &c.).

Huygens employed continued fractions for the purpose of obtaining in smaller numbers the approximate values of the ratios of large numbers.[1]

La Grange was the first who showed that the square root of any whole number, not a complete square, can be expressed in the form of a periodic continued fraction.[2]

Mr. James Gregory, in his "True Quadrature of the Circle and Hyperbola," published in the year 1667, gave a converging series which proceeds by pairs of terms, such that the difference between any pair is less than the difference of the preceding pair, and that after the same law that the second pair is formed from the first, the third is formed from the second, and so on; by which the differences successively decreasing, at length a difference becomes less than any given quantity that can be assigned, when the series is indefinitely continued. In that case the difference vanishes, and the two terms become equal, either of which is the quantity sought, whereby an approximation to the area of the hyperbola, as well as to that of the circle, may be made to any degree of accuracy.

Dr. Barrow was surpassed by few men of his age in clearness of perception, exactness of reasoning, and soundness of judgment.[3] He will ever stand among the first in mathematical learning, a perfect master of the ancient geometry, who considerably contributed to the advancement of the new calculus and its applications. Nor will his reputation be ever deemed less for his justness of thought in his works on theology, the deepest and the most important of all the sciences. His Mathematical Lectures were delivered at Cambridge in 1664 and the following years, as also his Geometrical Lectures. The latter were published in 1670, and the former after his death in 1683.[1]

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[1] See Opuscula Posthuma, Lugd. Bat. 1703. Descriptio Automati Planetarii, pp. 449, &c.

[2] At the end of the third edition of Euler's Algebra (London, 1822) are printed the additions of M. de la Grange, which consist of a valuable treatise on continued fractions, and the application of them to the solution of indeterminate problems. And in his "Traité de la Résolution des Equations Numeriques" is explained the manner of approximating to the numerical values of the roots of equations by means of continued fractions. Paris, 1st ed., 1798; 2nd ed., 1826.

[3] The following anecdote is related of Dr. Barrow, as as instance of his ready wit and judicious skill in repelling an impertinence:—

"On one occasion, while Dr. Barrow was at Court in discharge of his duties as the king's chaplain in ordinary, he encountered one day the witty Lord Rochester. Rochester, intending to banter him, with a flippant air and a low formal bow, gravely accosted him with the words, 'Doctor, I am yours to my shoe-tie.' Barrow, perceiving his drift, determined upon defending himself, and returned the salute with, 'My lord, I am yours to the ground.' Rochester, upon this, improving his blow, quickly returned it with, 'Doctor, I am yours to the centre;' which was smartly followed up by Barrow with, 'My lord, I am yours to the antipodes.' Upon this rejoinder, Rochester, disdaining to be foiled by a musty old piece of divinity, as he used to call him, exclaimed, 'Doctor, I am yours to the lowest pit of hell;' upon which Barrow, turning upon his heel, with a sarcastic smile, archly replied, 'There, my lord, I leave you.'"

He begins with the generation of magnitudes which involves the origin of mathematical hypotheses, and admits that magnitudes may be conceived to be generated or formed by local motion, which implies the idea of force which produces the motion, the time of its continuance, and the direction. Time may be measured by motion, and motion by time. As time is a quantity uniform in its continuity, and all its parts correspond to the respective parts of spaces passed over with equal or unequal motions, it may be represented to our minds by any geometrical magnitude between which and time there can be any analogy, such as a straight line or a circular arc. Now as time consists of parts altogether similar, it may be regarded as of one dimension only, whether considered as a continuous flux, or made up by the addition of moments. For this reason, *length only* is ascribed to time, and its quantity is determined by the length of the line passed over. As a line can be conceived to be the trace of a moving point, so time may be conceived as the trace of a moment continually moving on in a continuous flux. And like as the quantity of a line consists of length only following the motion, so the quantity of time pursues only one direction, which the length of the line or distance moved over shows, and determines the time.

The motive force by which magnitudes are supposed to be generated is a quantity, and subject to computation as other quantities. For when two bodies depart from the same place along the same line, if one moves over a greater distance than the other in the same time, then that body which moves more quickly is acted on by a greater force or motive power. Force, therefore, admitting the modification of greater and less, may be conceived to be divisible into any definite or indefinite parts. And in order to represent the quantity of this force justly to the mind, it is only necessary to assume some geometrical magnitude to represent it. The straight line will fulfil the required conditions both with respect to magnitude and direction. When this force comes under mathematical consideration it is called velocity, which is defined to be that power by which a point or a movable body can pass over a given space or distance in a given time. Hence it follows, that the quantity or measure of any velocity cannot be known either from knowing separately the space moved over, or from the time; but can be determined from the space and the time together; that is, from the space passed over in a given time.

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[1] The Mathematical Lectures were translated into English by the Rev. John Kirkby, and published in 1734, and the Geometrical Lectures were also translated into English by Edmund Stone, and published in 1735.

So, again, the quantity of time can be obtained from the quantity of space and the velocity together. The quantity of space is found in the same manner as is that of a superficies, simply by its dimensions. But the quantities of velocity and time are found exactly in the same way as, when a superficies and one of its dimensions are given, we thereby find the other. For to every moment of time there answers some degree of velocity which a movable body is then conceived to have, to which degree some length of space moved over corresponds.

When time flows equably, it will be most fitly represented by the straight line; and the several degrees of velocity, whether equal or unequal in each instant, may also be denoted by straight lines. And because these degrees of velocity do in any momemt of time pass over each independently, therefore if horizontal straight lines parallel to each other be drawn through all the points of the perpendicular line representing the time, the plane superficies thence resulting will exactly represent the aggregate of the degrees of velocity, and which superficies having its parts proportional to the respective parts of the space moved through, may properly represent that space. If the velocities corresponding to each instant of time be equal, this superficies will be a parallelogram; if unequal, a triangle; from the properties of the former are deduced all the properties of equable and uniform motion; and from the latter all those which concern equally accelerated motion. Moreover, any degrees of velocity increasing or decreasing by any law of force in any times being given, the measure of the representative space, or the quantity of space moved through, may be determined, and the contrary.

The Geometrical Lectures exhibit many important results relating to the properties of curves. Dr. Barrow's method of tangents[1] is the same as that of P. Fermat, simplified and generalised. He conceives the ordinate and the abscissa of a curve to receive each an infinitely small increment. These two increments and the small portion of the curve intercepted between them, form an infinitesimal triangle, similar to the triangle formed by the ordinate, subtangent, and tangent to the curve. From the equation to the curve he finds the relation between these two increments, and then concludes, that as the increment of the ordinate is to the increment of the abscissa, so is the ordinate to the subtangent, which gives the solution of the problem.

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[1] As an example, let $y^2=px$, the equation of the common parabola, be taken; and let e denote the increment of x, the abscissa, and a the increment of y, the ordinate, at any point of the curve. Then $(y+a)^2=p(x+e)$, or $y^2+2ay+a^2=px+pe$; but since $y^2=px$, there will remain $2ay+a^2=pe$; but since a is a quantity infinitely small, a^2 or $a\times a$ is infinitely smaller in comparison with a, and therefore a^2 may be rejected, and then $2ay=pe$, and $a:e::p:2y$; but $a:e::y$: the subtangent, hence $p:2y::y$: the subtangent; and the subtangent $=\frac{2y^2}{p}=\frac{2px}{y}=2x$, or the subtangent of the common parabola is equal to twice the abscissa.

ELEMENTARY ALGEBRA,
WITH BRIEF NOTICES OF ITS HISTORY.

SECTION III.

OF ALGEBRA *(continued)*.

BY ROBERT POTTS, M.A.,
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HON. LL.D. WILLIAM AND MARY COLLEGE, VA., U.S.

LONDON:
LONGMANS AND CO.,
1879.

OF ALGEBRA.

(*Continued.*)

The researches of Wallis, Barrow, and others, had already opened the way by their discoveries for the further development of new methods of reasoning in the science of mathematics.

Besides, towards the latter part of the seventeenth century, several ingenious treatises on the Science of Algebra were published which contributed to advance the knowledge of that science. One of these may be noted. In the year 1675, in two quarto volumes, appeared "Nouveaux Elemens des Mathematiques, par Jean Prestet, prêtre." A second edition was published in 1689, and a third in 1694. The work consists of an ample treatise on Arithmetic and Algebra, and bears the singular and peculiar dedication to Almighty God. Such a dedication may be thought to savour at least of presumption, if not of some more serious offence against that reverence which is generally considered due from the creature to the Creator. But it is more probable that the author, being under the orders of his church, would not have adventured so far without the special permission of infallible authority. This dedication is very different from that of Thomas Taylor, the Platonist, who dedicated his English translation of the Commentary of Proclus on the First Book of Euclid's Elements "To the Sacred Majesty of Truth."

The Algebra of Rhonius, in High Dutch, was translated into English by Thomas Branker, and published in a quarto volume in London, 1668. Several improvements in the work were received from Dr. John Pell, by whom the character ÷ was devised to denote the operation of division.

Elementa Arithmeticæ Numerosæ et Speciosæ, auctore Edw. Wells. 1698. Oxonii. 2nd Edition, 1726.[1] Mr. Wells also, in 1714, published in three volumes a work entitled, "The Young Gentleman's Course of Mathematics." The author remarks that no gentleman ought to think arithmetic below him who does not think an estate below him.

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[1] The late Professor De Morgan has thus described this work:—"This is an elegant work, in which Arithmetic and Algebra are exhibited to the university student in a simple form, accompanied by the historical learning which such a student ought to have. The works on Arithmetic were taking too commercial a turn to be all-sufficient for the purposes of liberal education; and Wells, without by any means rejecting commercial questions, writes in order that the student may not of necessity be driven to the works in which, as he says, 'Exempla non aliunde quam a Butyro et Caseo, Zingibere et Pipere, aliisque similibus.'"

The substance of the Arithmetica Universalis of Mr. Newton was orginally composed and delivered in his lectures as Lucasian Professor at Cambridge between the years 1669 and 1702. Mr. Whiston, who succeeded to the Lucasian Professorship, "thought it a pity so able and useful a work should be doomed to a college-confinement." He obtained leave from Sir Isaac Newton to supply what he had left undone, and from Dr. Halley to reprint his method of finding the roots of equations, and from both of them their sanction for the publication. Accordingly the work was printed at Cambridge in 1707, typis academicis, with the title of "Arithmetica Universalis, sive de Compositione et Resolutione Liber. Cui accessit Halleiana Æquationum Arithmetice inveniendi Methodus."

The following extract from the introduction explains the nature and object of the work:—"Computation is either performed by numbers, as in vulgar arithmetic, or by species, as is usual among algebraists. They are both built on the same foundations, and aim at the same end; namely, arithmetic definitely and particularly; algebra, indefinitely and universally: so that at most all expressions that are found out by this computation, and particularly conclusions, may be called theorems. But algebra is particularly excellent in this, that whereas in arithmetic questions are only resolved by proceeding from given quantities to the quantities sought; algebra proceeds in a retrograde order, from the quantities sought, as if they were given, to the quantities given, as if they were sought, to the end that we may, some way or other, come to a conclusion or equation, from which we may bring out the quantity sought. And after this way the most difficult problems are resolved, the resolutions whereof would be sought in vain from only common arithmetic."

An able and distinguished mathematician has made the following remarks on this work:—"It abounds with many great and useful improvements—particularly in the transformation of equations, the limits of their roots, the number of impossible roots, and the use of rational and surd divisors. It also contains a select collection of algebraical and geometrical problems, which are resolved in the most elegant and masterly manner; as likewise the linear construction of equations of different orders, and a general method of determining numeral equations by approximation."

The work was translated into English by Mr. Ralphson, and published in 1720. A second edition, revised and corrected by Mr. Cunn, was issued in 1728. A later edition was also published by Wilder in 1769, with a commentary in two volumes octavo.

An edition of the Arithmetica Universalis was published in three octavo volumes, with a commentary in Latin, by P. Antonio Secchi, at

Milan in 1752. Another edition was published in 1761 at Amsterdam, in two quarto volumes, with a commentary by Joannes Castillione.

It appears to be unquestionable that Newton was the first who assigned, in a general form, the law of the coefficients of the Binomial Theorem, and showed that the theorem was true when extended to fractional and negative indices. As he has described at considerable length in his letter addressed to Mr. Oldenburg, dated 24th October, 1676, how he was led to the discovery, the account will be best represented in the form it is given by Newton himself. He writes:— "About the beginning of my mathematical studies, as soon as the works of the celebrated Dr. Wallis fell into my hands, by considering the series by the interpolation of which Dr. Wallis had exhibited the areas of the circle and the hyperbola in his 'Arithmetic of Infinites,' namely, that in a series of curves, whose common base or axis is x, and the ordinates

$$(1-xx)^{\frac{0}{2}},\ (1-xx)^{\frac{1}{2}},\ (1-xx)^{\frac{2}{2}},\ (1-xx)^{\frac{3}{2}},\ (1-xx)^{\frac{4}{2}},\ (1-xx)^{\frac{5}{2}},\ \text{\&c.},$$

if the areas of every other of them, that is,

$$x,\ x-\tfrac{1}{3}x^3,\ x-\tfrac{2}{3}x^3+\tfrac{1}{5}x^5,\ x-\tfrac{3}{3}x^3+\tfrac{3}{5}x^5-\tfrac{1}{7}x^7,\ \text{\&c.}$$

could be interpolated, we should have the areas of the intermediate ones, of which the first $(1-xx)^{\frac{1}{2}}$ is the circle. To interpolate these, I observed that the first term in all of them was x, and that the second terms $\tfrac{0}{3}x^3$, $\tfrac{1}{3}x^3$, $\tfrac{2}{3}x^3$, $\tfrac{3}{3}x^3$, &c., were in arithmetical progression; and that therefore the first two terms of the series to be intercalated should be

$$x-\frac{\frac{1}{2}x^3}{3},\ x-\frac{\frac{3}{2}x^3}{3},\ x-\frac{\frac{5}{2}x^3}{3},\ \text{\&c.}$$

For interpolating the rest, I considered that the denominators, 1, 3, 5, 7, &c., were in arithmetical progression, and that nothing remained but to investigate the numeral coefficients of the numerators. Now these I saw, in the alternate areas that were known, to be figures which express the powers of the number 11, namely, 11^0, 11^1, 11^2, 11^3, 11^4, &c., that is, the first was 1; the second 1, 1; the third 1, 2, 1; the fourth 1, 3, 3, 1; the fifth 1, 4, 6, 4, 1, &c.

"Wherefore I inquired how, from the first two figures given in these series, the rest might be derived; and I found that if the second figure be called m, the rest might be produced by a continual multiplication of the terms of this series $\frac{m-0}{1}\times\frac{m-1}{2}\times\frac{m-2}{3}\times\frac{m-3}{4}\times$ &c.

"For example, let the second term $m=4$, then $4\times\frac{m-1}{2}$, that is, 6 will be the third term; and $6\times\frac{m-2}{3}$, that is 4, the fourth term; and $4\times\frac{m-3}{4}$, that is 1, the fifth; and $1\times\frac{m-4}{5}$, that is 0, the sixth term. With which the series, in this case, terminates.

"Therefore I applied this rule for interpolating the series. And since, for the circle, the second term was $\frac{\frac{1}{2}x^3}{3}$, I put $m = \frac{1}{2}$, and the terms arising were $\frac{1}{2} \times \frac{\frac{1}{2}-1}{2}$, or $-\frac{1}{8}$; $-\frac{1}{8} \times \frac{\frac{1}{2}-2}{3}$, or $+\frac{1}{16}$; $+\frac{1}{16} \times \frac{\frac{1}{2}-3}{4}$, or $-\frac{5}{128}$; and so on in infinitum. Whence I came to understand that the area of the circular segment, which I wanted was $x - \frac{\frac{1}{2}x^3}{3} - \frac{\frac{1}{8}x^5}{5} - \frac{\frac{1}{16}x^7}{7} - \frac{\frac{5}{128}x^9}{9}$, &c.

"And the like method may be made use of to interpolate other series, and that although they be deficient by the intervals of two or more terms together. Such was my first entry upon these studies, which had certainly escaped me, had I not accidentally cast my eye upon some loose papers that came in my way a few weeks ago. But having arrived at this length, I immediately began to consider that the terms

$$(1-xx)^{\frac{0}{2}},\ (1-xx)^{\frac{2}{2}},\ (1-xx)^{\frac{4}{2}},\ (1-xx)^{\frac{6}{2}},\ \&c.,$$

that is to say, 1, $1-x^2$, $1-2x^2+x^4$, $1-3x^2+3x^4-x^6$, &c.

might be interpolated after the same manner as the areas generated by them; and that nothing was required for this purpose but to omit the denominators 1, 3, 5, 7, &c., which are in the terms expressing the areas. That is to say, that the coefficients of the terms of the quantity to be interpolated $(1-xx)^{\frac{1}{2}}$ or $(1-xx)^{\frac{3}{2}}$, or in general of $(1-xx)^m$, arise by the continual multiplication of the terms of this series $\frac{m}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4}$, &c. And thus I perceived, for example, that

$(1-xx)^{\frac{1}{2}}$ was equivalent to $1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 -$ &c.
$(1-xx)^{\frac{1}{3}}$ was equivalent to $1 - \frac{1}{3}x^2 - \frac{1}{9}x^4 - \frac{5}{81}x^6 -$ &c.
$(1-xx)^{\frac{3}{2}}$ was equivalent to $1 - \frac{3}{2}x^2 + \frac{3}{8}x^4 + \frac{1}{16}x^6 +$ &c.

"And thus I was possessed of a general method of reducing radicals to infinite series by the rule I mentioned towards the beginning of my former letter[1], and that before I was acquainted with

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[1] This was the letter of Newton to Mr. Oldenburg of the date 13th June, 1676, and designated in the Commercium Epistolicum, Epistola Newtoni Prior. The following extract from the letter contains the rule referred to with the examples there given. "The extraction of roots is much shortened by this theorem :—

$$(P+PQ)^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ +, \ \&c.$$

In which $\frac{m}{n}$ denotes the index affirmative, or negative, P the first term of the binomial, Q the second term divided by the first, and A, B, C, D, &c., the first, second, third, fourth, &c., terms of the expansion with their proper signs.

Since Algebraists write a^2, a^3, a^4, &c., for aa, aaa, $aaaa$, &c., so I write $a^{\frac{1}{2}}$, $a^{\frac{3}{2}}$, the extraction of roots. But once this was known, I could not remain long ignorant of that other. For in order to prove these opera-

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$a^{\frac{5}{3}}$ for $\surd a$, $\surd a^3$, $\surd c.a^5$; and I write a^{-1}, a^{-2}, a^{-3}, &c., $\frac{1}{a}$, $\frac{1}{aa}$, $\frac{1}{aaa}$, &c. And so I write $aa \times (a^3+bbx)^{-\frac{1}{3}}$ for $\frac{aa}{\surd c(a^3+bbx)}$; and $aab \times (a^3+bbx)^{-\frac{2}{3}}$ for $\frac{aab}{\surd c:(a^3+bbx)\times(a^3+bbx)}$.

If the last expression $(a^3+bbx)^{-\frac{2}{3}}$ be compared with $(P+PQ)^{\frac{m}{n}}$, $P=a^3$, $Q=\frac{bbx}{a^3}$, $m=-2$, $n=3$. And in the terms of the expansion, $A=P^{\frac{m}{n}}$, $B=\frac{m}{n}AQ$, and so on for the other terms.

Ex. 1. $\surd(cc+xx)$ or $(cc+xx)^{\frac{1}{2}}=c+\frac{xx}{2c}-\frac{x^4}{8c^3}+\frac{x^6}{16c^5}-\frac{5x^8}{128c^7}+\frac{7x^{10}}{256c^9}-$, &c.

For in this case,

$$P=cc,\ Q=\frac{xx}{2c},\ m=1,\ n=2,\ A=c,\ B=\frac{m}{n}AQ=\frac{xx}{2c},\ C=\frac{m-n}{2x}BQ=-\frac{x^4}{8c^3},$$

and so on.

Ex. 2. $\surd 5:(c^5+c^4x-x^5)$ or $(c^5+c^4x-x^5)^{\frac{1}{5}}=c+\frac{c^4x-x^5}{5c^4}-\frac{2c^8x^2+4c^4x^6-2x^{10}}{25x^9}+$, &c., as will be clear by substituting in the Rule, 1 for m, 5 for n, c^5 for P, and $\frac{c^4x-x^5}{c^5}$ for Q.

Also $-x^5$ can be substituted for P, and $\frac{c^4x+c^5}{-x^5}$ for Q, and then

$$(c^5+c^{\frac{1}{4}}x-x^5)^{\frac{1}{5}}=-x+\frac{c^4x+c^5}{5x^4}+\frac{2c^8x^2+{}^4c^0x+c^{10}}{25x^9}+, \text{\&c.}$$

The former method is to be chosen if x be very small, the latter if it be very great.

Ex. 3. $\frac{N}{\surd c:(y^3-a^2y)}$ or $N\times(y^3-a^2y)^{-\frac{1}{3}}=N\times\left\{\frac{1}{y}+\frac{aa}{3y^3}+\frac{2a^4}{9y^5}+\frac{14a^5}{81y^7}+, \text{\&c.}\right\}$

For $P=y^3$, $Q=-\frac{aa}{yy}$, $m=-1$, $n=3$, $A=\frac{1}{y}$, $B=\frac{aa}{3y^3}$, &c.

Ex. 4. $(d+e)^{\frac{4}{3}}=d^{\frac{4}{3}}+\frac{4ed^{\frac{1}{3}}}{3}+\frac{2ee}{9d^{\frac{2}{3}}}-\frac{4e^3}{81d^{\frac{5}{3}}}+$, &c.

For $P=d$, $Q=\frac{e}{d}$, $m=4$, $n=3$, $A=d^{\frac{4}{3}}$, &c.

Ex. 5. $(d+e)^5=d^5+5d^4c+10d^3e^2+10^2e^3+5de^4+e^5$.

Here $P=d$, $Q=\frac{e}{d}$, $m=5$, $n=1$, $A=d^5$, $B=\frac{m}{n}AQ=5d^4e$.

And so $C=10d^3e^2$, $D=10d^2e^3$, $E=5dc^4$, $F=e^5$.

Ex. 6. $\frac{1}{d+e}$ or $(d+e)^{-1}=\frac{1}{d}-\frac{e}{d^2}+\frac{e^2}{d^3}-\frac{e^3}{d^4}+$, &c.

For here $P=d$, $Q=\frac{e}{d}$, $m=-1$, $n=1$, $A=d^{-1}$ or $\frac{1}{d}$, $B=\frac{m}{n}AQ=-\frac{e}{d^2}$, $C=\frac{e^2}{d^3}$, $D=\frac{e^3}{d^4}$, &c.

Ex. 7. $\frac{1}{(d+e)^3}$ or $(d+e)^{-3}=\frac{1}{d^3}-\frac{3e}{d^4}+\frac{6e^2}{d^5}-\frac{10e^3}{d^6}+$, &c.

tions, I multiplied $1-\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{1}{16}x^6-$ &c., into itself, and I found the product $1-xx$, the rest of the terms in infinitum vanishing by the continuation of the series.

"And thus likewise the series $1-\frac{1}{3}x^2-\frac{1}{9}x^4-\frac{5}{81}x^6-$ &c., multiplied twice into itself produced $1-xx$.

"Which, as it afforded a certain proof of the truth of these conclusions, so it led me to try whether, conversely, these series which thus appeared to be the roots of the quantity $1-x^2$, might not in an arithmetical manner be extracted out of it. The matter succeeded to my mind.

"Having discovered this, I entirely neglected the interpolation of series, and made use of these operations only, as a more genuine foundation. And as to reduction into infinite series by division, it did not lie long concealed, being a much easier affair. Yea, I immediately attempted the resolution of affected equations, and obtained it likewise. Whence the ordinates, the segments of the axes, and any other right lines, were at once known, from the areas or arcs of the curves being given.

"About that time the plague breaking out, obliged me to get hence (from Cambridge) and think of something else. Notwithstanding, I added to my other discoveries, at that time, a certain construction of logarithms deduced from the area of the hyperbola."

After shewing how this construction was effected, he proceeds thus:—

"I am ashamed to tell to how many places of figures I carried these computations, having no other business at that time; for I pleased myself too much with these discoveries. But as soon as that ingenious performance of Nicolaus Mercator appeared, I mean his Logarithmotechnia (who, I doubt not, first discovered the things he published), I began to cool a little upon these things, suspecting that either he understood the extraction of roots, as well as division of fractions; or at least that others, upon the discovery of division, would find out the rest before I could be of sufficient age for writing.

"Yet about the same time that performance appeared, I communi-

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Ex. 8. $\frac{N}{(d+e)^{\frac{1}{3}}}$ or $N\times(d+e)^{-\frac{1}{3}}=N\times\left\{\frac{1}{d^{\frac{1}{3}}}-\frac{e}{3d^{\frac{4}{3}}}+\frac{2e^2}{9d^{\frac{7}{3}}}-\frac{14e^3}{81d^{\frac{10}{3}}}+\text{, \&c.}\right\}$

Ex. 9. $\frac{N}{(d+e)^{\frac{3}{5}}}$ or $N\times(d+e)^{-\frac{3}{5}}=N\times\left\{\frac{1}{d^{\frac{3}{5}}}-\frac{3e}{5d^{\frac{8}{5}}}+\frac{12e^2}{25d^{\frac{13}{5}}}-\frac{52e^3}{125d^{\frac{18}{5}}}+\text{, \&c.}\right\}$

By the same rule, the raising of powers and the extraction of the higher roots of numbers can be readily effected, as well as the multiplication and division of powers and roots of quantities."

The originals of the letters, one of date 13th June, 1676, from which this note is extracted, and the other of 24th October, 1676, are printed in the Commercium Epistolicum, the former noted as Epistola Newtoni prior, and the latter as Epistola Newtoni posterior.

cated to Mr. Collins, by the intervention of our friend Dr. Barrow (then Professor of Mathematics at Cambridge), a certain compend of these series, in which I had signified that the areas and lengths of all curves, and the surfaces and contents of solids, might be determined from right lines given; and conversely, from these given, the others might be found. And I illustrated the method I pointed out by several series. After this a literary correspondence having arisen betwixt us, Mr. Collins, a person born for the advancement of Mathematical learning, insisted strongly upon my publishing these and other things. Accordingly, five years ago (1671), when at the persuasion of my friends I had resolved to publish a treatise on the refraction of light and of colours, which I had then ready by me, I began to turn my thoughts again to these series, and wrote likewise a treatise upon them, that I might publish them both together.

"But having written you a letter upon the subject of the reflecting telescope, in which I explained briefly my notions of the nature of light, something unexpected fell out, which made me imagine it concerned me to write to you speedily about printing that letter. Immediately upon the back of this I met with frequent interruption from the letters of several persons taken up with objections and other things, which frightened me entirely from executing my design, and made me find fault with myself for my imprudence, that, by catching at a shadow, I had been so far deprived of the peace and quiet of my own mind, which is a thing of more substantial worth.

"About that time James Gregory, from one certain series of mine, which Mr. Collins had sent him, after much thought and application (as he himself wrote back to Collins), lighted upon the same method, and hath left a treatise upon it, which we hope will be published by his friends. Since, considering his extraordinary genius, he must have added many new discoveries of his own, which would be a great loss to Mathematical learning, should they be lost.

"But as for my treatise I mentioned, I had not quite finished it when I gave over my design; and I have had no inclination to this very day to add the other things to it. For it wanted that part in which I had designed to explain the method of solving such problems as cannot be reduced to quadratures, although I had laid down something of the foundation of this also. However, the doctrine of infinite series made but a small part of that treatise.

"I put a good many things together, among which was the method of drawing tangents, which the sagacious Slusius communicated to you some two or three years ago; concerning which (at the desire of Mr. Collins) you wrote back that I was in the knowledge of the same. We fell upon it, it is true, in different ways. When one knows the foundation I proceed upon, he cannot draw tangents after any other manner, unless he designedly go out of the right way.

"Moreover there is no stop, whatever way the equations be affected with radicals, affecting one or both the indefinite quantities; but without any reduction of such equations (which would render the work frequently insupportable) the tangent is directly drawn. And the thing is much the same in the questions of Maxima and Minima, and some others which I shall not here mention.

"The foundation of these operations, which obvious enough in itself (because I cannot pursue the explication of it), I choose rather to conceal in an anagram thus:

6accdæ13eff7i3l9n4o4qrr4s9t12vx.[1]

"Upon this foundation I have endeavoured likewise to render the speculation concerning the quadrature of curves more simple; and have arrived at some more general theorems. . . ."

It appears from the preceding account that the Binomial Theorem having been discovered at first by the method of trial and induction, Newton gave no general demonstration of it beyond the verification of its truth in particular instances, by multiplication and actual extraction.

A proof of the Binomial Theorem, in the case where the index is a positive integer, was devised by James Bernoulli, founded on the principle of combinations and figurate numbers. It was printed in 1714, in his "Ars Conjectandi," a work on which he was engaged at the time of his death, which happened in 1705. His mathematical works were collected and published at Geneva in the year 1744, in two quarto volumes.

Besides the proof of the Binomial Theorem depending on the principle of combinations, there is another proof, in which it is shown that, whether the index be integral, fractional, or negative, the first term of the expansion of $(a+b)^n$ will always be a^n, and the expansion will proceed by positive and integral powers of b in successive terms beginning with the first power in the second term, and that the coefficient of the second term is always equal to the index.

Euler's more general proof of the Binomial Theorem first appeared in the New Commentaries of St. Petersburg for the year 1774.

Since the time of Newton numerous attempts have been made to devise general proofs of the Binomial Theorem. Most of these demonstrations have been considered faulty or unsatisfactory. The Multinomial or Polynomial Theorem is an extension of the Binomial Theorem; a general formula for expressing in a series the expansion of a quantity consisting of any number of terms, whether the index be integral or otherwise. This theorem was first investigated by De Moivre, and will be found printed in the volume for 1697 of the Philosophical Transactions of the Royal Society.

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[1] The anagram contains the letters of the following sentence :—"Data equatione quotcunque fluentes quantitates involvente, fluxiones invenire; et vice versa." That is: An equation being given involving any number of flowing quantities, to find the fluxions, and conversely.

The invention of the method of fluxions and fluents by Newton, and that of the differential and integral calculus by Leibnitz, constitute one of the most important discoveries in the mathematical sciences. The method of fluxions, or flowing quantities, is distinguishable from the calculus of Leibnitz in its conception and notation, but in all other respects the two methods are identical in reference to their ultimate objects.

The principles of this calculus are applicable for the investigation of questions relating to all variable magnitudes whatever which admit of increase or decrease.

Leibnitz first published in the Leipsic Acts for October, 1684, his paper in Latin, which contained "A New Method for Maxima and Minima, and likewise for Tangents, which is affected neither by fractions nor irrational quantities, and a peculiar kind of calculus for them." In 1685 he published the inverse method of his calculus, and in 1686 another tract, in which the principles of his methods were further developed.

Newton published his great work[1] under the title of "Philosophiæ

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[1] The following is an extract from the preface to Dr. Pemberton's view of Sir Isaac Newton's Philosophy :—

"The first thoughts which gave rise to the Principia, he had when he retired from Cambridge in 1666, on account of the plague. As he sat alone in a garden, he fell into a speculation on the power of gravity: that as this power is not found sensibly diminished at the remotest distance from the centre of the earth to which we can rise, neither at the tops of the loftiest buildings, nor even on the summits of the highest mountains, it appeared to him reasonable to conclude that this power must extend much farther than was usually thought. 'Why not as high as the moon?' said he to himself; and if so, her motion must be influenced by it; perhaps she is retained in her orbit thereby. However, though the power of gravity is not sensibly weakened in the little change of distance at which we can place ourselves from the centre of the earth, yet it is very possible that, so high as the moon, this power may differ much in strength from what it is here. To make an estimate what might be the degree of this diminution, he considered with himself that, if the moon be retained in her orbit by the force of gravity, no doubt the primary planets are carried round the sun by the like power. And, by comparing the periods of the several planets with their distances from the sun, he found that if any power like gravity held them in their courses, its strength must decrease in the duplicate proportion of the increase of distance. This he concluded by supposing them to move in perfect circles concentrical to the sun, from which the orbits of the greatest part of them do not much differ. Supposing, therefore, the power of gravity, when extended to the moon, to decrease in the same manner, he computed whether that force would be sufficient to keep the moon in her orbit. In this computation, being absent from books, he took the common estimate in use among geographers and our seamen before Norwood had measured the earth, that sixty English miles were contained in one degree of latitude on the surface of the earth. But as this was a very faulty supposition, each degree containing about 69½ of our miles, his compu-

Naturalis Principia Mathematica," in the year 1687, in which he applied the principles of his new calculus to some of the most important problems relating to the system of the world. It may be observed that priority of publication does not necessarily determine priority of discovery. An inventor may have kept the secret of his discovery long before he made it known publicly, in order to investigate its possible and probable consequences.

Newton, in a letter to Collins of the date 10th December, 1672, shows that he had found a method of drawing tangents to curves, and applies it to an example without giving the demonstration. He afterwards remarks that it is only a corollary of another general method which he has for drawing tangents, squaring curves, finding their lengths and centres of gravity, &c.; involving no difficulties arising from radical quantities. The evolution of radicals into series was a considerable step made by Newton in his generalised form of the Binomial Theorem.

In the beginning of the year 1673 Leibnitz first visited London, and became acquainted with Mr. Oldenburg, with whom he commenced

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tation did not answer expectation; whence he concluded that some other cause must at least join with the action of the power of gravity on the moon. On this account he laid aside, for that time, any further thoughts upon this matter. But some years after, a letter which he received from Dr. Hooke put him on inquiring what was the real figure in which a body let fall from any high place descends, taking the motion of the earth round its axis into consideration. Such a body, having the same motion, which by the revolution of the earth the place has whence it falls, is to be considered as projected forward, and at the same time drawn down to the centre of the earth. This gave occasion to his resuming his former thoughts concerning the moon, and Picard, in France, having lately measured the earth, by using his measures the moon appeared to be kept in her orbit purely by the power of gravity; and consequently, that this power decreases, as you recede from the centre of the earth, in the manner our author had formerly conjectured. Upon this principle he found the line described by a falling body to be an ellipsis, the centre of the earth being one focus. And the primary planets moving in such orbits round the sun, he had the satisfaction to see that this inquiry, which he had undertaken merely out of curiosity, could be applied to the greatest purposes. Hereupon he composed near a dozen propositions relating to the motion of the primary planets about the sun. Several years after this, some discourse he had with Dr. Halley, who, at Cambridge, made him a visit, engaged Sir Isaac Newton to resume again the consideration of this subject, and gave occasion to his writing the treatise which he published under the title of 'Mathematical Principles of Natural Philosophy.' This treatise, full of such a variety of profound inventions, was composed by him from scarce any other materials than the few propositions before mentioned, in the space of a year and a half."

At the beginning of a letter of Newton to Dr. Bentley, written in 1692, are these words:—"When I wrote my treatise about our system, I had an eye upon such principles as might work with considering men for the belief of a Deity, and nothing can rejoice me more than to find it useful for that purpose. But if I have done the public any service in this way, it is due to nothing but industry and patient thought."

a correspondence. In one of his letters written in London, Leibnitz states that he had discovered a method of summing certain series by means of their differences. Dr. Pell alleged that this method had been already published in a work by Mouton, Canon of St. Paul's, at Lyons, "On the Diameters of the Sun and Moon." Leibnitz maintained that he had invented another method, which he explained, of forming the differences, and thence deducing the sums of series of fractions whose numerators are unity, and whose denominators are formed by certain laws. And he further relates, that on the further consideration of his former remarks on the differences of numbers in connection with his recent meditations on geometry, he hit upon this calculus about the year 1676.

Newton published his Opticks, or a treatise of the reflexions, refractions, inflexions, and colours of light, 4to, in English, in 1704. Also in the same volume, two treatises (in Latin) of the Species and Magnitudes of Curvilinear Figures.

In the preface he remarks:—

"In a letter written to Mr. Leibnitz in the year 1676, and published by Dr. Wallis, I mentioned a method by which I had found some general theorems about squaring curvilinear figures, or comparing them with the conic sections, or other the simplest figures with which they may be compared. And some years ago I lent out a manuscript containing such theorems, and having since met with some things copied out of it, I have on this occasion made it public, prefixing to it an *Introduction*, and subjoining a solution containing that method. And I have joined with it another small tract concerning the curvilinear figures of the second kind, which was also written many years ago, and made known to some friends, who have solicited the making it public."

The following is the "*Introduction*" referred to in the preface to the "Tractatus de Quadratura Curvarum":—

"I consider mathematical quantities in this place not as consisting of very small parts, but as described by a continued motion. Lines are described, and therefore generated not by the apposition of parts, but by the continued motion of points; superficies by the motion of lines; solids by the motion of superficies; angles by the rotation of the sides; portions of time by a continual flux; and so in other quantities. These geneses really take place in the nature of things, and are daily seen in the motion of bodies. And after this manner the ancients, by drawing moveable right lines along immoveable right lines, taught the genesis of rectangles. Therefore considering that quantities, which increase in equal times, and by increasing are generated, become greater or less according to the greater or less velocity with which they increase and are generated, I sought a method of determining quantities from the velocities of the motions or increments with which they are generated; and calling these velocities of the motions or increments *fluxions*, and the generated quantities *fluents*,[1] I fell by degrees, in the years 1665 and 1666, upon the method of fluxions, which I have made use of here in the quadrature of curves."

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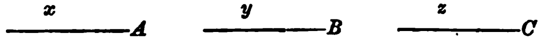
[1] Early notice of Fluxions, Nov. 13, 1665, from an original paper in Newton's handwriting belonging to the Earl of Macclesfield, first printed in the Appendix No. II. of the "Historical Essay on the first publication of Sir Isaac Newton's Principia, by Stephen Peter Rigaud, M.A., F.R.S., Savilian Professor of Astronomy, Oxford, 8vo, 1838":—

"An equation being given expressing the relation of two or more lines x, y, z, &c., described in the same time by two or more moving bodies, A, B, C, &c., to find the relation of their velocities, p, q, r, &c.

Resolution.

Set all the terms on one side of the equation that they become equal to nothing.



And first multiply each term by so many times $\frac{p}{x}$ as x hath dimensions in that term.

Secondly, multiply each term by as many times $\frac{q}{y}$ as y hath dimensions in it.

Thirdly, multiply each term by so many times $\frac{r}{z}$ as z hath dimensions in it, &c.

The sum of all these products shall be equal to nothing: which equation gives the relation of p, q, r, &c.

Or more generally thus:

Order the equation according to the dimensions of x, and, putting a and b for any two numbers, whether rational or not, multiply the terms of it by any part of this progression, viz.:

$$\frac{ap-3bp}{x},\ \frac{ap-2bp}{x},\ \frac{ap-bp}{x},\ \frac{ap}{x},\ \frac{ap+bp}{x},\ \frac{ap+2bp}{x},\ \frac{ap+3bp}{x},\ \&c.$$

Also, order the equation according to y, and multiply the terms of it by this progression, viz.:

$$\frac{aq-2bq}{y},\ \frac{aq-bq}{y},\ \frac{aq}{y},\ \frac{aq+bq}{y},\ \frac{aq+2bq}{y},\ \frac{aq+3bq}{y},\ \&c.$$

Also order it according to the dimensions of z, and multiply its terms by this progression:

$$\frac{ar-3br}{z},\ \frac{ar-2br}{z},\ \frac{ar-br}{z},\ \frac{ar}{z},\ \frac{ar+br}{z},\ \frac{ar+2br}{z},\ \frac{ar+3br}{z},\ \&c.$$

The sum of all these products shall be equal to nothing: which equation gives the relation of p, q, r, &c.

Example 1.—If the propounded equation be $x^3-2xxy+4x^2+7xyy-y^3-103=0$, by the precedent rule the first operation will produce $3xxp-4xyp+8xp+7yyp$;
the second produceth $-2xxq+14xyq-3yyq$;
which two added together make
$3xxp-4xyp+8xp+7yyp-2xxq+14xyq-3yyq=0$.

Example 2.—If the equation be $x^3-2a^2y+zzx-yyx+zyy-z^3=0$,
the first operation will produce $3pxx+pzz-pyy$;
the second produceth $-2aaq-2yxq+2zyq$;

In the letter from Collins to Newton dated 5th March, 1677, it is stated that Leibnitz, having spent a week in London in the month of October, 1676, had placed in the hands of Collins some papers of which copies or extracts should be sent to Newton immediately.

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the third, $2zxr+yyr-3zzr$. The sum of which is
$$3pxx+pzz-pyy-2aaq-2yxq+2zyq+2zxr+yyr-3zzr=0.$$

Demonstration.

If two bodies *A* and *B* move uniformly, the one from *A* to *C*, *E*, and *G*; the other from *B* to *D*, *F*, and *H*, in the same lines; then are the lines *AC* and *BD*, *CE* and *DF*, *EG* and *FH*, as their velocities *p* and *q*:

A *C* *E* *G* *B* *D* *F* *H*
|————|——|—| |————|——|——|

and though they move not uniformly, yet are the infinitely little lines, which each moment they describe, as their velocities are, which they have while they describe them.

As, if the body *A*, with the velocity *p*, describe the infinitely little line *o* in one moment, in the same moment the body *B*, with the velocity *q*, will describe the line $\frac{oq}{p}$. For $p:q::o:\frac{oq}{p}$. So that if the described lines be x and y in one moment, they will be $x+o$ and $y+\frac{oq}{p}$ in the next.

Now, if the equation expressing the relation of the lines x and y be $rx+xx-yy=0$, I may substitute $x+o$ and $y+\frac{oq}{p}$ into the place of x and y, because they, as well as x and y, do signify the lines described by the bodies *A* and *B*.

By doing so, there results $rx+ro+xx+2xo+oo-yy-\frac{2qoy}{p}-\frac{qqoo}{pp}=0$.

But $rx+xx-yy=0$, by supposition:

there remains, therefore, $ro+2xo+oo-\frac{2qoy}{p}-\frac{qqoo}{pp}=0$,

and dividing it by o, 'tis $r+2x+o-\frac{2qy}{p}-\frac{oqq}{pp}=0$.

Also those terms in which o is, are infinitely less than those in which it is not. Therefore blotting them out, there rests $r+2x-\frac{2qy}{p}=o$: or $pr+2px=2qy$.

Hence may be observed: first, that those terms ever vanish in which o is not, because they are the propounded equation; secondly, the remaining equation being divided by o, those terms also vanish in which o still remains, because they are infinitely little; thirdly, that the still remaining terms will ever have that form, which by the first preceding rule they should have.

The rule may be demonstrated after the same manner, if there be three or more unknown quantities, x, y, z, &c."

Hitherto we have copied the words of this discourse. In the same discourse there follows the application of this method to the drawing of tangents, by finding the determination of the motion of any point which describes the curve; and also to the finding the radius of curvity of any curve at any point, by making the perpendicular to the curve move upon it at right angles, and finding that point of the perpendicular which is in least motion. For that point will be the centre of the curvity of the curve at that point upon which the perpendicular stands.

In another leaf of the same waste-book the same method is set down in other

Collins makes no note of the subject of these papers, and they do not appear in the Commercium Epistolicum. The letter of date 5th March, 1677, is printed complete in the works of Dr. Wallis, p. 646, vol. iii.

Sir Isaac Newton's treatise on the Method of Fluxions was translated into English by John Colson, F.R.S., Master of Sir Joseph Williamson's Free Mathematical School at Rochester, and published with a commentary in a quarto volume in 1736, before the Latin original (which was written most probably before the end of the year 1676) had been made public.

In the first lemma of the first section of the Principia, Newton lays down the principle of his new calculus: namely, that—

"Quantities and the ratios of quantities which in any finite time continually tend to equality, and which before the end of that time approach nearer the one to the other than by any given difference, become ultimately equal."

The calculus of fluxions is founded on the idea that all magnitudes may be generated by a continued flux or motion, as, for instance, of geometrical magnitudes, a line by the motion of a point; a superficies by the motion of a line; and a solid by the motion of a surface; and these motions may be supposed to commence at any point of a line, or at any line of a surface, or at any surface of a solid.

In this method quantities are not considered in reference to their parts, but in reference to the velocities with which they are formed, or rather by the quantity of motion by which they are generated. Thus the velocities of the generating point, line, and surface are named the fluxions of the line, the surface, and the solid respectively. And the variable or flowing line, surface, and solid are called the fluents of the respective fluxions.

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words, and fluxions applied to their fluents are represented by pricked letters. And this paper is dated May 20, 1665.

In another leaf of the same waste-book the method of fluxions is described without pricked letters, and dated May 16, 1666.

In a small tract, dated in October, 1666, the same method is again set down without pricked letters, and how to proceed in equations involving facts or surds; and several rules are given for returning back from the fluents to the fluxions; and in doing this the areas of curves are represented by prefixing rectangles to the ordinates. As if x be the absciss, and a, b represent given quantities, and $\frac{axx-x^3}{ab+xx}$ be the ordinate, the area or fluent of the ordinate is represented by $\square\,\frac{axx-x^3}{ab+xx}$.

After this, the method is applied in this tract to the solving of problems concerning tangents, curvatures of curves, the greatest or least curvatures, the squaring of curvilinear figures, comparing their areas with the areas of simpler curves, finding the lengths of curves, finding such curves whose lengths may be defined by equations, or by the lengths of other curves, and about the centres of gravity of figures. And, for finding the curvature of curves without calculation, the following rule is set down: Let X signify the given equation. * * * *

The fundamental proposition is this:—that the fluxions of quantities are proportional to the nascent or ultimate increments or decrements of those quantities; and it consists of two parts. One is concerned in showing how to find the relation of the increments of variable quantities, having given the relations of the quantities themselves; and the other is concerned in finding the relations of the variable quantities themselves freed from the consideration of their increments. Or more briefly, to find the fluxions, having given the fluents; and conversely, to find the fluents, having given the fluxions.

If two variable quantities begin to increase and cease in any finite time, the quantities so generated have a ratio to one another, and this ratio may be considered under different circumstances. The increments of the magnitudes produced in finite portions of time are supposed of finite magnitude, but may be gradually diminished so far as to become so small as may be called evanescent quantities, and that a notion may be formed, not of *the absolute*, but of *the relative* magnitude of the two quantities on the point of vanishing, but which have not actually vanished. The nascent or ultimate ratio of two variable quantities which begin or cease to increase at the same moment of time is the ratio of their initial or final velocities; and their nascent or ultimate ratio is the ratio of their velocities at the same moment they begin or cease to increase.

Newton represented known and determinate quantities by the first letters of the alphabet, a, b, c, &c., and variable quantities, which he called *fluents*, by the last letters, z, y, x, &c. And the velocities by which fluents are increased by their generating motions he named the *fluxions* of such quantities, and represented them by the same letters with a point placed over each letter, as the fluxion of x is denoted by $\dot{x}$. And if the fluxions were variable he considered the velocities with which they increased the same as the second fluxion of the first quantity, and the fluxion of $\dot{x}$ was denoted by $\ddot{x}$, and in similar way the fluxion of $\ddot{x}$ by $\dddot{x}$, and so on for the higher fluxions.

The moments of flowing quantities, that is their indefinitely small parts, by the accession of which in indefinitely small portions of time they are continually increased, are as the velocities of their flowing or increasing. If the moment of any one, as x, be represented by the product $\dot{x}o$ of the velocity $\dot{x}$ into an indefinitely small quantity o, the moment of y will be represented by $\dot{y}o$, because $\dot{x}o$, $\dot{y}o$ are to each other as $\dot{x}$ to $\dot{y}$. Since the moments $\dot{x}o$ and $\dot{y}o$ are the indefinitely small accessions of the flowing quantities x and y, by which these quantities are increased through the indefinitely small intervals of time, it follows that these quantities x and y, after any indefinitely small interval of time, become $x+\dot{x}o$ and $y+\dot{y}o$. And therefore an equation which expresses the relations of the flowing quantities will express the relation between $x+\dot{x}o$ and $y+\dot{y}o$, as between x and y.

§ 3

In the second lemma to the seventh proposition in the second book of the Principia, Newton gives the following explanation of the moment of a generated quantity when it consists of more terms than one:—

"I call every quantity generated which is not made by addition or subduction of divers parts, but is generated or produced in arithmetic by the multiplication, division, or extraction of the root of any terms whatsoever; in geometry by the invention of contents and sides, or of the extremes and means of proportionals. Quantities of this kind are products, quotients, roots, rectangles, squares, cubes, square and cubic sides, and the like. These quantities I here consider as variable and indetermined, and increasing or decreasing as it were by a perpetual motion or flux; and I understand their momentaneous increments or decrements by the name of moments; so that the increments may be esteemed as added or affirmative moments, and the decrements as subducted or negative ones. But take care not to look upon finite particles as such. Finite particles are not moments, but the quantities generated by the moments. We are to conceive them as the first nascent principles of finite magnitudes. Nor do we in this lemma regard the magnitude of the moments but their first proportion as nascent. It will be the same thing if, instead of moments, we use either the velocities of the increments and decrements (which may also be called the motions, mutations, and fluxions of quantities), or any finite quantities proportional to these velocities. The coefficient of any generating side is the quantity which arises by applying the quantity generated to that side.

"Wherefore the sense of the lemma is, that if the moments of any quantities, A, B, C, &c., increasing or decreasing by a perpetual flux, or the velocities of the mutations which are proportional to them, be called a, b, c, &c., the moment or mutation of the generated rectangle AB will be $aB+bA$; the moment of the generated volume ABC, will be $aBC+bAC+cAB$; and the moment of any power or root A^m will be maA^{m-1}."

In the method of fluxions the object is to find the fluxion of any variable quantity, which is a direct process, and in general is easily and readily effected. But where it is required to find the fluents or variable quantities of given fluxions, the inverse process cannot be followed in all cases. For though every variable quantity has its peculiar fluxion, yet every fluxion has not its peculiar fluent in a finite form. This circumstance renders the inverse process more difficult and more operose than the direct, except in such cases where the inverse process is directly possible. But in cases where the fluxion is not reducible to known forms, the fluents may be found approximately by means of infinite series or other methods.

In the method of Leibnitz his process is purely algebraical.

Quantities are not considered as generated by the velocities of the motions by which they are supposed to be formed, but by the addition or subtraction of parts of finite magnitude to or from the variable quantities; and these parts are called increments or decrements of the quantities to which they belong, because that by the addition or subtraction of them the variable quantities are increased or diminished. And the object is the same as in the method of fluxions—namely, to find the limiting ratio of the increments or decrements of the variable quantities. The method of increments or differences also consists of two parts, one being concerned in finding the increments from the given quantities, called integrals; the converse, in finding the integrals, having given the limiting ratio of the increments.

As in the method of fluxions there are first, second, third, &c., fluxions, so in the method of increments there are first, second, third, &c., differences or differentials.

For the science of indefinite quantity recently discovered, which is chiefly concerned with sums and differences, new signs were devised by Leibnitz. He states that dx signifies an element, the momentaneous increment or decrement of the quantity x itself continually increasing or decreasing. It is also named *the difference* between two successive values of x, when the difference is so small that it cannot be assigned. In the same manner $d(xy)$ is such an element of the quantity xy, which when expanded gives $xdy+ydx$. Moreover ddx denotes the difference of a difference; for the quantity dx itself is not always constant, but in general increases or decreases continually. And in the same way $dddx$ or d^3x denotes the third difference; and in general d^ex may arise when the exponent of the difference is indeterminate.

The reverse of *the difference* is *the sum*. Since while a variable quantity is continually decreasing until it vanishes, the quantity itself is always the sum of all the successive differences, so that $d\int ydx$ means the same as ydx. But $\int ydx$ signifies an area which is composed of all the rectangles, the length of every one of which is y and breadth, the increment dx. So also $\int dz\int ydx$ denotes a volume which is composed of all the elemental surfaces or areas, such as ydx multiplied into the element dz corresponding to it.[1]

And in a letter to Dr. Wallis of date 30th March, 1699, Leibnitz states that it is convenient to imagine quantities infinitely small, such as when an assignable ratio between them is sought, they cannot be accounted as nothing, but that they may be rejected as often as they are added to quantities incomparably greater according to the lemmas he had printed in the Leipsic Acts. As if $x+dx$ stand by itself, dx is rejected. If otherwise is sought the difference $(x)-x$ or $x+dx-x$, then the assignable quantity vanishes. And in like manner xdx and $dxdx$ cannot stand together in the product of $x+dx$ and dx. Hence if xy be differentiated and be written $(x)(y)-xy$; putting $x+dx$ for (x) and $y+dy$ for (y), then $(x)(y)-xy$ is equal to $xdy+ydx+dxdy$, from which the assignable quantity vanishes, and $dxdy$ is to be rejected as incomparably less than $xdy+ydx$, and $d(xy)$ becomes equal to $xdy+ydx$.

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[1] G. G. Leibnitii Opera Omnia, Genevæ, 1768. Tom III., p. 416-420. No. lxxv.

It has been thought by some persons, because fluxions are proportional to the nascent increments of quantities, that Newton has introduced the consideration of infinitely small quantities into his method of fluxions. In this supposition they are mistaken; for he considers only the nascent or ultimate ratios of quantities when they begin to exist or when they vanish, not just after they have become something, or just before they vanish, but at the very moment they do so.

And further, it may be added, that the principles of the new calculus have been scrupulously examined and, by some clever men, ignominiously rejected, perhaps not fully understood from the conciseness with which they are explained. A discourse was published, about five years after Newton's death, addressed to "an infidel mathematician," by a personage bearing the titles of learned and right reverend. Having assumed the garb of "the minute philosopher," in sarcastic criticisms he attempted to expose the new calculus by representing the doctrine of fluxions as founded on inconceivable principles, and replete with false and specious reasonings.[1] He pretended not to object to the conclusions of the science, but to its principles and logic, and ironically describes a fluxion as "the ghost of a departed quantity."

In a scholium at the end of the first section of the Principia, Newton anticipated the objections which might be made to the nascent and ultimate ratios of evanescent quantities. He remarks:—

"Perhaps it may be objected that there is no ultimate proportion of evanescent quantities, because, before the quantities have vanished, they have not reached their ultimate proportion; and when they have vanished there is no proportion at all. But by the same argument it may be alleged, that a body arriving at a certain place and there stopping, has no ultimate velocity, because the velocity, before the body comes to the place, is not the ultimate velocity; and when it has arrived, there is no velocity. The answer is easy. For by the ultimate velocity is meant that with which the body is moved, neither before it arrives at its last place and the motion ceases, nor after, but *at the very instant* it arrives; that is, that velocity with which the body arrives at its last place, and with which the motion ceases.

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[1] One of the most important papers on the subject which appeared at that time was written by Mr. Benjamin Robins, entitled, "A Discourse concerning the Nature and Certainty of Sir Isaac Newton's Method of Fluxions, and of Prime and Ultimate Ratios." It is printed in the second volume of his Mathematical Tracts, which were published in 1761.

The most complete answer, however, to the Irish bishop's objections was made by Colin Maclaurin, in his Treatise on Fluxions, published in 1742, in two quarto volumes. In this work all the objections against the doctrine of fluxions are completely refuted, and whatever metaphysical doubts may be started hereafter about the nature of the new calculus, the mathematician has no more concern with them than with the famous sophisms about space and motion.

"And in like manner, by the ultimate ratio of evanescent quantities is to be understood the ratio of the quantities, not before they vanish, nor afterwards, but the ratio with which they do vanish. In like manner the first ratio of nascent quantities is that with which they begin to be. And the first or last sum is that with which they begin and cease to be (or to be augmented or diminished). There is a limit which the velocity at the end of the motion may attain, but not exceed. This is the ultimate velocity. And there is the like limit in all quantities and proportions that begin and cease to be. And since such limits are certain and definite, to determine the same is a problem strictly geometrical. But whatever is geometrical we may be allowed to use in determining and demonstrating every other thing that is likewise geometrical.

"It may also be objected that if the ultimate ratios of evanescent quantities are given, their ultimate magnitudes will also be given; and so all quantities will consist of indivisibles, which is contrary to what Euclid has demonstrated concerning incommensurables in the tenth book of his Elements. But this objection is founded on a false supposition. For those ultimate ratios with which quantities vanish are not truly the ratios of ultimate quantities, but limits towards which the ratios of quantities decreasing without limit do always converge, and to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to, till the quantities are diminished *in infinitum*. This thing will appear more evident in quantities infinitely great. If two quantities whose difference is given be augmented *in infinitum*, the ultimate ratio of these quantities will be given—namely, a ratio of equality; but it does not thence follow that the ultimate or greatest quantities themselves, whose ratio that is, will be given. Therefore, when mention is made of quantities as least, or evanescent, or ultimate, it must not be supposed that quantities of any determinate magnitude are meant, but such as are conceived to be always diminished without end."

And not only was the truth of the principles of the new calculus doubted, and by many able men rejected, but also so early as the year 1699 the question was raised whether Leibnitz rather than Newton was not the original inventor of this new method of mathematical reasoning. A long and angry controversy was carried on in favour of the claims of Leibnitz, supported on one side by the editors of the Leipsic Acts, and on the other in favour of Newton by the English mathematicians. It is painful to remark, such was the rancour and animosity excited in this controversy, that it became evident that men of the highest intellectual powers are not always exempt from the common infirmities of human nature.

A similar question arose between Briggs and Napier with respect to the priority of a discovery of an improvement in logarithms. Briggs, in his lectures at Gresham College, first pointed out the advantage of making 1 the logarithm of the ratio of 10 to 1, instead of 2.30258... as Napier had done. Napier, in his Rabdologia, laid claim to the improvement, and stated that he had committed the execution of it to Briggs. Robert Napier, in the posthumous work of his father, passed over in silence the claim of Briggs, who afterwards, in the preface to his Arithmetica Logarithmica, declared that he had first suggested the improvement in his lectures. Here the same claim is made by both; and it is probable that each may have arrived at the same knowledge independently of the other. Napier did not, but Briggs did state at what time it first became known to him.

It may here be remarked that the work of James Gregory on the quadrature of the circle and hyperbola contained several important facts connected with the discoveries which Newton had made. Collins communicated Newton's discoveries to Leibnitz, Gregory, and other eminent mathematicians. The letters of Gregory supply satisfactory evidence that he had divined Newton's method, and consequently had the merit of discovering in some degree the calculus of fluxions. But he declined to publish anything on the subject, as he has stated in one of his letters, that he might not interfere with the original inventor.

In the year 1671, when Leibnitz was in London, he published two tracts, one of which he dedicated to the Royal Society, and in the dedication mentioned his correspondence with Mr. Oldenburg. In February, 1672-3, though Dr. Pell showed Leibnitz that his method was the same as the differential method of Mouton, Leibnitz persisted in maintaining the method to be his own invention, by reason that he had found it himself without knowing what Mouton had done before, and that his own method was a great improvement on that of Mouton. Before his first visit to England, Leibnitz dedicated to the Academy of Sciences at Paris a paper entitled "Theoria motus abstracti." In this treatise may be found some indications of his early ideas relating to the calculus.

In the year 1684 Leibnitz published in the Leipsic Acts the rules for certain cases of his calculus; but there is evidence that he had employed this method as early as 1677; and, in some respects, it resembled Barrow's method of tangents, which was published in

1670. Newton communicated his method of tangents to Collins in 1672. An abstract of the letter containing this account was sent to Leibnitz by Mr. Oldenburg, who afterwards also sent to Leibnitz another letter of Newton written in 1676. It is clear from these facts that Leibnitz had seen Newton's letters before the appearance of anything of his own on the subject.

On the 10th April, 1695, Dr. Wallis from Oxford wrote to Newton and informed him that he had received an intimation from Holland that Newton's notions of fluxions passed there with great applause, and that Leibnitz was considered as the inventor of the differential calculus. Wallis further remarks:—

"I had this intimation when all but part of the preface to this volume was printed off, so that I could only insert, while the press stayed, that short intimation thereof which you will find there. You are not so kind to your reputation (and that of the nation) as you might be, when you let things of worth lie by you so long till others carry away the reputation that is due to you. I have endeavoured to do you justice in that."

In a second letter, of date 30th April, 1695, Dr. Wallis again wrote to Newton:—"Consider that it is now about thirty years since you were master of these notions about fluxions and infinite series, but you have never published aught of it to this day. It is true, I have endeavoured to do you right in that point. But if I had published the same or like notions without naming you, and the world possessed of another *calculus differentialis* instead of your fluxions, how should this or the next age know of your share therein? . . . And even yet I see you make no great haste to publish these letters [on fluxions][1]

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[1] These letters on fluxions are printed in pp. 391—396 in the second volume of the works of Wallis.

There is in the Savilian Library a copy of the works of Wallis, in which he has inserted many additions and corrections. In the margin of p. 654, vol. iii., he has written: "Ad Algebræ meæ cap. 95, p. 396, lin. 15, fit mentio methodi differentialis Leibnitii; sed ea sunt Newtoni verba; et quorum ego non memineram neque sciebam quæ sit ea methodus."

These words are written against the following extract of a letter of Wallis to Leibnitz of the date of 1696:—

"Neque Calculi Differentialis vel nomen audivisse me memini, nisi postquam utrumque volumen absolverint operæ, eratque præfationis (præfigendæ) postremum folium sub prelo, ejusque typos jam posuerunt typothetæ. Quippe tum me monuit amicus quidam (harum rerum gnarus) qui peregre fuerat, tum talem methodum in Belgio prædicari, tum illam cum Newtoni Methodo Fluxionum quasi coincidere. Quod fecit ut (transmotis typis jam positis) id monitum interseruerim."

But in the second volume (p. 396) Wallis has stated: "Huic Methodo affinis est Methodus Differentialis Leibnitii."

It is noticed in the Commercium Epistolicum that Newton in 1692, at Wallis's request, sent him the first proposition of the book of quadratures, with his examples of first, second, and third fluxions, "id quod videre est in Wallisii operum tomo which are to be my vouchers for what I say of it. And even these letters at first were rather extorted from you than voluntary. . . . I own that modesty is a virtue, but too much diffidence (especially as the world now goes) is a fault."

It should not be forgotten that Leibnitz had, without contradiction, enjoyed, on the Continent, this reputation for a period of fifteen years from the time he made public his discovery in the Leipsic Acts of 1684.

In the year 1699 Nicolas Fatio de Duillier, in a treatise printed in London, made the following remarks respecting the inventor of the new calculus:—"Perhaps the celebrated Leibnitz will inquire how that calculus, which I use, became known to me. I found out by my own powers its fundamental principles and most of its rules in the year 1687, about the month of April and the months following, and at other times afterwards, when I thought that no one used such a calculus besides myself. Nor would it have been less known to me if Leibnitz had not then been born. He therefore may boast of other disciples, of me certainly he cannot. And this would be sufficiently evident if the letters which formerly had passed between the most celebrated Huygens and me were made public. Compelled by the evidence of facts, I acknowledge Newton the first, and by many years the earliest inventor of this calculus. Whether Leibnitz, its second inventor, has borrowed anything from any one, I prefer to my own the judgment of those who have seen the letters of Newton and his writings. Neither the more modest silence of Newton, nor the zealous activity of Leibnitz, assigning to himself everywhere the invention of this calculus, will impose on any persons, who are conversant with the documents which I myself have examined."[1]

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secundo pag. 391, 392, 393 et 396." From this it must have been clearly understood that Newton furnished the materials from what appears in the place referred to; but the truth goes still further; for Wallis has written against the 18th line of p. 390, "Quæ hic sequuntur sunt ipsius Newtoni verba, ab ipso scripta, atque ad me missa, eo animo ut hic inserantur, sed quasi meo nomine, usque ad pag. 396, lin. 19." Hence we see the motives of Newton's reserve. He does not appear to have hesitated in complying with Wallis's request . . . he could not induce himself to communicate his own, unless under conditions which might exempt him from the danger of any personal controversy.—*Rigaud, pp.* 22, 23.

[1] "Quæret forsan Cl. Leibnitius, unde mihi cognitus sit iste calculus, quo utor. Ejus equidem fundamenta, ac plerasque regulas, proprio marte, anno 1687, circa mensem Aprilem et sequentes, aliisque deinceps annis inveni; quo tempore neminem eo calculi genere præter meipsum, uti putabam. Nec mihi minus cognitus foret, si nondum natus esset Leibnitius. Aliis igitur glorietur discipulis, me certe not potest. Quod satis patebit, si olim literæ, quæ inter clarissimum Hugenium meque intercesserunt, publici juris fiant. Newtonum tamen primum ac pluribus annis vetustissimum hujus calculi inventorem, ipsa rerum evidentia coactus agnosco: a quo utrum quicquam mutuatus sit Leibnitius, secundus ejus inventor, malo eorum quam meum sit judicium, quibus visæ fuerint Newtoni literæ, aliisque ejusdem manuscripti codices. Neque modestioris Newtoni silentium, aut prona Leibnitii sedulitas

Leibnitz printed a reply to the statements of Fatio in the Leipsic Acts of 1700, of which the following is an extract:—

"I have freely declared his (Newton's) distinguished merits as occasion offered; and he himself very well knows, and has publicly admitted, when he published his 'Mathematical Principles of Natural Philosophy,' in the year 1687, that some new geometrical methods had been discovered which were common to himself and to me, and that neither of us was indebted to light received from the other, but that each was indebted to his own cogitations, and that they had been made public by me now ten years ago. When I published my 'Elements of the Calculus,' in 1684, no other matter was certain to me concerning his discoveries in this subject than what he himself had formerly stated in his letters,—that he could find tangents without removing irrational quantities, which Huygens afterwards made known to me that he could also do, while as yet he was unacquainted with the rest of that Calculus. But I fully understood that Newton had achieved far greater discoveries after I had seen his Principia. I did not learn, however, that a calculus so like to the differential had been practised by him before the first and second volumes of that distinguished geometer, John Wallis, were put forth, and Huygens copied the extract relating to Newton and promptly sent it to me. But after so great benefits conferred on the public, though it may be unreasonable to make any requirements from Mr. Newton that may demand fresh labour of investigation, I cannot, however, restrain myself, but that on the present occasion I may publicly request a mathematician of the highest genius that, mindful of human weakness and of the common good, he would no longer conceal the rest of his excellent and mature cogitations, by which he can further illustrate not only the mathematical sciences, but also especially the laws of nature. But if the renown of so great objects does not move him (although scarcely any addition can be made to the renown which he has already gained), he may at least consider, that no consideration is of greater concern to a generous mind than to be deemed worthy of the highest esteem of the human race."

Newton published his Optics with the treatise on quadratures in the year 1704, and in it repeated the fact which Dr. Wallis had published nine years before, to the effect that the method of fluxions was invented in the years 1665 and 1666. The editors of the Leipsic Acts, in their account of this work in the part for January, 1705, stated that M. Leibnitz was the first inventor of this method, and that Mr. Newton had substituted fluxions for differences; just as

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inventionem hujus calculi sibi passim tribuentis, ullis imponet, qui ea pertractarint, quæ ipse evolvi, instrumenta."—*Investigatio Geometrica solidi rotundi, in quod minima fiat resistentia.* Londini, 1699, p. 18.

Fabri in his synopsis of geometry had substituted movement for the indivisibles of Cavallerius. This claim, and the charge against Newton, opened the controversy,[1] which continued till the death of Leibnitz in 1716.

In the Philosophical Transactions of the Royal Society for 1708, Dr. Keill published a letter addressed to Dr. Halley, "On the Laws of Centripetal Forces," in which he remarks:—

"All these things follow from the now much celebrated method of fluxions, of which our Newton was doubtless the first inventor, as will be evident to any one who shall read his letters published by Dr. Wallis. Yet afterwards the same method was published by M. Leibnitz in the Acta Eruditorum, only changing *the name and manner of notation.*"

M. Leibnitz, understanding Dr. Keill's words[2] in a stronger sense than the writer intended, regarded them as a calumny. In the year 1711 M. Leibnitz made a complaint to the Royal Society that Dr. Keill had accused him of publishing "The Method of Fluxions" invented by Mr. Newton, under other forms and denominations; adding, that nobody knew better than Newton himself that he had borrowed nothing from him; and required that Dr. Keill should publicly disavow the ill construction which might be put on his words.

When the letter of Leibnitz was read at a meeting of the Society, Dr. Keill justified the language he had employed. At another meeting of the Society on April 5, 1711, Newton, from the chair, gave an account of his discovery, and Dr. Keill was requested "to draw up an account of the matter in dispute, and to set it in a just light." The account was drawn up and read to the Society, and a copy was sent to Leibnitz. On receiving this letter he was not content, but wrote again to the secretary a letter of the date of December 29, 1711, and requested the Royal Society to silence Dr. Keill's vain and unjust clamours, of which he expressed his belief Newton did not approve.

The Royal Society, thus appealed to, on March 6, 1712, appointed a committee[3] to examine the letters, papers, and other documents relating to the matter in dispute; and after examination of the evidence, the committee concluded by affirming their opinion that "Dr. Keill, in asserting Mr. Newton to be the first inventor, has been in no ways injurious to M. Leibnitz." The Royal Society printed the determination of the question,[1] together with the letters and documents

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[1] "Pro differentiis igitur Leibnitianis D. Newtonus adhibet, semperque adhibuit Fluxiones, quæ sunt quam proxime ut Fluentium augmenta æqualibus temporis particulis quam minimis genita; iisque tum in suis Principiis Naturæ Mathematicis, tum in aliis postea editis eleganter est usus, quemadmodum et Honoratus Fabricius in sua synopsi Geometrica, motuum progressus Cavallerianæ methodo substituit."—*Acta Lipsiensia, Januarii* 1705.

[2] The words of Dr. Keill are: "Fluxionum Arithmeticam sine omni dubio primus invenit D. Newtonus, ut cuilibet ejus Epistolas a Wallisio editas legenti facile constabit. Eadem tamen Arithmetica postea mutatis nomine et notationis modo a Domino Leibnitio in Actis Eruditorum edita est."

[3] It is only fair, however, to remember that Newton was President at the time, and that the members of the committee, Dr. Arbuthnot, Mr. Hill, Mr. Jones, Mr. Machin, and Mr. Burnet, being all zealous and intimate friends of Newton, can

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hardly have been considered unbiased. On March 20, Mr. Robarts was added to the committee; M. Bonet, the Prussian minister, on the 27th; and Mr. De Moivre, Mr. Aston, and Dr. Brook Taylor on the 17th of April.

[1] The following is the decision of the Committee:—

"We have consulted the letters and letter-books in the custody of the Royal Society, and those found among the papers of Mr. John Collins, dated between the years 1669 and 1677 inclusive; and showed them to such as knew and avouched the hands of Mr. Barrow, Mr. Collins, Mr. Oldenburg, and Mr. Leibnitz; and compared those of Mr. Gregory with one another, and with copies of some of them taken in the hand of Mr. Collins; and have extracted from them what relates to the matter referred to us; all which extracts herewith delivered to you, we believe to be genuine and authentic: and by these letters and papers we find,

"I. That Mr. Leibnitz was in London in the beginning of the year 1673, and went thence in or about March to Paris, where he kept a correspondence with Mr. Collins by means of Mr. Oldenburg, till about September, 1676, and then returned by London and Amsterdam to Hanover. And that Mr. Collins was very free in communicating to able mathematicians what he had received from Mr. Newton and Mr. Gregory.

"II. That when Mr. Leibnitz was the first time in London, he contended for the invention of another differential method properly so called; and notwithstanding that he was shown by Dr. Poll that it was Mouton's method, persisted in maintaining it to be his own invention, by reason that he had found it by himself, without knowing what Mouton had done before, and had much improved it. And we find no mention of his having any other differential method than Mouton's, before his letter of the 21st of June, 1677, which was a year after a copy of Mr. Newton's letter of the 10th of December, 1672, had been sent to Paris to be communicated to him; and above four years after Mr. Collins began to communicate that letter to his correspondents; in which letter the method of fluxions was sufficiently described to any intelligent person [idoneo harum rerum cognitori].

"III. That by Mr. Newton's letter of the 13th of June, 1676, it appears that he had the method of fluxions above five years before the writing of that letter. And by his 'Analysis per Æquationes numero terminorum infinitas,' communicated by Dr. Barrow to Mr. Collins in July, 1669, we find that he had invented the method before that time.

"IV. That the differential method is one and the same with the method of fluxions, excepting the name and mode of notation. Mr. Leibnitz calling those quantities differences, which Mr. Newton calls moments or fluxions; and marking them with the letter *d*, a mark not used by Mr. Newton. And therefore we take the proper question to be, not who invented this or that method, but who was the first inventor of the method. And we believe that those who have reputed Mr. Leibnitz the first inventor, knew little or nothing of his correspondence with Mr. Collins and Mr. Oldenburg long before; nor of Mr. Newton's having that method above fifteen years before Mr. Leibnitz began to publish it in the Acta Eruditorum of Leipsick.

"For which reasons we reckon Mr. Newton the first inventor; and are of opinion that Mr. Keill, in asserting the same, has been in no ways injurious to Mr. Leibnitz. And we submit to the judgment of the Society, whether the extracts of letters and relating to it, under the title of "Commercium Epistolicum de Analysi promota," 8vo, London, 1712. A reprint of the Commercium Epistolicum was published with some alterations and additions at London in 1722, where Newton, in the preface and annotations, particularly answered all the objections which had been made since the Commercium Epistolicum[1] first made its appearance.

Newton himself put forth the following statement:—

"In my letters of the 13th of June and the 24th of October, 1676, I affirmed that I had the method of Fluxions some years before; but I never allowed that Mr. Leibnitz had the differential method before the year 1677; nor in those days did I know more of his pretences to it than what he represented that year in his letter of the 21st of June; nor did I allow the method of transmutations to be a general method of series; nor then knew that the series which he sent to me was sent to him by Mr. Oldenburg the year before, and invented by Mr. James Gregory in the year 1671. The method of transmutations is not a method of series, but a particular theorem for transmutations of figures into one another, like those of Gregory and Barrow. And as for the Scholium upon the second lemma[2] of the second book of the Principia

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papers now presented to you, together with what is extant to the same purpose, in Dr. Wallis's third volume, may not deserve to be made public."

[1] A new edition of this work has been recently published, entitled, "Commercium Epistolicum J. Collins et Aliorum de Analysi promota, etc. Ou Correspondance de J. Collins et d'autres savants célèbres du XVII^e Siècle, relative à l'Analyse Supérieure, réimprimée sur l'édition originale de 1712, avec l'indication des variantes de l'édition de 1722, complétée par une collection de pièces justificatives et de documents, et publiée par J. B. Biot, membre de l'Institut, et F. Lefort, ingénieur en chef des ponts et chaussées. Paris, 1856."

[2] In the first edition of the Principia, published in 1687, in page 253, is a scholium at the end of the seventh proposition of the second book, in substance as follows:—"In a correspondence in which I was engaged with the very learned geometrician, M. Leibnitz, ten years ago [through Mr. Oldenburg], having informed him that I was acquainted with a method of determining the maxima and minima, drawing tangents, and doing other similar things, which succeeded equally in rational equations and radical quantities, and having concealed this method by transposing the letters of the words, which signified: 'an equation containing any number of flowing quantities being given, to find the fluxions, and inversely:' that celebrated gentleman answered, that he had found a similar method; and this, which he communicated to me, differed from mine only in the enunciation and notation."

In the second edition of the Principia, published in the year 1714, at the end of the scholium is added, "and in the idea of the generation of quantities."

In the third edition, published in 1726, this scholium was withdrawn, probably on account of the unreasonable proceedings of Mr. Leibnitz's friends, who persistently, but unjustly, affirmed him to have been the first inventor, and the following was substituted in its place:—

"In a letter of mine to Mr. J. Collins, dated December 10, 1672, having described a method of tangents, which I suspected to be the same with Slusius's

Philosophiæ Mathematica, which is so much wrested against me, it was written, not to give away that lemma to Mr. Leibnitz, but, on the contrary, to assert it to myself. Whether Mr. Leibnitz invented it after me, or had it from me, is a question of no consequence, for second inventors have no right. Dr. Wallis, by his letter of December 1, 1696, printed in the third volume of his works, gave notice to Mr. Leibnitz of the paragraph in his preface to the first volume above cited. And Mr. Leibnitz denied not that in the year 1676 I explained to him the method found by me ten years before or above, nor complained of the Doctor for saying this: *De te autem queri*, saith he, in his letter of the 29th of March, 1697, *nunquam mihi in mentem venit; quem facile apparet nostra, in Actis Lipsiensibus prodita, non satis vidisse.* He allowed that the methods were of like nature, as the Doctor had affirmed, and said in his letter of the 28th of May, 1697, that he therefore called them both by the common name of the *Infinitesimal Method;* but added, that as the Analysis of Vieta and Cartes were both called by the common name of *Analysis Speciosa*, and yet differed in some things, so my *Infinitesimal Analysis* and his might differ in some things; meaning some improvements made by him to this method; and excused the Doctor for not mentioning these improvements, because he had not sufficiently seen them in the 'Acta Lipsiensia;' and declared that it never came into his mind to complain of him for anything else."[1]

Mr. S. P. Rigaud, in his Essay on the Principia, thinks it may be thought very presumptuous to question the accuracy of the Committee which drew up the documents of the Commercium Epistolicum. He remarks there are difficulties well worth considering respecting the date, 1683, in the following extract of their statement in No. LXXI:—"Anno scilicet 1684 in iisdem Actis Lipsicis pro mense Octobri, Calculi Differentialis Elementa primum, edidit D. Leibnitius, literis G. G. L. designatus. Anno antem 1683 ad finem vergente, D. Newtonus propositiones principales, earum quæ in Philosophiæ Principiis Mathematicis habentur, Londinum misit: eædemque cum Societate Regia mox communicatæ sunt, annoque 1686 liber ille ad Societatem

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method, which at that time was not made public; I subjoined these words:—'This is one particular, or rather a corollary, of a general method, which extends itself, without any troublesome calculation, not only to the drawing of tangents to any curve lines, whether geometrical or mechanical, or anyhow respecting right lines or other curves, but also to the resolving other abstruse kinds of problems about the crookedness, areas, lengths, centres of gravity of curves, &c., nor is it (as Hudden's Method de Maximis et Minimis) limited to equations which are free from surd quantities. This method I have interwoven with that other of working in equations by reducing them to infinite series. And these last words relate to a treatise I composed on that subject in the year 1671. The foundation of that general method is contained in the preceding lemma.' "

[1] Isaaci Newtoni Opera quæ extant omnia, commentariis illustrabat Samuel Horsley, LL.D., R.S.S. Tom. iv., p. 616. Londini, 1782.

missus est ut imprimeretur, proximoque anno mense Martio lucem vidit: et exemplar ejus D. Nicolao Fatio datum est ut ad Leibnitium mitteretur."

"The same date of 1683 occurs in both editions of the Commercium Epistolicum (in 1712 and 1722), and it cannot be a press error, for the argument with reference to Leibnitz would fall to the ground if 1684 were substituted for 1683. Newton wrote a letter dated Feb. 23, 1684-5, to Aston, one of the secretaries, thanking him for 'entering his notions about motion' in the Register; and added, 'I designed them for you before now, but the examining several things has taken a greater part of my time than I expected, and a great deal of it to no purpose;' and a minute of this letter appears in the Journal, dated Feb. 25, 1684.

"A loose paper preserved in Lord Macclesfield's collection seems to offer the most probable solution of the difficulty. It is in Newton's handwriting, and is entitled, 'Animadversiones in Schediasma Leibnitii, de resistentia medii et motu projectorum gravium in medio resistente, in Actis Lipsiensibus anno, 1689 mense Feb. impressum.' It is not complete, but rather the rough draught for the beginning of some regular remarks. Newton is well known to have been in the habit of repeatedly transcribing his papers, and copying them with alterations, and accordingly we find on the same sheet of paper both the entries. The first begins with the words, 'Newtonus principales de motu planetarum propositiones anno 1683 Londinum misit cum philosophis communicandis. Anno 1656 Principia Mathematica ad Regiam Societatem misit ut in lucem emitteretur. . . .' The second paper begins with, 'Anno 1684 Newtonus propositiones principales earum, quæ in philosophiæ Principiis Mathematicis habentur, cum Societate Regia communicare coepit, annoque 1686 liber ille MS. ad Societatem Regiam missus est. . . .'

"This paper was certainly in the possession of Jones, and contains several of the expressions used in the Commercium Epistolicum, with the same date, in the first instance, annexed to them. The writers, however, seem not to have observed more than the first entry, the second being on the opposite side of the sheet; it could otherwise have hardly failed to strike them that the year had there also at first been written 1683, since the last figure, having evidently been altered to a 4, is much blacker than the rest. They would then have either had at once the correct period, or would have been led to ascertain which of the two they were to adopt. If the doubt had been once suggested, there could have been no difficulty in resolving it."

It has been remarked[1] that some doubts have been entertained whether a certain letter of Newton was actually sent to Leibnitz with other documents, as is asserted in the Commercium Epistolicum, No. XLVI. "In hoc collectione habetur et Epistola D. Newtoni ad D. Collins, 10 Decemb. 1672 data, in qua Newtonus se Methodum generalem habere dicit ducendi tangentes exponit."

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[1] P. xlvii. Correspondence of Sir Isaac Newton and Professor Cotes, including letters of other eminent men, now first published from the originals in the Library of Trinity

The manuscripts, papers, and letters left by Leibnitz, and preserved in the Royal Library of Hanover, have of late years been printed,[1] and it may be ascertained what documents Leibnitz did actually receive from Oldenburg, and the dates of them. Among these papers has not been found a copy of Newton's letter of Dec. 10, 1672. It appears that Oldenburg wrote to Leibnitz from London on July 26, 1676; and the letter itself shows that he did not send the parcel of letters (including Newton's) which he had received from Collins, but gave his own account of what they contained. And with respect to Newton's letter of Dec. 10, 1672, he gave the substance, but neither gave the example[2] contained in the letter of Newton, nor

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College, Cambridge; together with an appendix containing other unpublished letters and papers of Newton; with notes, synoptical view of the Philosopher's life, and a variety of details illustrative of his history, by J. Edleston, M.A., Fellow of Trinity College, Cambridge. London, 1850.

[1] Die Entdeckung der Differentialrechnung durch Leibniz mit Benutzung der Leibnizischen Manuscripte auf der Koniglichen Bibliothek zu Hannover dargestellt von Dr. C. J. Gerhardt. Halle, 1848, 1849, 1850, 1855—63.

Historia et origo calculi Differentialis a G. G. Leibnitio, conscripta. Zur zweiten Säcularfeier des Leibnizischen geburtstages aus den Handschriften der Königlichen Bibliothek zu Hannover herausgegeben von Dr. C. J. Gerhardt. Hannover, 1846.

The claim of Leibnitz to the invention of the Differential Calculus by Dr. H. Sloman. Translated from the German with considerable alterations and New Addenda by the author. 4to. Cambridge, 1860.

[2] The following is the passage omitted of Newton's letter. The original is in the collection of Lord Macclesfield:—

"I am heartily glad at the acceptance which our Reverend friend Dr. Barrow's Lectures find with foreign mathematicians, and it has pleased me not a little to understand that they are fallen into the same method of drawing tangents with me. What I guess their method to be you will apprehend by this example. Suppose *CB*, applied to *AB* in any given angle, be terminated at any curve line *AC*, and calling *AB*, x and *BC*, y; let the relation between x and y be expressed by any equation, as $x^3-2xxy+bxx-bbx+byy-y^3=0$, whereby this curve is determined. To draw the tangent *CD*, the rule is this:—Multiply the terms of the equation by any arithmetical progression, according to the dimensions of y,

$$\text{suppose thus } \begin{array}{cccccc} x^3 & -2xxy & +bxx & -bbx & +byy & -y^3; \\ 0 & 1 & 0 & 0 & 2 & 3 \end{array}$$

also according to the dimensions of x,

$$\text{suppose thus } \begin{array}{cccccc} x^3 & -2xxy & +bxx & -bbx & +byy & -y^3. \\ 3 & 2 & 2 & 1 & 0 & 0 \end{array}$$

The first product shall be the numerator, and the last divided by x the denominator of a fraction, which expresseth the length of *BD*, to whose end *D* the tangent *CD* must be drawn. The length of *BD* therefore, is

$$-2xxy+2byy-3y^3 \text{ divided by } 3xx-4xy+2bx-bb."$$

made any allusion to it. It can occasion no surprise that Leibnitz declared he had no recollection of having received Newton's letter of Dec. 10, 1672.

It appears to be unquestionable that two men who separately make the same important discovery, and show the use of it, have an equal claim to merit and to public gratitude. As the methods of Newton and Leibnitz lead to the same results, it has been affirmed that the method of Leibnitz was derived from that of Newton; but this Leibnitz has firmly denied. With equal reason it might be asserted that Newton's method was taken from that of Leibnitz. Whether the method of fluxions might have its origin from the method of differences, or the converse, is not the question at issue; but whether, in point of fact, each system had a separate and independent origin in the minds of the inventors. It is evident that already several approaches had been made towards the new calculus, especially by Dr. Wallis and by Dr. Barrow, so that the principles which formed the basis of the new calculus were well known among mathematicians at that period. It can afford no matter for surprise that two men of resplendent genius, such as Newton and Leibnitz, should independently have originated a calculus which, though different in some respects, leads to the same results. The publications of Wallis afford satisfactory proof that Newton had been in possession of his method of fluxions and had developed it before any notices had been given by Leibnitz of the nature of his discoveries in the method of differences. And it may be added that the letters communicated to Leibnitz contained nothing which can prove that he borrowed his calculus from Newton, but, that from the description given in passages of these letters, he divined the advantages of the calculus which Newton's letters had described.

James Bernoulli, on a somewhat similar occasion, once exhorted his brother to moderate his pretensions, and to attach less importance to discoveries which the instrument rendered easy, and to acknowledge that "as quantities in geometry increase by degrees, so every man, furnished with the same instrument, would find by degrees the same results."

And Professor De Morgan has justly affirmed:—"I have come to a settled conclusion that *great points belong to those who made great points of them.* The history of mathematical discovery is vexed with never-ending disturbances arising out of claims of priority, which mean that this person threw the thing away before that person used it. In many cases it is by no means certain that this person ever saw in his own words or formula what that person enables *us* to see. Giving due moral blame to any one who consciously suppresses a hint which he knows he has taken, I consider that an inventor who is the first user has a position from which a hundred previous inventors cannot dislodge him, nor do anything but enhance his merit as the *inventor of the use*, most often the more difficult invention of the two."[1]

Montucla, the historian of the Mathematical Sciences, has thus expressed his opinion:—

"It is incontestable that Newton was the first inventor of the calculus of fluxions. The proofs of it are as clear as the light. But is Leibnitz guilty of having published as his own a discovery which he had drawn out of the writings of Newton? I do not think so. In the two letters of Newton communicated to Leibnitz there appear only the results of the method, or of the two methods employed by Newton, but not the explanation of them. Leibnitz was a man endowed with transcendent talents. Could he not have been moved to search out the means employed by Newton and to achieve success, as Fermat, Barrow, and Wallis had opened the way? In fact, if it be considered how little there was to pass from their methods to the Differential Calculus, it will appear, I think, superfluous to search elsewhere for the origin of the latter."

Upon the whole view of the circumstances of the case, it appears that both parties to the dispute were to blame. Leibnitz should have been more ready to acknowledge the aid he obtained from Newton's communications, although this aid bore only indirectly on the discovery of the calculus, and Newton ought to have seen and acknowledged that Leibnitz's notation was more systematic than his own, and better adapted to promote the progress of the calculus. There are sufficient grounds for the belief that the discoveries of both were independent. Leibnitz affirmed that he owed nothing to Mr. Newton; and Newton, on the other hand, declared that whether Mr. Leibnitz invented the calculus after him, or had it from him, is a question of no consequence, for second inventors have no right. Both owed much to those who had gone before them. No doubt Newton's was earlier in point of time, but then he did not publish it, nor did he develop his method in the same systematic way as was done by Leibnitz and his followers.[2]

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[1] Prof. De Morgan, note, p. 200, vol. xi., Cambridge Philosophical Transactions.

[2] The fluxional notation was adopted at Cambridge, and was continued from the time of Newton until a recent period. It was used in the Senate House examination papers for mathematical honours until the year 1817, when Mr. Peacock, Fellow of Trinity College, was junior moderator. He adventured on the innovation of introducing the differential notation of Leibnitz, instead of the fluxional notation of Newton, into his papers in that year, while his colleague, Mr. White, Fellow of Caius College, the senior moderator, employed the fluxional notation in his papers. In 1818 the fluxions resumed their place in the Senate House examination papers, but in 1819, when Mr. Peacock was again moderator, he

A few brief notes here respecting these two distinguished men may not appear to be altogether without interest.

The intellectual labours of Leibnitz and Newton were not confined to mathematical and physical researches. Leibnitz, endowed with a vigorous and versatile genius, had acquired habits of indomitable industry. His contributions on almost all subjects of human knowledge display an extraordinary intellectual energy.. "His powers," it has been truly remarked, "were dissipated by the multiplicity of his pursuits. He attempted more than he could finish, and designed more than he could execute;" and, as an independent thinker, he was more discursive but less profound than Newton. In his attempt to grasp the whole circle of the sciences, he exemplified the saying that has been applied to a distinguished man of more modern times, that "if *Science* was his forte, *Omniscience* was his foible."

The tendency of his mind was rather to disregard the obvious conclusions of experience and common sense, and "to regard all difficult things as easy, and all easy things as difficult," according to his own account of himself. He was, as his father, a member of the Lutheran church, and faithful to the Confession of Augsburg. He advocated toleration, and earnestly desired the reconciliation of the Lutheran and Roman Catholic communions. His more enlightened Lutheran friends urged that the Popes and their agents had corrupted the pure truths of the gospel and had usurped a tyrannical control over the Christian church; that the persecution of the Huguenots in France supplies evidence how little reliance could be placed on the written or verbal promises and engagements of Roman Catholics; and that, so long as they hold that their church cannot err, that their Pope is in-

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repeated his innovation. This practice was very strongly deprecated at the time, but the superior advantages of the notation of Leibnitz became appreciated, and after the year 1819 the use of the fluxional notation entirely disappeared in the examination papers. The following extract of a letter to Mr. J. F. W. Herschel, of date 6th March, 1817, from Mr. Whewell, Fellow of Trinity, may be read with interest:—

"You have, I presume, seen Peacock's examination papers. They have made a considerable outcry here, and I have not much hope that he will be moderator again. I do not think he took precisely the right way to introduce the true faith. He has stripped his analysis of its applications, and turned it naked among them. Of course all the prudery of the university is up and shocked at the indecency of the spectacle. The cry is, 'Not enough philosophy.' Now, the only way to prevent such a clamour would have been to have given good, intelligible, but difficult physical problems, things which people would see they could not do their own way, and which would excite their curiosity sufficiently to make them thank you for your way of doing them. Till some one arises to do this, or something like it, they will not believe even though one were translated to them from the French." —P. 16, vol. ii., Account of the Writings of W. Whewell, D.D., Master of Trinity College, Cambridge, with Selections from his Literary and Scientific Correspondence. By I. Todhunter, M.A., F.R.S., Honorary Fellow of St. John's College.

fallible in his decision of religious controversies, so long would all proposals for union between the Lutheran and Romish churches[1] be fruitless and ineffective. Notwithstanding these objections, Leibnitz had the courage to attempt the impossible problem of effecting the union of the Lutheran and Roman Catholic churches in Hanover. He utterly failed, as might have been foreseen, in an attempt to reconcile irreconcilable contradictions.

There is an unfinished treatise, entitled "Systema Theologicum,"[2] in the handwriting of Leibnitz, preserved in the Royal Library of Hanover, and was probably written about the year 1684. The unscriptural dogmas of the Church of Rome in this treatise are represented in a light more favourable than might have been expected from a Lutheran. From one of the letters of Leibnitz still extant, it appears certain that the treatise was composed with the view of promoting the reconciliation of the two churches.

In a letter to his friend Anthony Arnauld, the leader of the Jansenists, he stated that he had anxiously sought out and read with the utmost attention every distinguished antagonist of Christianity and all its most eminent defenders. He honestly and vigorously contended against the enemies of the Christian revelation, and his opinions on this subject will appear in the introductory lines of his "Systema Theologicum."[3]

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[1] Leibnitz was engaged by Ernest Augustus, the Elector of Hanover, to write the history of the House of Brunswick. In order to execute this task, he travelled abroad to collect materials for the purpose. While he was in Italy he met with an adventure which, in a small way, illustrates a Catholic principle. As Leibnitz was passing in a small barque from Venice to Mesola a storm arose; the pilot, believing the German to be a heretic, looked on him as the cause of the tempest. This good Catholic, supposing he was not understood by a German, proposed to strip him of his clothes and money, and throw him overboard. Leibnitz, hearing this proposal of the pilot, without discovering the least emotion, drew a set of beads from his pocket, and began to turn them over with great seeming devotion. The artifice succeeded. One of the sailors observed to the pilot, that since the man was no heretic he ought not to be drowned.

[2] This fragment of Leibnitz has been printed with the title of "A System of Theology by Godfrey William Von Leibnitz, translated with an Introduction and Notes by Charles William Russell, D.D., Professor of Ecclesiastical History in St. Patrick's College, Maynooth." 8vo. London, 1850.

[3] "After a long and mature examination of the controversies on the subject of religion, in which I have invoked the Divine assistance, and divested myself, as far, perhaps, as is possible for man, of party feeling, as though I came from a new world, a neophyte unattached to any party, I have at length fixed in my own mind, and, after full consideration, resolved to adopt the following principles, which, to an unprejudiced man, will appear to carry with them the recommendation of sacred Scripture, of pious antiquity, and even of right reason and the authority of history.

"In the first place, then, I believe in the existence of a Most Perfect Substance, One, eternal, omnipresent, omniscient, and almighty, which we call GOD; by whom all other things have been created in a most beautiful order, and are preserved by a

This truly great man, Sir Isaac Newton, ended his life and his labours on the 28th March, 1727, in the eighty-fifth year of his age, having retained the use of his faculties unimpaired till the last. He united a modest simplicity of character with a strong aversion to engage in disputes and controversies. Courteous and affable, his exemplary life was a continued series of generosity, goodness, and patient labour. His modest opinion of himself appears in attributing his discoveries in science "as due to nothing but industry and patient thought." He was firmly attached to the cause of liberty and the constitution, and was chosen one of the delegates to represent the University before the High Commission Court, to answer for their contempt of the mandamus of King James II., to admit a papist to the degree of M.A. without taking the oaths prescribed by the statutes of the University. In 1688 he represented the University in the Con-

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sort of continued production. The doctrine, therefore, of those who conceive God to be corporeal, finite, circumscribed by place, and ignorant of all future contingent events, whether absolute or conditional, is utterly intolerable; and hence I strongly reprobate certain sectaries, who have not spared even this first principle of faith, and who entertain most unworthy conceptions of God.

"Now this Supreme Intelligence created, in order to His own glory, other intellectual beings, whom He governs with most perfect justice; insomuch that any one who could understand the whole order of the Divine economy, would find therein a model of the most perfect form of commonwealth, in which it would be impossible for a philosopher to discover a single want, or to supply anything in desire. Hence we must avoid those opinions which represent God as a certain supreme power, from which all things, although they emanate, yet emanate indiscriminately, by a kind of necessary existence, and without any selection of the beautiful or the good; as if these notions were arbitrary, or had no foundation in nature, but only in the imagination of men. For God is not only the supreme Creator of all existing things, but He is also the beneficent Prince of all intellectual beings, and in some sense their legislator; a legislator, however, who requires nothing from His subjects but souls actuated by sincere affection, animated with a right intention, persuaded of the beneficence, the consummate justice, the beauty, and goodness, of the most amiable of all lords; and therefore not merely fearing His power as that of a supreme and all-seeing Monarch, but also confiding in His benevolence, and, in fine, glowing with the love of Him above all things—a sentiment which comprises all the rest.

"For those who are impressed with such sentiments, who fix them deeply in their souls, and evince in their lives the sincerity of their convictions, never murmur against the Divine will, being well assured that all things must conduce to the good of those who love God: and, as they are content with the past, so, in what concerns the future, they seek always to act in conformity with that which they presume to be the will of God. Now all that God, in proposing rewards and punishments, requires of each of us is, that he labour for the fulfilment of his own especial duties; that, like the first man, he cultivate the garden in which he has been placed, and that, in imitation of the Divine goodness, he diffuse his beneficence on every object around him, but especially within the due proportions which justice requires, on all those with whom he may be thrown into intercourse, as being his neighbours; because, among the creatures which come within our sphere, there is none more excellent than man, none whose perfection is more grateful to God."

vention Parliament, and was a second time returned in 1701. His theological writings evince his sincere belief in the Records of the Christian Revelation, and a passage[1] in the general scholium at the

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[1] The following is a translation of the passage :—

"This most beautiful system of the sun, planets, and comets, could only proceed from the counsel and dominion of an intelligent and powerful Being. And if the fixed stars are the centres of other like systems, these, being formed by the like wise counsel, must be all subject to the dominion of One, especially since the light of the fixed stars is of the same nature with the light of the sun, and from every system light passes into all other systems. And lest the systems of the fixed stars should, by their gravity, fall on each other mutually, He has placed these systems at immense distances one from another.

"This Being governs all things, not as the soul of the world, but as Lord over all. And on account of his dominion He is wont to be called Lord God, παντοκράτωρ, or Universal Ruler. For God is a relative word, and has a respect to servants ; and Deity is the dominion of God, not over His own body, as those imagine who fancy God to be the soul of the world, but over servants. The supreme God is a Being eternal, infinite, absolutely perfect ; but a being, however perfect, without dominion, cannot be said to be Lord God ; for we say, my God, your God, the God of Israel, the God of gods, and Lord of lords ; but we do not say, my Eternal, your Eternal, the Eternal of Israel, the Eternal of gods ; we do not say, my Infinite, or my Perfect : these are titles which have no respect to servants. The word God usually signifies Lord ; but every lord is not a god. It is the dominion of a spiritual being which constitutes a god ; a true, supreme, or imaginary dominion makes a true, supreme, or imaginary god. And from His true dominion it follows that the true God is a Living, Intelligent, and Powerful Being ; and from His other perfections, that He is supreme or most perfect. He is eternal and infinite, omnipotent and omniscient ; that is, His duration reaches from eternity to eternity ; His presence from infinity to infinity ; He governs all things, and knows all things that are or can be done. He is not eternity nor infinity, but eternal and infinite ; He is not duration or space, but He endures and is present. He endures for ever, and is everywhere present ; and by existing always and everywhere, He constitutes duration and space. Since every particle of space is always, and every indivisible moment of duration is everywhere, certainly the Maker and Lord of all things cannot be never and nowhere. Every soul that has perception is, though in different times and in different organs of sense and motion, still the same indivisible person. There are given successive parts in duration, co-existent parts in space, but neither the one nor the other in the person of a man, or his thinking principle ; and much less can they be found in the thinking substance of God. Every man, so far as he is a thing that has perception, is one and the same man during his whole life, in all and each of his organs of sense. God is the same God, always and everywhere. He is omnipresent, not *virtually* only, but also *substantially* ; for virtue cannot subsist without substance. In Him are all things contained and moved ; yet neither affects the other : God suffers nothing from the motion of bodies ; bodies find no resistance from the omnipresence of God.

"It is allowed by all that the supreme God exists necessarily ; and by the same necessity He exists always and everywhere. Whence also He is all similar, all eye, all ear, all brain, all arm, all power to perceive, to understand, and to act ; but in a manner not at all human, in a manner not at all corporeal, in a manner utterly unknown to us. As a blind man has no idea of colours, so have we no idea of the manner by which the all-wise God perceives and understands all things. He is utterly void of all body and bodily figure, and can therefore neither be seen, nor end of the second edition of his great work, exhibits his conception of the Deity as the Creator and Ruler of the universe.

In the year 1715, Dr. Brook Taylor, of St. John's College, Cambridge, published his "Methodus Incrementorum Directa et Inversa" in a quarto volume. He gives to the differences of two variable quantities the name of increments or decrements, whether considered finite or infinitely small, of two consecutive terms in a series formed according to a given law. When these differences are indefinitely small, the calculus of them, both direct and inverse, belongs to the Differential Calculus. But when the differences are finite, the method of finding the relations they bear to the quantities that produce them, forms a new kind of calculus, the first principles of which were given by Dr. Brook Taylor himself.

His work is extremely concise and somewhat obscure. M. Nicole, a French mathematician, has clearly unfolded the method for resolving finite differences, which may be read in the Memoirs of the French Academy of Sciences for 1717 and 1728; and these articles may be considered as the first elementary treatise on the Integral Calculus with finite differences. In one of the corollaries to the seventh proposition in his work, Dr. Brook Taylor has given a new theorem of the highest importance and of very extensive use in mathematical reasonings. It consists of a series in ascending powers of the increment, to express what any function of a variable quantity becomes, when the variable quantity receives any increment whatever. This famous theorem is called "Taylor's Theorem," from the name of its inventor. It was originally expressed in the fluxional notation; but in later times it has been expressed in the notation of the Differential Calculus, and extended to functions of two or more variable quantities.

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heard, nor touched; nor ought He to be worshipped under the representation of any corporeal thing. We have ideas of His attributes, but what the real substance of anything is, we know not. In bodies we see only their figures and colours, we hear only the sounds, we touch only their outward surfaces, we smell only the smells, and taste the savours; but their inward substances are not to be known, either by our senses, or by any reflex act of our minds; much less then have we any idea of the substance of God. We know Him only by His most wise and excellent contrivances of things, and final causes; we admire Him for His perfections, but we reverence and adore Him on account of His dominion. For we adore Him as His servants, and a God without dominion, providence, and final causes, is nothing else but fate and nature. Blind metaphysical necessity, which is certainly the same always and everywhere, could produce no variety of things. All that diversity of natural things which we find, suited to different times and places, could arise from nothing but the ideas and will of a Being necessarily existing. But by way of allegory, God is said to see, to speak, to laugh, to love, to hate, to desire, to give, to receive, to rejoice, to be angry, to fight, to frame, to work, to build. For all our notions of God are taken from the ways of mankind, by a certain similitude which, though not perfect, has some likeness, however. And thus much concerning God; to discourse of whom from the appearances of things, does certainly belong to natural philosophy."

In the first chapters of his "Institutions," Euler gave a lucid explanation of the principles of this subject; it was also treated by Emerson in his Method of Increments, which was published in 1763.

The formula which is generally known as Maclaurin's Theorem, will be found in p. 610 of the second volume of his treatise on Fluxions, published in 1742. It may, however, be noted that the same formula appears in p. 102 of Stirling's "Methodus Differentialis," which was published in 1730. This theorem may easily be deduced from Taylor's Theorem (of which it is a particular case), and is useful in finding the expansion of any function of a variable in a series of ascending powers of that variable.

The labours of the Bernoullis in connection with Leibnitz contributed very largely to the progress of the new calculus on the Continent. Soon after the early essays of Leibnitz appeared in the Leipsic Acts, in which the art and method of his calculus were not fully disclosed, James Bernoulli and his brother John, having read these papers, saw the advantages of his method, and were induced to attempt to search out the full meaning of Leibnitz. Their efforts ended in such success, that Leibnitz himself is reported to have declared that the invention belonged to them as much as to himself.

The contributions of John Bernoulli are printed in the Leipsic Acts and in the Memoirs of the French Academy. His collected works were published at Geneva in four quarto volumes in 1742. The valuable papers of James Bernoulli on the calculus and various other subjects were printed in the Leipsic Acts. His works, that had been previously printed, were collected and published at Geneva in two quarto volumes, in 1744. His papers on the Integral Calculus give an excellent exposition of the subject, especially in the formation and integration of differential equations. Daniel Bernoulli, one of the sons of John Bernoulli, was an eminent mathematician, and during his lifetime he was successful in bearing off ten prizes from the Academy of Sciences, which were contended for by the greatest mathematicians of Europe. In the year 1734 he divided the honour of one of them with his father, which caused an estrangement between them.

John de Rond D'Alembert was one of the most eminent scholars and mathematicians of his age. He was born in 1717, and died in 1783. His literary and scientific labours embraced numerous subjects on philology, history, and philosophy. His new principle of the calculus of differences he exemplified in the discourse on the general theory of the winds, for which the prize medal of the Academy of Berlin was adjudged in 1746. He also applied the same principle to the problems of vibrating chords and the propagation of sound. His writings contain some valuable papers on the Integral Calculus. The mathematical writings of D'Alembert were published under the title of "Opuscules Mathematiques, ou Memoires sur differens sujets de

Géométrie, de Méchaniques, d'Optiques, d'Astronomie," in eight quarto volumes, between the years 1761 and 1780.

Euler published his "Institutiones Calculi Differentialis" in 1755, and his "Institutiones Calculi Integralis" in 1768-70. In the preface to the former he describes the Differential Calculus as "the art of finding the vanishing increments that any functions acquire, when we attribute to the variable quantity of which they are functions a vanishing increment." His "Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes" was published at an earlier period, in the year 1744.

It has been seen that there is a marked difference between the original conceptions of Newton and Leibnitz with respect to the foundations of the calculus. The differentials of Leibnitz were indefinitely small quantities added to, or subtracted from the variable quantity. Newton introduced into his method of fluxions the conception of time and motion—subjects foreign to geometry, and belonging to the science of dynamics. Thus he employed motion as the means of connecting its principles with ordinary Algebra; whereas Leibnitz considered the increase or decrease of a variable to arise from the continued addition or subtraction of independently small parts without any reference to time. Mathematicians have differed in opinion as to the most scientific method of laying down the principles of the calculus, and different attempts have been made to substitute other methods than those of Newton and Leibnitz, but none of them has been more favourably considered than that of La Grange.

La Grange proposed, in the Memoirs of the Berlin Academy for 1772, to make the calculus dependent on the ordinary principles of Algebra, divesting the subject of the consideration of the limiting ratio of indefinitely small increments. He subsequently extended his views of the subject, and published in 1797 his method in a volume entitled "Théorie des Fonctions Analytiques, contenant les principes du Calcul Différentiel, dégagés de tout consideration d'infiniment petits, d'évanouissants; de limits et de fluxions et reduits a l'analyse Algebrique des quantitiés finies." He further elucidated and completed his views in another work which he published in 1801, with the title, "Leçons sur le Calcul des Fonctions."

He makes the basis of his method Taylor's Theorem, the form of which he obtains by the processes of ordinary Algebra. As in the method of finite differences, he considers differentials as finite quantities, always indeterminate, so that they can be made as small as we please. He has also given theorems for the determination of the limits between which lies the remainder of Taylor's series after a finite number of terms has been found.

La Grange devised a formula that bears his name, which is useful for the development of implicit functions. It first appeared in the Berlin Memoirs for the year 1770.

A more general formula for the developement of implicit functions was discovered by La Place, and was first printed in the Memoirs of the French Academy of Sciences for 1777. La Grange's theorem is only a particular case of the theorem of La Place.

The calculus of variations, an important improvement in the calculus, is due to La Grange. His papers on the subject are printed in the second and fourth volumes of the Miscel. Taur. 1760 to 1769.

The principal object of this calculus is to solve generally certain questions of maxima and minima. In the ordinary questions of the subject, it is required to find the values which must be assigned to different variables which enter into a proposed finite function of these variables, so that this function may attain its greatest or least possible value. In the calculus of variations, on the contrary, it is required to find the ratios between the variables, in order to satisfy the condition of a maximum or minimum. Besides, the function which must be a maximum or minimum is not, as in ordinary questions, solely composed of finite quantities, but it must be the integral simply marked of a differential function which is not to be integrated.

In 1797, M. Carnot published his "Reflexions on the Metaphysical Principles of the Infinitesimal Analysis." It was translated into English by the Rev. W. R. Browell, M.A., Fellow of Pembroke College, Oxford, and published at Oxford in 1832.

Every expression of Algebra must necessarily give an idea rather of a set of operations to be performed upon quantities than of the quantity itself which results from performing these operations. Nothing is more difficult than the necessity of considering a complicated literal expression as the *value* of an unknown quantity instead of the exhibition of the processes by which the value is to be obtained, so soon as the subjects of operation shall have received specific values.[1]

Many of the calculations with which we are familiar consist of two parts, a direct and an inverse; as when we consider an exponent of a quantity. For instance, to raise any number to a given power is the direct operation; to extract a given root of any number is the inverse method. The Differential Calculus, which is a direct method, naturally gave rise to the Integral, which is its inverse; the same remark is applicable to finite differences. In all these cases the inverse method is by far the most difficult, and it might perhaps be added the most useful.

If an unknown quantity be given by means of an equation, it becomes a question *to determine its value;* similarly, if an unknown function be given by means of any functional equation, it is required *to assign its form.* In the first case, it is *quantity* which is to be determined; in the second, it is *the form assumed* by quantity, that becomes the subject of investigation. In the one case, the various powers of the unknown quantity enter into the equation; in the other, the different orders of the function are concerned."[1]

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[1] Encyclop. Metropol., ii.; Babbage, p. 366.

When Harriott transposed the absolute term of an equation to the same side as the other terms which involve the unknown quantity, he was enabled to show that an equation of any degree could be formed with simple equations; and conversely, that every equation has as many roots as are denoted by the index of the highest power of the unknown quantity. From that time the problem for the solution of equations became the discovery of some general method by which the roots could be readily found. Solutions of certain forms of cubic and biquadratic equations had been devised by Tartaglia and Ferrari; but a like success has not attended the attempts of mathematicians to solve equations of the fifth and higher degrees.

In his researches in numerical equations Newton was led to seek the limits within which the real roots were situated, and then to determine their value by a method of approximation which bears his name. The difficulties and imperfections of this method were pointed out by La Grange in his "Traité de la Resolution des Equations Numériques de tous les degrés," which was first published in 1798; and he exhibited a method whereby the roots could be found in the form of continued fractions.

During the present century numerous attempts have been made to discover a general method of finding the roots of any equation, at the same time short and easy in its application. The valuable contributions of M. Budan were published in 1803, and those of M. Fourier, after his death in 1831, by M. Navier.

In this brief sketch it would not be possible to notice all the ingenious and curious methods which have been proposed to simplify or effect the solution of numerical equations. There are, however, two methods which appear to have superseded all preceding attempts in this direction.

Mr. Horner's method of approximating to the roots of numerical equations first appeared in the Transactions of the Royal Society for the year 1819, and afterwards in other scientific publications. It consists of a general process, simple and easy, of approximation, for finding the roots of numerical equations. As in the extraction of the roots of numbers, so in this general evolution of the roots of equations; the first figure of a root is found by trial, and the other figures, one by one, are determined by successive steps. The first figure of the root sought, with its algebraical sign, is first required to be known before the method can be applied.

An important theorem of M. Sturm was printed in the sixth volume for 1835 of the "Mémoires présentés par divers Savans à l'Académie Royale des Sciences de l'Institut de France."

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[1] Phil. Trans., 1815; Babbage, p. 339.

The mathematical prize for 1834 of the French Academy was awarded to M. Sturm on account of this discovery. His memoir exhibits a method of separating the real from the imaginary roots of numerical equations, the negative as well as the positive, and of finding the integral limits between which the roots are situated.

It will hence appear that, by a combination of the methods of Mr. Horner and M. Sturm, the number and situation of the real roots of any numerical equation can be ascertained, and approximate numerical results may be found to any degree of accuracy required.

Some of the recent extensions and additions to the science of Algebra, on account of their importance, require not to be passed over in silence. Dr. Salmon, at the end of the third edition of his work, entitled, "Lessons introductory to the Modern Higher Algebra," has added an historical account of the names and methods, and of the formations and properties of these functions. He states that the first idea of determinants, one class of these functions, originated with Leibnitz, and the method fell into oblivion until Cramer, in 1750, rediscovered the idea, and exhibited the determinants arising from linear equations. The subject since that time has engaged the attention of mathematicians, and considerable advancements have been made at different times. The more recent discoveries of Professor Cayley and Professor Sylvester have contributed largely to the completion and perfection of these new methods, as may be seen in their contributions which have appeared in the Philosophical Transactions and other scientific journals. Besides Dr. Salmon's "Lessons," other elementary treatises on this subject have been published, of which may be mentioned Spottiswoode's Elementary Theorems relating to Determinants, London, 1851; Brioschi on the same subject, Pavia, 1854; and Baltzer's work, Leipzig, 1857.

In closing these brief notices respecting the history of Algebra, it may not be altogether useless to add a few remarks for the consideration of intelligent students who are desirous of securing the advantages to be gained from the study of the mathematical sciences. It has been remarked by a distinguished writer, "that mathematical studies, judiciously pursued, form one of the most effective means of developing and cultivating the reason."

There is a necessary connection between cause and effect, and that the same causes uniformly produce the same effects, and that the same effects will uniformly follow the action of the same causes in the future as have been observed in the past, are admitted as the fundamental principles of all reasonings, whether deductive or inductive.

All reasoning must rest upon facts or primary intuitive conceptions which are admitted from the evidence contained in themselves as obvious to the senses and the reason; and are simply admitted, and believed as true, because it is impossible to disbelieve them without manifest absurdity.

There are also distinctions between mathematical and moral evidence, as there are between conclusive and cumulative arguments; but the modes of reasoning, and the forms of argument, and the laws of inference, are the same in all subjects of human inquiry.

In the appropriate language of Dr. Whewell—

"The chains of the logician generally consist only of two or three links. In the mathematics, on the contrary, every theorem is an example of an extended progressive chain; every proof consists of series of assertions, of which each depends on the preceding, but of which the last inferences are no less evident, or no less easily applied, than the simplest first principles. The language contains a constant succession of short and rapid references to what has been proved already, and it is justly assumed that each of these brief movements helps the reasoner forward in a course of infallible certainty. Each of these hasty glances must possess the clearness of intuitive evidence and the certainty of mature reflection, and yet must leave the reasoner's mind entirely free to turn instantly to the next step of his progress."

The late Professor Sedgwick, in a discourse he delivered in the College Chapel in 1831, on the occasion of the annual commemoration of the founder and benefactors of Trinity College, exhorted the junior members of the college in the following words with respect to their mathematical studies:—

"A study of the laws of nature for many years has been, and I hope ever will be, held up to honour in this venerable seat of the discoveries of Newton. But in this, as in every other field of labour, no man can put aside the curse pronounced on him—that by the sweat of his brow he shall reap his harvest. Before he can reach that elevation from whence he may look down and comprehend the mysteries of the natural world, his way is steep and toilsome, and he must read the records of creation in a strange, and to many minds, a repulsive language, which, rejecting both the senses and the imagination, speaks only to the understanding. But when this language is once learnt, it becomes a mighty instrument of thought, teaching us to link together the phenomena of past and future times; and gives the mind a domination over many parts of the material world, by teaching it to comprehend the laws by which the actions of material things are governed. To follow in this track, first trodden by the immortal Newton—to study this language of pure unmixed truth, is to be regarded not only as your duty, but your high privilege. It is no servile task, no ungenerous labour. The laws by which God has thought good to govern the universe are surely subjects of lofty contemplation; and the study of that symbolical language by which alone these laws can be fully deciphered, is well deserving of your noblest efforts."[1]

Lord Brougham has pointed out, in the following expressive language, how a student may acquire some certain knowledge of one of the systems of worlds in the universe—that in which the sun is the central orb around which revolve the earth and the other planets of the system:—

"The reader of the Principia, if he be a tolerably good mathematician, can follow the whole chain of demonstration by which the universality of gravitation is deduced from the fact, that it is a power acting inversely as the square of the distance from the centre of attraction. Satisfying himself of the laws which regulate the motion of the bodies in trajectories around given centres, he can convince himself of the sublime truths unfolded in that immortal work, and must yield his assent to this position, that the moon is deflected from the tangent of her orbit round the earth by the same force by which the satellites of Jupiter are deflected from the tangent of theirs, the very same force which makes a stone unsupported fall to the ground. The reader of the 'Mécanique Céleste,' if he be a still more learned mathematician, and versed in the modern improvements of the calculus which Newton discovered, can follow the chain of demonstration by which the wonderful provision made for the stability of the universe

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[1] Pp. 10, 11, "A Discourse on the Studies of the University of Cambridge," by Adam Sedgwick, M.A., F.R.S., Woodwardian Professor, and Fellow of Trinity College. The Fifth Edition, with additions, and a Preliminary Dissertation. Cambridge, 1850.

The occasion of the first publication of this discourse of Professor Sedgwick is not without some interest. The writer can never forget the electric effect of the discourse on his own mind, and on the minds of some other undergraduates who were present on that occasion. We were all so deeply impressed with the excellency of the counsels that the professor had delivered in his eloquent and expressive language, that we consulted together, and agreed to request the professor to print his discourse. A requisition was drawn up and signed by some of the undergraduates who heard the discourse, and it was placed in the hands of Mr. Whewell, one of the tutors of the college, to present to the professor. It appears that when the requisition was placed in Mr. Whewell's hands, the professor had left college for the Christmas vacation. Mr. Whewell, however, lost no time in communicating with the professor, as will appear from the following extract of a letter of the date, Trinity College, Dec. 23, 1831:—"My dear Sedgwick,—When you had scribbled down the last sentence of your sermon after the bell had stopt, and had succeeded by a sort of miracle in reading your pothooks without spectacles, omitting, however, half the sentences, and a quarter of the syllables of those which remained, I dare say you thought you had done marvellously well, and had completed, or more properly had ended, your task. In this, however, you were mistaken, as I hope soon to make you acknowledge. The rising generation, who cannot err, inasmuch as they will disis deduced from the fact, that the direction of all the planetary motions is the same—the eccentricity of their orbits small, and the angle formed by the plane of their orbits with that of the ecliptic acute. Satisfying himself of the laws which regulate the mutual actions of these bodies, he can convince himself of a truth yet more sublime than Newton's discovery, though flowing from it, and must yield his assent to the marvellous position that all the irregularities occasioned in the system of the universe by the mutual attraction of its members are periodical, and subject to an eternal law which prevents them from ever exceeding a stated amount, and secures through all time the balanced structure of a universe composed of bodies whose mighty bulk and prodigious swiftness of motion mock the utmost efforts of the human imagination. All these truths are to the skilful mathematician as thoroughly known, and their evidence is as clear, as the simplest proposition of Arithmetic is to common understandings. But how few are those who thus know and comprehend them!"[1]

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course most wise and true sentences when you and I are laid in the alluvial soil, declare that their intellectual culture requires that you should print and publish your sermon. I will give you a list on the other side of the names [25] of the persons who have joined in expressing this wish. I undertook very willingly to communicate this their desire to your reverence, inasmuch as I thought your sermon full of notions, as the Americans speak, which it will be very useful and beneficial to put in their heads; or rather to call them out, for a great number of those good thoughts are already ensconced in the excellent noddles of our youngsters, like flies in a bookcase in winter, and require only the sunshine of your seniorial countenance to call them into life and volatility. I do not know anything which will more tend to fix in their minds all the good they get here, than to have such feelings as you expressed, at the same time the gravest and the most animating which belong to our position, stamped, upon a solemn and official occasion, as the common property of them and us. And I also think it of consequence that, when they on their side proffer their sympathy in such reflections, we, on ours, that is, in the present case, your dignified self, should not be backward in meeting them, by giving to all parties the means of returning to and dwelling upon these reflections. Such is my thinking about this matter, and therefore I have undertaken to urge their request; and I hope you will be able to extract from some abyssmal recess your manuscript, and to place it before the astonished eyes of the compositor. It is probable that he will look, as Dante says the ghosts looked when they peered at him, like an old cobbler threading his needle,—but never mind that. The fronts of compositors were made to be corrugated by good sentences written in most vile hands; so let him fulfil his destiny without loss of time."

This extract is taken from Mr. Todhunter's account of the writings and selections from Dr. Whewell's correspondence, pp. 149, 150, vol. ii.

Professor Sedgwick printed his discourse and presented a copy of it to each of the undergraduates who signed the requisition.

[1] Pp. 172, 173, vol. ii., "Dissertations on Subjects of Science connected with Natural Theology: being the concluding volumes of the new edition of Paley's Natural Theology." By Henry Lord Brougham, F.R.S., and Member of the National Institute of France. 2 vols. London. 1839.

In the year 1869 an important work was published by Professor Challis, bearing the title of "Notes on the Principles of Pure and Applied Calculation; and Applications of Mathematical Principles to Theories of the Physical Forces." In the introduction the author states that "the first part[1] is not immediately subservient to the second, excepting so far as results obtained in it are applicable in the latter. In the second part, the mathematical reasoning rests on hypotheses. It does not concern me to inquire whether these hypotheses are accepted, inasmuch as they are merely *put upon trial.* They are proved to be true, if they are capable of explaining all phenomena, and if they are contradicted by a single one, they are proved to be false. . . Some may think that I have deferred too much to Newton's authority. I do not feel that I have need of authority; but I have a distinct perception that no method of philosophy can be trustworthy which disregards the rules and principles laid down in Newton's Principia. . . In the philosophy I advocate there is nothing speculative. *Speculation,* as I understand it, consists in personal conceptions, the truth of which does not admit of being tested by mathematical reasoning; whereas *theory,* properly so called, seeks to arrive at results comparable with experience, by means of mathematical reasoning applied to universal hypotheses intelligible from sensation and experience."

The learned Dr. Isaac Barrow, the first Lucasian Professor, in his fourth mathematical lecture remarks:—

"It may be observed of mathematicians, that they only meddle with such things as are certain, passing by those which are doubtful or unknown. They profess not to know all things, neither do they affect to speak of all things. What they know to be true, and can make good by invincible arguments, that they publish and insert among their theorems. Of other things they are silent and pass no judgment at all, choosing rather to acknowledge their ignorance than affirm anything rashly. They affirm nothing among their arguments or assertions which is not most manifestly known and examined with the utmost rigour, rejecting all probable conjectures and little witticisms. They submit nothing to authority, indulge no affection, detest the subterfuges of words, and declare their sentiments, as in a court of judicature, without passion, without apology; knowing that their reasons, as Seneca testifies of them, are not brought to persuade, but to compel."

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[1] In consequence of ambiguity in the employment of the word *equal* and the symbol $=$, in making no distinction between *an identity* and an *equation*, Professor Challis has proposed the altered mark ⌶ of equality to denote an *identity;* the mark $\equiv$ has also been proposed to be employed for the same purpose. Either of these notations will serve the purpose of marking an identity; but there are other ambiguities in the usage of the word equal, which will not be removed by the employment of this symbol. On the whole it appears desirable to retain the mark $=$, and define or explain its exact meaning where it may be necessary, as different from the ordinary sense.

And in his inaugural speech before the University on his election to the Lucasian Professorship, he thus addressed the younger members of the University:—

"But if nothing else can compel you to apply yourselves vigorously to these disciplines, and labour diligently in this exercise, at least that spur of noble minds, the thirst of following all great examples, which always sinks deep into generous dispositions, should the more strongly provoke and inflame you. Especially since you cannot merit the name nor sustain the dignity of an university upon any other account, than by preserving to yourselves a knowledge more than common in all kinds of science becoming a generous mind; and by taking away every occasion of calumny from all who either envy your happiness or emulate your fame. Not spending your time like infants in learning the languages, nor neglecting the care of searching after truth for digging up foolish fables out of the rubbish of obsolete antiquity; not affecting the vain trappings of words, and the delusions of a painted speech, while the nature of things lies unregarded, and the use of plain reason is set aside; nor, lastly, abusing your leisure, playing away your time, and miserably wresting and torturing your wits with sophistical trifles and empty jargon, engaging yourselves in barren disputes, insisting upon uncertain conjectures, and venting of doubtful opinions.

"These reproaches you may easily wipe off, or entirely avoid, by only applying yourselves to the study of the mathematics with that diligence which is requisite. The mathematics, I say, which effectually exercises, not vainly deludes nor vexatiously torments studious minds with obscure subtleties, perplexed difficulties, or contentious disquisitions; which overcomes without opposition, triumphs without pomp, compels without force, and rules absolutely without the loss of liberty; which does not privately overreach a weak faith, but openly assaults an armed reason, obtains a total victory, and puts on inevitable chains; whose words are so many oracles, and works as many miracles; which blabs out nothing rashly, nor designs anything from the purpose, but plainly demonstrates and readily performs all things within its verge; which obtrudes no false shadows of science, but the very science itself, the mind firmly adhering to it as soon as possessed of it, and can never after desert it of its own accord, or be deprived of it by any force of others. Lastly, the mathematics which depend upon principles clear to the mind, and agreeable to experience; which draws certain conclusions, instructs by profitable rules, unfolds pleasant questions, and produces wonderful effects; which is the fruitful parent of, I had almost said, all arts, the unshaken foundation of sciences, and the plentiful fountain of advantage to human affairs."

ELEMENTARY ALGEBRA,
WITH BRIEF NOTICES OF ITS HISTORY.

SECTION IV.

ADDITION AND SUBTRACTION,
MULTIPLICATION AND DIVISION.

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ALGEBRA.

EXPLANATIONS AND DEFINITIONS.

Art. 1. As all our knowledge of the external world is subject more or less to the conditions of number and space, the sciences which treat of those subjects are of high importance.

The mathematical sciences may be considered as speculative or practical, and are conversant with all subjects which admit of increase or decrease. The speculative view regards the truth of the propositions, and the practical, the application of them to purposes of utility. The truth of the propositions depends on the definitions laid down as being intelligible, and the postulates and axioms assumed, as unquestionable. In the pure sciences of space and number no first principles are admitted, but such as are supported by their own strength and clear in their own light.

Arithmetic consists of methods for the performance of numerical calculations; Algebra is a generalisation and extension of Arithmetic.

The principles of Elementary Algebra have their origin in those of Elementary Arithmetic. In the latter, the symbols employed have each an intrinsic and local value, and the results of computation are definite; but in the former, general symbols are employed to denote quantities, unrestricted or limited in value; and operations and results only are indicated and not performed as in Arithmetic, but which may be performed in any case where numerical values are given to the general symbols.

The word *quantity* is assumed to mean both whatever can be made a subject of mathematical reasoning, and also the symbol which represents it, whether it be concrete or abstract. Two classes of symbols are employed, which may be described as "symbols of quantity" and "symbols of operation."

2. The letters of alphabets are assumed to represent numbers and the magnitudes of all quantities, whether concrete or abstract.

The early letters a, b, c, &c., are generally assumed to denote known quantities, and the final letters z, y, x, &c., unknown quantities. Besides these assumptions, other conventions are frequently employed, either to facilitate the solution of a problem, or to simplify the process of an investigation.

When a series of quantities is required to be expressed, instead of denoting them by the successive letters of the alphabet, they can be more conveniently denoted by the same letter repeated, with marks attached to it; as, for instance, instead of writing a, b, c, d, &c., a, a', a'', a''', &c., may be written; or still more conveniently a_0, a_1, a_2, a_3, &c.

The series of the natural numbers beginning from 0, as 1, 2, 3, 4, 5, 6 ..... n ..... may be supposed to be continued to any assignable number n, and the series may be further extended beyond that limit to any extent. The symbol ∞ is assumed to denote a number greater than can be assigned.

3. In the science of Arithmetic numbers are considered as simply absolute, whether abstract or concrete; whereas in the science of Algebra both numbers and the general symbols of quantity are considered to have, besides their assumed values, some specific relations of an opposite or contrary nature. This idea of contrariety was originally denoted by a mark written above the symbol, and signified that it was to be considered in some way, of a contrary nature to one which was not so marked.

The marks + and − prefixed* to numbers and symbols denoting magnitude, are assumed to indicate opposite qualities, so that the symbols so marked are no longer absolute quantities, but quantities having some specific relation of an opposite nature to others with which they may be connected.

This idea of contrariety or opposition will be different according to the nature of the quantities considered, whether they are concrete or abstract quantities.

If a denote the length of a straight line, and if $+a$ be assumed to denote a line of a units drawn in any direction; then $-a$ may denote a line of the same length drawn in a contrary direction.

If θ denote the magnitude of an angle generated by the motion of a moveable straight line round a point in a fixed straight line; and if $+\theta$ denote an angle traced out by the moving line in one direction in a plane from the fixed line, then $-\theta$ may denote an equal angle traced out by the line moving in a contrary direction.

If f denote the intensity of any force, and if $+f$ be assumed to indicate a force acting in any direction; then $-f$ may represent an equivalent force acting in an opposite direction.

In local motion, *progression* may be called affirmative, and *regression*, negative; and they may be denoted in the same manner; as if $+s$ denote a distance moved over in one direction, $-s$ may denote the same distance moved over in the contrary direction.

Periods of time before and after any fixed epoch may be indicated in the same manner; thus, if $+a$ be assumed to denote a number of years *after* any fixed epoch, $-a$ may consistently imply a period of the same number of years *before* that epoch.

If t denote any degrees of temperature, measured from some fixed point, as the freezing point. Under this supposition, if $+t$ denote t degrees above the freezing point; then $-t$ may denote the same number of degrees below that point.

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* The symbol +, by writers on Algebra in the Latin language, was named *plus*, *more*, and when it was placed before any number or symbol of quantity, was translated "increased by;" also the symbol − was called *minus*, *less*, and when prefixed to any quantity, signified "*lessened by*" or "*diminished by*."

The sign × was a substitute for "*ductum in*," *multiplied into*;" and ÷ was assumed for *applicatum ad*, and explained to mean "*divided by*."

In the regulation of mercantile transactions, the same method of illustration is manifestly possible; as, for instance, between the *creditor* and *debtor* side of an account, as if $+a$ denote the credits or sums possessed, then $-a$ may denote the debts or sums owed; also profits and losses can be denoted in the same manner, for if $+a$ denote profits, $-a$ may denote losses, the contrary of profits.*

Abstract numbers as well as concrete may also be considered as having a contrary affection to other numbers.† One abstract number can be combined with another abstract number, or taken from it; and in the former case it is *additive*, and in the latter *subtractive;* and the marks + and − may respectively indicate these operations. And this contrariety may be signified by the same marks as are prefixed to concrete quantities; as, if $+a$ denote a number to be added, $-a$ may consistently denote a number to be subtracted, or one number having a contrary affection to another.

It may be remarked that sometimes the affirmative sign is omitted, when no ambiguity is likely to arise in the meaning of the number or quantity to which it should rightly be prefixed.

The coefficient of a quantity is the number or symbol prefixed to it, and indicates what multiple or what part of the quantity is signified. Numerical coefficients may be supposed to arise from the successive additions of the same quantity; thus $a+a$ is assumed to be denoted by $2a$, $a+a+a$ by $3a$, $a+a+a+a$ by $4a$, and generally $a+a+a+a+\ldots$ to m terms by ma, where 2, 3, 4, m are the coefficients, and m stands for any assignable number. A quantity without any coefficient prefixed to it, is supposed to have 1 or unity, as a is the same as $1a$.

4. The operation of multiplication is denoted by the sign ×; and the product of two or more quantities is assumed to be expressed by connecting the symbols which denote them by the sign × as, $a \times b$ denotes the product of the quantities a and b.

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* The simple fact, that *opposite relations* with respect to concrete quantities can be conceived to exist, and do exist, suggests the necessity of exhibiting these relations in a science where the symbols denote real quantities. These invariable relations may be denoted by the marks + and −, and may fairly be considered as a consequence arising from relations which exist in nature.

† The mark of contrariety or opposition was originally employed as an adjective, and the conversion of the use of words so employed into the sense of an active process, is among the most familiar contrivances of all systematic languages. Number and quantity in Algebra are never made a subject of reasoning but under such a supposition; and hence, the signs or marks which indicate opposite qualities, may also be made to indicate active operations. As instances in English, the adjectives black, white, hard, soft, give the verbs, to make black, white, hard, soft respectively, or to blacken, to whiten, to harden, to soften.

The sign × is sometimes omitted, and a point is placed instead, at the lower part between the factors, as $a.b$; but generally when no ambiguities are likely to arise, both the sign and the point are omitted, and the factors are written together. Thus, ab denotes the product of a and b, and abc that of a, b, and c.

The terms which form the product of two or more quantities are called the *factors* of that product; as a and b are the factors of the product ab, and a, b, c are the factors of the product abc; also a and bc, b and ac, or c and ab, may be considered the factors of the product abc.

When a product consists of several factors, it is generally indifferent in what order the factors are written, as far as the result is concerned; as ba is the same as ab, and acb, bac, are each the same as abc.

The operation of the division of one quantity by another is denoted by placing the dividend above the divisor with a line between them, as $\frac{a}{b}$, denotes that the quantity a is to be divided by the quantity b.

The mark ÷ also is employed for the same purpose, as $a \div b$.

A multiple of any quantity is some exact number of times that quantity, as $12a$ is a multiple of $3a$; and a submultiple of any quantity is any part of that quantity, as $3a$ is a submultiple of $12a$. When one quantity is exactly divisible by another, the latter is called a divisor or a measure of the former.

5. The product which is obtained by multiplying together several quantities each equal to one another, is called a power of that quantity, and the number of equal factors in the product constitutes the degree or order of the power.

The products of two, three, four, &c., equal quantities, as aa, aaa, $aaaa$, &c., are respectively named the second, third, fourth, &c., powers of the quantity a.

The power of any quantity is assumed to be denoted by placing a small figure at the upper part on the right of the quantity.

This figure is called the *index* or *exponent* of the power, and points out the number of equal factors that are multiplied together to produce that power; thus, aa is denoted by a^2, aaa by a^3, and $aaaa$ by a^4, &c.; and a^2, a^3, a^4, &c., represent respectively the second, third, fourth, &c., powers of the quantity a, which are said to be of two, three, four, &c., dimensions.

The first power of any quantity may be written with unity for its index, as a^1; but this index is generally omitted, except in cases where ambiguity or other inconvenience might occur.

The second power of any quantity is also called the square of that quantity, and the third power is also called the cube. These terms bear an analogy to the square and cube in geometry, one consisting of two and the other of three dimensions; the straight line being considered to be of one dimension.

The term *dimension* is used in an extended sense to denote the number of simple factors in any product, not considering the coefficients, whether numeral or literal, and the sum of the indices of the factors indicates the dimensions of any term, as a^2, ab, $3b^2$, are each of two dimensions; and a^2b^2, ab^3, $abcd$, are each of four dimensions. When an expression consists of several terms, if each term is of the same dimensions, the expression is said to be homogeneous.

The original quantity from which a *power* is produced is called a *root* of that *power*, and in the reverse operation, the degree or order of the root depends upon the number of equal factors by which the power was produced; as a^2 is called the second power or the square of a, so a is the second root or the square root of a^2; and in a similar manner, a^3 is the third power or the cube of a, and a is the third root or the cube root of a^3.

A square number or quantity is one which has an exact square root; and a cubic number or quantity is one which has an exact cube root; and similarly of all higher powers and roots.

The second root, or square root of any quantity is denoted by prefixing the mark $\surd$, (the initial letter of the word *radix* somewhat altered) as $\sqrt{a}$ denotes the square root of a. Also $a^{\frac{1}{2}}$ denotes the square root of a. Likewise $\sqrt[3]{a}$, $\sqrt[4]{a}$, and $a^{\frac{1}{3}}$, $a^{\frac{1}{4}}$, denote respectively the third and fourth roots of a; and similarly for the higher roots.

Quantities are considered as rational or irrational. The word irrational or surd (*surdus*) is applied to numbers and quantities whose roots cannot be exactly determined, as the square root of 5 and the cube root of a, which are denoted by $\sqrt{5}$ and $\sqrt[3]{a}$ respectively, but whose exact values cannot be exactly determined.

6. Simple quantities are those which consist of one term only. Compound quantities are those which consist of more terms than one, connected by the signs $+$ and $-$.

A binomial quantity consists of two terms, as $a+b$; a trinomial of three terms, as $a+b-c$; a quadrinomial of four terms, as $a+b+c-d$, and so on; and a polynomial or multinomial of many terms.

Two or more monomials are said to be *like* or *unlike*, according as they consist or do not consist of the like symbols or combination of symbols, as $3a$ and $5a$ are like quantities; but $3a$ and $5n$ are unlike quantities, the coefficients not being considered.

7. There are several marks and characters employed in mathematical reasonings.

When two or more quantities are to be considered as one quantity, a straight line, called a vinculum, is drawn above them, or they may be included within parentheses or brackets; as when $a+b-c$ is to be considered a single quantity in any process, it is written thus, $\overline{a+b-c}$, $(a+b-c)$, or $\{a+c-b\}$.

The mark $\therefore$ is employed for the word "therefore" or "consequently," and indicates the conclusion of an argument or a process of reasoning. And $\because$, or $\therefore$ inverted, is used for "because," and points out something assumed or granted as a ground of argument.

The symbol $>$ placed between two quantities is assumed for the words "is greater than," as $5 > 3$ means that the number 5 is greater than 3; and $x>y$ means that the quantity denoted by x is greater than that denoted by y. In the same manner $<$ is used for the words "is less than," as $3<5$ means that 3 is less than 5, and $y<x$, that y is less than x.*

The mark $=$ is the sign of equality, and is used for the words "is equal to," as $2+3=5$ means that the sum of 2 and 3 is equal to 5; and $x=a+b$ denotes that the quantity x is equal to the sum of the quantities a and b.

Here, however, it may be remarked that in reasonings on the generalised forms of number, this mark will be found to bear other meanings besides that of the simple equality of two numbers.

Since this sign was first employed to denote numerical equality, in the subsequent improvements of symbolical language it is employed to express the identity of meaning of different conventional notations; as also the equivalence of mathematical operations signified in involved expressions, and the results produced by the performance of the processes indicated, without any regard whether the results admit of any contingent or necessary Arithmetical equality.

When two expressions are connected by the sign of equality, the

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* These marks for "greater than" and "less than" have not the same meaning in the sciences of Arithmetic and Algebra. In Arithmetic numbers are considered absolutely, and one number is greater than, or less than another, according as the former contains a greater or less number of units than the latter. But the Algebraical meaning of these terms "greater than" and "less than" will be different when the numbers are considered as having contrary affections.

As the science of Algebra admits both the series of natural numbers and general symbols of numbers including the character ∞; the series of positive numbers may be considered as commencing from 0, each *increasing* by the successive addition of unity; and the series of negative numbers may be also considered as commencing from 0, each *decreasing* by the successive subtraction of unity, so that these two series can be united as one continued series of Algebraic numbers, each number successively increasing by unity, if reckoned from the left towards the right, and decreasing successively by unity, if reckoned from the right towards the left, thus :—

$$\cdot -\infty \ldots -x, \ldots -4, -3, -2, -1, 0, +1, +2, +3, +4, \ldots +x \ldots +\infty.$$

Hence it is obvious that any number in the series is always *greater than* any other which stands in any place of the series to the *left* of it; and *less than* any other which stands in any place to the *right* of it: as $+6>+2$, and $+2<+6$, which are also Arithmetically true. It also appears that $+1>0$ and $-1<0$; also $-2>-5$ and $-5<-2$, which apparent anomalies arise from the *Algebraical meanings* of the terms "*greater than*" and "*less than*."

two expressions so connected constitute an equation which may involve both known and unknown quantities.

The equation is called an identical equation or an identity when the two expressions are always equal, whatever numerical values are assigned to the symbols employed, as $(a+b)+(a-b)=2a$, denotes generally, that the difference of two numbers added to their sum is equal to twice the greater number: and $(a+b)\times(a-b)=a^2-b^2$, also denotes, that the product of the sum and difference of any two numbers is equal to the difference of their squares.*

8. The names of the elementary processes of Arithmetic and Algebra are the same; Addition and Subtraction, Multiplication and Division, Involution and Evolution; Subtraction being the reverse of Addition, Division of Multiplication, and Evolution of Involution. There is, however, this distinction to be observed in Algebra: the processes are indicated in general terms, and numerical results can be obtained when numbers are substituted for the general symbols. All the elementary processes of Algebra are performed with general symbols in the same manner as the ordinary processes of abstract numbers in Arithmetic, no distinction being made between the symbols of abstract and concrete quantities. In the final result of the processes only is this distinction required to be considered.

The propositions of pure Mathematics admit of a species of proof applicable only to the abstractions of number and space. The term *demonstration* implies an appeal to the sense of sight, and is, in fact, the showing of the truth of a proposition; and the meaning of the word *proof* has been aptly described as "testing by trial."† The process consists of a connected series of syllogisms, or rather enthymemes, and the final result is a single inference out of a chain of necessities. That single result must be true if the others be true; those others are, therefore, proved to be true by tracing in order each step until we arrive at the last step, which completes the demonstration.

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* In Professor Challis's "Notes on the Principles of Pure and Applied Calculation," p. xii., he has proposed the mark $\vdash\!\dashv$ to signify that the two sides of an equality are identical for all values whatever of the general symbols. The mark $\equiv$ has also been proposed to be employed for the same purpose.

† The final term of all proof is an appeal to the senses. Every proposition which relates in any way to the objects of sense must seek its final proof in this conviction. Whatever may be the imperfection of our senses, this is the law of our nature. If the senses do not produce a certain conviction to the mind of the existence of matter, not even geometry has any foundation beyond the region of fancy. It may also be remarked that in all past ages the objects of material nature have uniformly produced the same general impressions upon the senses (when in a healthy state) and have created the same ideas in the mind; and hence, in respect of our bodily and mental constitution, we must either take the impressions we so receive as true, or must cease to reason altogether.

Sometimes an inquiry is directed to the investigation of relations between quantities, whether they are equal or unequal; and in the latter case, what is the exact difference, or what are the limits of the inequality. Also the inquiry may relate to the forms of expressions respecting identity or equivalence.

The results of all processes of reasoning must be in strict accordance with the first principles. *If a direct mathematical process* be possible, *the reverse process* may not be always possible. In cases of this kind, results may arise involving some ambiguity, contradiction, or impossibility, which the definitions and elementary processes did not anticipate. When such cases occur, it will be necessary to extend the meaning of the definitions, or to devise new ones, but such as shall not be inconsistent with, nor in contradiction to, the principles and processes already adopted.

The human mind is so constituted as to admit certain general propositions as true as soon as they are understood. These elementary truths, or axioms as they are commonly called, constitute the bases of all proof and demonstration.

AXIOMS.

1. If the same quantity or equal quantities be added to equal quantities, the sums are equal.

2. If the same quantity or equal quantities be taken from equal quantities, the remainders are equal.

3. If the same quantity be added to and subtracted from another, the value of the latter is unaltered.

4. If the same quantity or equal quantities be added to unequals, the sums are unequal.

5. If the same quantity or equal quantities be taken from unequals, the remainders are unequal.

6. If equal quantities be multiplied by the same quantity, or by equal quantities, the products are equal.

7. If equal quantities be divided by the same quantity, or by equal quantities, the quotients are equal.

8. If any quantity be both multiplied and divided by the same quantity, the value of that quantity is not altered.

9. If unequals be multiplied by the same quantity, or by equal quantities, the products are unequal.

10. If unequals be divided by the same quantity, or by equal quantities, the quotients are unequal.

11. The same power or equal powers of equal quantities are equal.

12. The same root or equal roots of equal quantities are equal.

13. If any quantity be raised to any power and the same root be extracted, the quantity is not altered.

ADDITION AND SUBTRACTION.

Art. 9. By Addition is meant the process of connecting together two or more quantities into one sum or aggregate; and by Subtraction is meant the process of taking one quantity from another.

Addition and Subtraction are processes, one the reverse of the other; as, if 2 be added to 3, the sum is 5, and if 2 be taken from 5, the remainder is 3. Also, if b be added to a, the sum is $a+b$, and if b be taken from $a+b$, the remainder is a. Also, if b be taken from a, the difference can only be indicated as $a-b$, but not performed so long as the values of a and b are unassigned.

Prop. *To explain the processes of Algebraical Addition and Subtraction.*

The operations of Addition and Subtraction simply consist *in putting to* and *taking away* one thing from another, and the assumed meaning of the symbols employed suggest these operations, and the rules for the performance of them follow directly from the assumed meaning of the symbols. As all quantities are considered to have contrary affections of some kind, as positive or negative, additive or subtractive, it will be at once obvious that the addition of one quantity to another does not always *produce* increase, nor that the subtraction of one quantity from another always imply *decrease*.

Since every quantity is affected with the positive or negative sign, or the additive or subtractive sign, as they are also called, it is evident that the addition of one quantity to another will be denoted by connecting any quantity to be added with *its proper sign* to the other quantity, as the addition of $+b$ to $+a$ is denoted by $+a+(+b)$ or $a+b$, and the addition of $-b$ to $+a$ by $+a+(-b)$ or $a-b$.

As the operation of Subtraction is the reverse of that of Addition; the subtraction of one quantity from another will be denoted by connecting the quantity to be subtracted with *its contrary sign* to the other quantity; as the subtraction of $+b$ from $+a$ is denoted by $a-(+b)$ or $a-b$, and the subtraction of $-b$ from $+a$ is denoted by $a-(-b)$ or $a+b$.

Hence it appears that $+a+(+b)$ and $+a-(-b)$ are identical expressions, each being equivalent to $+a+b$, so that $a+(+b)=a+b$ and $a-(-b)=a+b$; and the sum $a+b$ may be considered to result either from the addition of $+b$ to $+a$ or from the subtraction of $-b$ from $+a$: that is, the *addition* of a positive quantity is equivalent to the *subtraction* of a negative quantity.

Also, that $a+(-b)$ and $a-(+b)$ are identical expressions, each being equivalent to $a-b$, so that $a+(-b)=a-b$ and $a-(+b)=a-b$, and the difference $a-b$ may be considered to result from the addition of $-b$ to $+a$, or from the subtraction of $+b$ from $+a$, or that the *addition* of a negative quantity is equivalent to the *subtraction* of a positive quantity.

In the performance of the operations of addition and subtraction, the sums and differences of *like* quantities can be incorporated into single terms by the addition and subtraction of their coefficients. But the sums and differences of *unlike* quantities can only be expressed by connecting the several terms by the signs with which they are individually affected. As the sum of $3a+4b$ and $4a-3b$ is thus expressed: $(3a+4b)+(4a-3b)=3a+4b+4a-3b=7a+b$; and the difference arising from $4a-3b$ taken from $3a+4b$ is $(3a+4b)-(4a-3b)=3a+4b-4a+3b=-a+7b$.

When quantities with literal coefficients are to be added or subtracted, the aggregate of the coefficients connected with their proper signs under a vinculum or between brackets, will denote the coefficient of the common quantity, as the aggregate of $ma+na-pa$ is denoted by $(m+n-p)a$, and $-ma+na-pa$ is equivalent to $-(m-n+p)a$.

As in Arithmetic, so in Algebra, only concrete quantities of the same kind can be added to, or subtracted from, one another; but abstract quantities are not subject to this restriction.*

When several quantities, like and unlike, with contrary signs, are to be added together, it will be found convenient to arrange like quantities with their proper signs under each other, as numbers are arranged for addition in ordinary arithmetic, with this difference in Algebra—the process of aggregation may begin either with the right-hand or left-hand column.

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* The following examples of addition and subtraction express two important theorems :—

If a and b denote any two numbers whatever, then $a+b$ will denote their sum and $a-b$ their difference.

And $(a+b)+(a-b)=a+b+a-b=2a$, which expressed in words is equivalent to this theorem :—

If the difference of any two numbers be added to their sum, the result is equal to twice the greater number.

Also $(a+b)-(a-b)=a+b-a+b=2b$, denotes the theorem :—

If the difference of any two numbers be subtracted from their sum, the result is equal to twice the less number.

Also, if the expressions $(a+b)+(a-b)=2a$ and $(a+b)-(a-b)=2b$, be each divided by 2, there results

$$\frac{a+b}{2}+\frac{a-b}{2}=a, \text{ and } \frac{a+b}{2}-\frac{b-a}{2}=b;$$

The former denotes the theorem :—Half the difference of any two numbers added to half their sum, is equal to the greater number. And the latter :—Half the difference of any two numbers subtracted from half the sum, is equal to the less number.

EXERCISES.

I.

1. Write a brief account of the most ancient treatise on Algebra known to be extant.

2. When was the science of Algebra first cultivated by the Arabians? What is the meaning of the word Algebra?

3. Give a brief account of the extant writings on Algebra in the Arabic language. What subjects do they embrace?

4. When did Diophantus flourish? Give some account of his Arithmetics.

5. When did the science of Algebra become known in Italy and in other countries of Europe?

6. Give the names of the writers of modern times by whom the science of Algebra has been improved and extended.

7. What marks or symbols were first devised and assumed for denoting the operations of addition and subtraction; also for denoting the contrariety of affection of quantities? Was the employment of these symbols contemporaneous, or did their use for one purpose precede that of the other by any interval of time?

8. What improvements were introduced into the language of Algebra during the seventeenth century?

II.

1. Explain the meanings of the signs $+$, $-$, and $=$; and exemplify how they are employed.

2. What is the distinction between Geometrical and Arithmetical equality? Express in words $a-b=0$ and $\frac{a}{b}=1$; and show that these are equivalent tests of the absolute equality of the two finite quantities a and b.

3. Add $+5a$ to $+7a$, and from $+12a$ subtract $+5a$.

4. Add $-5a$ to $-7a$, and from $-12a$ subtract $-5a$.

5. Add $-5a$ to $+7a$, and from $+2a$ subtract $-5a$.

6. Add $a-b$ to $a+b$, and subtract $a-b$ from $a+b$.

7. Show that $a-(b-c)=a-b+c$, a being greater than $b-c$, and b greater than c; also a being less than $b-c$, and b less than c.

8. Explain how $+a+(+b)$ and $+a-(-b)$ are each equivalent to $+a+b$.

9. Find the value of $(3+4+5)-(3+4-5)+(3+5-4)-(4+5-3)$.

10. Find the aggregate of $+(m-n+p)$, $+(m+n+p)$, $-(m-n-p)$, and $+(m-p+n)$ in the simplest form.

11. What is the sum and difference of $ma+na$ and $ma-na$?

12. Explain how (1) $ma+na=(m+n)a$, (2) $-ma-na=-(m+n)a$, (3) $ma-na=(m-n)a$, and (4) $-ma+na=-(m-n)a$.

III.

Simplify the following expressions:—

1. $a+(a+b)+(a+b+c)$.
2. $a-(a-b)-(a-b+c)$.
3. $a+\{(a+b)-(a+b+c)\}$.
4. $a-\{(a-b)-(a-b+c)\}$.
5. $a-\{a-b-(a-b-c)\}$.
6. $a-(a-b)+(a-b-c)-(a-b-c-d)$.
7. $a-\{(a-b)+(a-b-c)-(a-b-c-d)\}$.
8. $2(a+b)-3(c-d)-(a+b)+4(c-d)$.
9. $2(a+b)-3\{c-d-(a+b)+4(c-d\}$.
10. $\{a+b-(c+d)\}-\{b+a-(d-c)\}-\{a-b+(c-d)\}$ $+\{d-b-(a-c)\}$.
11. $3(a-b+c)-5(a-2b+3c)+4(a-3b+2c)-2(a-7b-2c)$.
12. $12a-[\{3b-(2b-c)\}-4c+\{2a-\overline{3b-c-2b}\}]$.

IV.

Reduce each of the following expressions to the simplest form:—

1. $(ax+by)+(x+y)+(a-1)x-(b+1)y$.
2. $3(a+b-c)+7(b+c-a)+9(a+c-b)$.
3. $(a+x)-\{b+y-(a-x-\overline{b-2y})\}$.
4. $(c+b)x+(b+c)y-\{(a-b)x-(b-c)y\}$.
5. $\{(a+b)x+(a+c)y\}+\{(b-c)x+(b-a)y\}+\{(c-a)x+(c-b)y\}$.
6. $\{(a-b)x+(b-c)y+(c-a)z\}-\{a(x-y)+b(y-z)+c(z-x)\}$ $-\{ax+by+cz\}$.

V.

Simplify the following expressions:—

1. $3x-7y-(2x-4y)-(-7x+2y)$.
2. $x-\{2y-(2z-3x-\overline{y-x})\}-\{2x-3(y-x)\}$.
3. $-\{x-(y-z)\}-\{y-(z-x)\}-\{z-(x-y)\}$.
4. $2x-3y-[(5x+4y)+3x+\{y-9x-(2y-x)+(x-y)\}]$.
5. $44x+\{48y-(6z+3y-7z)+4z\}-\{48y-8x+2z-(4x+y)\}$.
6. $2(x+y+z)+3(x+y-z)+5(x+z-y)+7(z+y-x)$.

VI.

1. If a, b, c be three numbers, a being greater than b and b greater than c, which of the following differences is positive, and which negative, $a-b$, $a-c$, $b-a$, $b-c$, $c-a$, $c-b$?

2. Distinguish *like* and *unlike* quantities, and give examples.

3. Express the aggregate of $a-(b-c)-\{b-(c-a)\}+\{c-(b-a\}$, first as a positive quantity, and secondly as a negative quantity.

4. Write down all the sums and differences which can be formed with $\pm a$, $\pm b$, when taken two at a time.

5. Find all the aggregates which can be formed from $\pm a$, $\pm b$, and $\pm c$, when taken three at a time, but so that the same symbol shall not occur more than once in each expression.

6. Express in symbols the quotient arising from the product of two quantities divided by the product of their sum and difference.

7. Express $x^4 - 2a(a-b)x^2 + (a^2+b^2)(a-b)x - a^2b^2$ with numerical coefficients when $a = 4$ and $b = 8$.

8. Simplify $\{x^3+y^3-(3x^2y+3xy^2)\}-\{(x^3-3x^2y)-(3xy^2-y^3)\}$.

9. Arrange $a^2(x+y-z)-ab(z+x-y)+b^2(z+y-x)$ so that the coefficients of x, y, z may consist of a^2, ab, b^2.

10. Arrange $(x+y+z)a_1+(x+y-z)a_2+(x+z-y)a_3+(y+z-x)a_4$ in three terms involving x, y, z respectively, with coefficients consisting of a_1, a_2, a_3, a_4.

11. Why are the same signs + and − used to denote operations and also signs of affection? Can we interpret the meaning of such signs of affection independently of a knowledge of the specific nature of the magnitudes represented by the symbols to which they are attached?

VII.

1. Find the sum of $a+b+c$, $a+b-c$, $a+c-b$, $b+c-a$, and its numerical value when $3a = 6b = 12c = 5$.

2. If $x = a+2b+3c$, $y = b+2c+3a$, and $z = c+2a+3b$, show that $x+y+z = 6(a+b+c)$.

3. Simplify the expression $16x-10-[7-\{8x-(9x-\overline{3-6x})\}]$, and find the least integral value of x that will render the result negative.

4. Find the sum of $5x+6y+4z$ and $2x-y+2z$, and the difference between $y-4z-3x$ and $z-7x+3y$; express the sum of the results in the simplest form, and determine the value when $x = 10$, $y = 100$, and $z = 1000$.

5. Find the sum of $3(x-2y+3z)$, $4(y-2z+3x)$ and $5(z-2x+3y)$, and its value in terms of x, when $x = 10y$ and $y = 100z$.

6. Determine the sum of $5x+3y+z$, $3z+5y+x$, and $y+3x+5z$, and its value when $100x = 10y = z = 1$.

7. Show that the value of $x^5-x^4+x^3-x^2+x+1$ is $\frac{1}{32}$ when $x = -\frac{1}{2}$.

8. If $a = \cdot 02$, $b = \cdot 04$, $c = \cdot 08$, find the values of $a^3+b^3+c^3 \pm 3\,abc$.

9. Add together $4x^3+3x^2y-y^3$, $4x^2y-3x^3$, $7xy^2+9y^3-2x^2y$, and find what must be subtracted from the sum to leave the remainder $2x^3-3x^2y+y^3$.

10. Take $ax^5+bx^4+cx^3+dx^2+ex+1$
from $px^6-qx^5+rx^4-sx^3+tx^2-vx+1$.

11. If $+6 > +2$, and $+2 < +6$, how is it explained that $-6 < -2$, and $-2 > -6$?

MULTIPLICATION AND DIVISION.

Art. 10. The product of any two quantities a and b has been assumed to be denoted by $a \times b$, $a.b$, and ab; and the product ab may be considered to be produced by a multiplied by b, or by b multiplied by a.

The quotient of one quantity a, divided by another b, has been assumed to be denoted by $\frac{a}{b}$ or $a \div b$.

The processes of Multiplication and Division are, one the reverse of the other; as the product of 2 multiplied by 3 is 6, and the quotient of 6 if divided by 3 is 2, but if by 2 the quotient is 3. In the same manner, the product a multiplied by b is ab, and the quotient of ab if divided by b is a, but if divided by a is b.

When a product consists of two simple factors, both of them may be positive or both negative, or one of them may be positive and the other negative.*

First. If the signs of the factors are both positive, the product $+a \times +b$ implies that $+a$ is to be added as many times as there are units in $+b$; and since the sum of any number of equal positive quantities is positive, it follows that the product $+a \times +b$ is $+ab$, or $+a \times +b = +ab$: that is, the product of two positive quantities is positive.

Secondly. If one factor is negative and the other positive, the product $-a \times +b$ denotes that the negative quantity $-a$ is to be added as many times as there are units in $+b$; and since the sum of any number of equal negative quantities is negative, it follows that the product $-a \times +b$ is $-ab$, or $-a \times +b = -ab$.

The product of $+a$ multiplied by $-b$ will produce the same result, as the multiplicand may be made the multiplier, and the multiplier the multiplicand, and then $-b \times +a$ denotes that $-b$ is to be added a times, and the result is the same as in the former case; that is, the product of a positive and negative quantity is negative.

Thirdly. If both the factors are negative. When a negative quantity $-a$ is to be multiplied by a negative quantity $-b$, the product $-a \times -b$ denotes that the negative quantity is to be subtracted as many times as there are units in $-b$; but the subtraction of a negative quantity is the same process as the addition of a positive quantity. It follows that the product $-a \times -b$ is $+ab$, or $-a \times -b = +ab$, that is, the product of two negative quantities is negative.

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* These explanations may perhaps become more intelligible by considering particular numbers instead of general symbols of number.

1. $+4 \times +3 = 4+4+4$ or 4 added 3 times, so that $+4 \times +3 = +12$.

2. $-4 \times +3 = (-4)+(-4)+(-4)$ or -4 added 3 times, that is $-4 \times 3 = -12$; also $-3 \times +4 = (-3)+(-3)+(-3)+(-3)$ or -3 added 4 times.

3. $-4 \times -3 = -(-4)-(-4)-(-4)$ or -4 subtracted 3 times,
or $-4 \times -3 = -(-4-4-4) = -(-12) = +12$.

Hence, generally, the product of two quantities having like signs is positive, and having unlike signs is negative.

11. In the division of one quantity by another, both quantities may be positive or negative, or one of them may be positive and the other negative. As the process of division is the reverse of multiplication; in the process of division the product becomes the dividend, and if the multiplicand be made the divisor, the multiplier is the quotient; but if the multiplier be made the divisor, the multiplicand is the quotient.

In dividing one quantity by another, the sign of the quotient may be inferred from the signs of the dividend and divisor.

First. If the dividend and divisor are both positive.

Since $+a \times +b = +ab$, then $\frac{+ab}{+a} = +b$, the quotient; or, if a positive quantity is divided by a positive quantity, the quotient is positive.

Secondly. If the dividend and divisor are both negative.

Since $-a \times +b = -b$, then $\frac{-ab}{-a} = +b$, the quotient; or, when a negative quantity is divided by a negative quantity, the quotient is positive.

Thirdly. If the dividend is negative and the divisor is positive. Since $+a \times -b = -ab$, then $\frac{-ab}{+a} = -b$, the quotient; or, if a negative quantity is divided by a positive quantity, the quotient is negative.

Also $-a \times -b = +ab$, then $\frac{+ab}{-a} = -b$, the quotient; or, if a positive quantity be divided by a negative quantity, the quotient is negative.

Hence, generally, when the dividend and divisor have like signs, the sign of the quotient is positive; but when they have unlike signs, the sign of the quotient is negative.

12. The product of two or more quantities has been assumed to be denoted by writing the factors together with the coefficient prefixed, and its proper sign, as determined by the signs of the factors.

But the product of two or more *equal* quantities, called a power, is denoted by placing at the right of one of the factors a small figure as an index or exponent, showing how many equal factors compose the product.

The index of the product of two or more powers of the same quantity will be denoted by the sum of the indexes of the factors; as the product of a^2 and a^3 is a^{2+3} or a^5, as will appear from the assumed notation of powers.

For $a^2 = a.a$, and $a^3 = a.a.a$, $\therefore$ $a^2 \times a^3 = (a.a) \times (a.a.a) = a.a.a.a.a = a^5$.

And the index of the quotient arising from the division of two powers of the same quantity, will be denoted by the difference arising from subtracting the index of the divisor from the index of the dividend; as the quotient arising from the division of a^5 by a^2 will be a^3.

For $a^5 = a.a.a.a.a$, and $a^2 = a.a$, $\therefore \dfrac{a^5}{a^2} = \dfrac{a.a.a.a.a}{a.a} = a.a.a$ or a^3.

And these consequences may be shown to hold true generally when the indices are denoted by any numbers m and n whatever.

For $a^m = a.a.a.a \ldots$ to m factors, and $a^n = a.a.a \ldots$ to n factors,

$$\begin{aligned} \therefore\ a^m \times a^n &= (a.a.a.a \ldots \text{ to } m \text{ factors}) \times (a.a.a \ldots \text{ to } n \text{ factors}) \\ &= a.a.a.a.a \ldots \text{ to } (m+n) \text{ factors} \\ &= a^{m+n}, \text{ by the assumed notation;} \end{aligned}$$

or, the product of any powers of the same quantity is denoted by a power the index of which is equal to sum of the indices of the factors.

$$\begin{aligned} \text{Also } \frac{a^m}{a^n} &= \frac{a.a.a.a \ldots \text{ to } m \text{ factors,}}{a.a.a \ldots \text{ to } n \text{ factors}} \\ &= a.a.a \ldots \text{ to } (m-n) \text{ factors, supposing } m > n. \\ &= a^{m-n}, \text{ by the assumed notation;} \end{aligned}$$

or, the quotient of any powers of the same quantity is denoted by a power the index of which is equal the difference arising from subtracting the index of the divisor from the index of the dividend.

If $n = m$, then $\dfrac{a^m}{a^n} = \dfrac{a^m}{a^m} = 1$, and $a^{m-m} = a^0$, $\therefore a^0 = 1$.

Hence, every quantity having zero for its index is equal to unity, and a^0 may be considered as the algebraical symbol for 1, as it arises from the division of equal powers of the same quantity.

If $m < n$, since $\dfrac{a^m}{a^n} = a^{m-n}$; dividing these equals by a^m,

$$\therefore \frac{1}{a^n} = \frac{a^{m-n}}{a^m} = a^{m-n-m} = a^{-n}, \text{ or } a^{-n} = \frac{1}{a^n},$$

or any power of a quantity with a negative index is equivalent to the reciprocal of the same power with its index positive.

Also, since $a^{-n} = \dfrac{1}{a^n}$, $\therefore a^{-n}.\,a^n = 1$, and $a^n = \dfrac{1}{a^{-n}}$,

or any power of a quantity with a positive index may be expressed by the reciprocal of the same power with its index negative.

Hence, when a power forms a factor in the numerator or denominator of a fraction, this factor may be removed from the numerator to the denominator, or from the denominator to the numerator, by changing the sign of the index.

Prop. *To explain the process of Algebraical Multiplication.*

In performing the process of multiplication the product of two or more monomials is denoted by prefixing the proper sign to the product of the coefficients and the indicated product of the quantities; as the product of $3a^2bx^3$ and $-5a^3x$ is $-15a^5bx^4$.

When the multiplicand is a compound quantity and the multiplier a monomial; the product is found by multiplying each term of the multiplicand by the multiplier, and connecting the terms of the product by their proper signs; as the product of $5a^2-4ax-3x^2$ and $-2ax$ is $-10a^3x+8a^2x^2+6ax^3$.

When both the multiplicand and multiplier are compound quantities; the product is found by multiplying every term of the multiplicand by each term of the multiplier, and the sum or aggregate of the partial products will be the product of the two compound quantities.*

As examples, multiply $a+b$ by $a+b$, $a-b$ by $a-b$, and $a+b$ by $a-b$.

$$\begin{array}{lll}
\begin{array}{l} a+b \\ a+b \\ \hline a^2+ab \\ \quad +ab+b^2 \\ \hline a^2+2ab+b^2 \\ \hline \end{array}
&
\begin{array}{l} a-b \\ a-b \\ \hline a^2-ab \\ \quad -ab+b^2 \\ \hline a^2-2ab+b^2 \\ \hline \end{array}
&
\begin{array}{l} a+b \\ a-b \\ \hline a^2+ab \\ \quad -ab-b^2 \\ \hline \qquad a^2-b^2 \\ \hline \end{array}
\end{array}$$

These products, with their factors, express the three following identical equations :—

$$(a+b)\times(a+b) \text{ or } (a+b)^2=a^2+2ab+b^2, \qquad (1)$$
$$(a-b)\times(a-b) \text{ or } (a-b)^2=a^2-2ab+b^2, \qquad (2)$$
$$(a+b)\times(a-b) \qquad =a^2-b^2, \qquad (3)$$

and denote in symbols three important numerical theorems :

(1) The square of the sum of any two numbers, is equal to the sum

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* In the product of two compound quantities both of which increase or decrease regularly by ascending or descending powers of the same symbol; the operation may be performed by omitting the powers of the symbol, and writing the coefficients only of the terms with their proper signs, and placing 0 for the coefficient of any term which may be wanting.

Let x^2+2x-3 be multiplied by x^2-3x-2.

$$\begin{array}{ll}
\begin{array}{l} x^2+2x-3 \\ x^2-3x-2 \\ \hline x^4+2x^3-3x^2 \\ \quad -3x^3-6x^2+9x \\ \qquad -2x^2-4x+6 \\ \hline x^4-x^3-11x^2+5x+6 \\ \hline \end{array}
&
\begin{array}{l} 1+2-3 \\ 1-3-2 \\ \hline 1+2-3 \\ \quad -3-6+9 \\ \qquad -2-4+6 \\ \hline 1-1-11+5+6 \\ \hline \end{array}
\end{array}$$

If it be required to multiply x^2+2x-3 by x^2-2, where the second term of the multiplier is wanting; 0 must be placed as the second coefficient of the multiplier, and the process will stand thus—

$$\begin{array}{ll}
\begin{array}{l} x^2+2x-3 \\ x^2-2 \\ \hline x^4+2x^3-3x^2 \\ \qquad -2x^2-4x+6 \\ \hline x^4+2x^3-5x^2-4x+6 \\ \hline \end{array}
&
\begin{array}{l} 1+2-3 \\ 1+0-2 \\ \hline 1+2-3 \\ \qquad -2-4+6 \\ \hline 1+2-5-4+6 \\ \hline \end{array}
\end{array}$$

The reverse process of division may also be performed in the same manner by omitting the powers of the symbol and writing only the coefficients of the terms in order with their proper signs.

of the squares of the two numbers together with twice their product. (See Euc. ii. 4.)

(2) The square of the difference of any two numbers, is equal to the difference between the sum of their squares and twice their product.

(3) The product of the sum and difference of two numbers, is equal to the difference of the squares of the two numbers. (See Euc. ii. 5. Cor.)

And the first and second have this connection:

$(a+b)^2=(a-b)^2+4ab$, $(a-b)^2=(a+b)^2-4ab$; that is,

The square of the sum of two numbers, is equal to the sum of the square of the difference and four times the product of the two numbers.

The square of the difference of two numbers, is equal to the difference between the square of the sum and four times the product of the numbers.

The product of three or more factors is found by multiplying the product of the first two by the third factor, and this product by the rest, if there be more than three factors; as the product of the factors $a+b$, $a+b$, $a+b$, and of $a-b$, $a-b$, $a-b$ can be found and expressed with their factors by the identical equations, as

$$(a+b)\times(a+b)\times(a+b)=a^3+3a^2b+3ab^2+b^3$$
$$\text{or } (a+b)^3=a^3+b^3+3ab(a+b),$$
$$\text{and } (a-b)\times(a-b)\times(a-b)=a^3-3a^2b+3ab^2-b^3$$
$$\text{or } (a-b)^3=a^3-b^3-3ab(a-b).$$

These two identical equations may be thus expressed in words:

The cube of the sum of two numbers, is equal to the sum of the cubes of the two numbers, increased by three times the product of the two numbers multiplied by their sum.

The cube of the difference of two numbers, is equal to the difference of the cubes of the two numbers, diminished by three times the product of the two numbers multiplied by their difference.

The sum and the difference of the cubes of two numbers can both be denoted by the product of two factors:

$$\text{Thus, } a^3+b^3=(a+b)\times(a^2-ab+b^2), \text{ and}$$
$$a^3-b^3=(a-b)\times(a^2+ab+b^2).$$

These may be expressed in words as follows:

The sum of the cubes of two numbers, is equal to the product of the sum of the two numbers and the difference between the sum of their squares and their product.

The difference of the cubes of two numbers, is equal to the product of the difference of the numbers and the sum of their squares increased by their product.

When the factors of a product are of the form of $x+a$ and $x-b$, where a and b are any integral numbers, the products of two and of three such factors may be written without performing the process of multiplication.[*]

Thus the products of two such factors are of the forms

$$(x+a)(x+b)=x^2+(a+b)x+ab, \text{ suppose } a>b,$$
$$(x-a)(x-b)=x^2-(a+b)x+ab,$$
$$(x+a)(x-b)=x^2+(a-b)x-ab,$$
$$(x-a)(x+b)=x^2-(a-b)x-ab.$$

Also the product of three factors,

$$(x+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(ab+ac+bc)x+abc.$$

When one, two, or all three of the factors are of the form $x-a$,[†] the products are:

$$(x-a)(x-b)(x-c)=x^3-(a+b+c)x^2+(ab+ac+bc)x-abc.$$
$$(x-a)(x+b)(x-c)=x^3+(b-a-c)x^2+(ac-ab-bc)x+abc.$$
$$(a-a)(x+b)(x+c)=x^3-(a-b-c)x^2+(bc-ab-ac)x-abc.$$

Prop. When a product is composed of a number of rational binomial factors of the forms $x+a$ and $x-b$, and arranged according to the descending powers of x; if the signs of the product be written in order, it will be found there will be as many continuations of the signs + + and — — as there are factors of the form $x+a$, and as many changes of sign + — and — + as there are factors of the form $x-b$ in the product.

Hence, conversely. When the terms of a product arising from binomial factors are arranged according to the descending powers of x, and the signs of the terms are written in order, the number of continuations of the signs + + and — — will indicate the number of factors of the form $x+a$ in the product, and the number of changes of signs + — and — + will indicate the number of factors of the form $x-b$.

This property will become evident from the product of two, three, or any number of such factors.[‡]

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[*] The following examples of these general forms of products may be found:

$$(x+4)(x+3)=x^2+7x+12.$$
$$(x-4)(x-3)=x^2-7x+12.$$
$$(x+4)(x-3)=x^2+x-12.$$
$$(x-4)(x+3)=x^2-x-12.$$

Also $(x+3)(x+2)(x+1)=x^3+6x^2+11x+6.$

$$(x-3)(x-2)(x-1)=x^3-6x^2+11x-6.$$
$$(x-3)(x+2)(x+1)=x^3\pm 0x^2-7x-6 \text{ or } x^3-7x-6.$$
$$(x-3)(x+2)(x-1)=x^3-2x^2+5x+6.$$

[†] Factors of the form $x-a$ may be written in the form $x+\bar{a}$, by writing the negative sign over the second term of the factor and connecting it by the positive sign to the first. By means of this artifice the product of binomial factors, some of form $x+a$, others of form $x-a$, may be exhibited as the product of factors of the form $x+a$. The product can then be reduced to the ordinary form by replacing $\bar{a}$ by $-a$.

[‡] The truth of this property may be shown in the examples of the products of two and of the three factors given in the note above marked *.

First. When the products consist each of two factors. The signs of the first pro-

13. Prop. *To explain the process of Algebraical division.*

The quotient of one monomial divided by another is determined by dividing the coefficient and factors of the dividend by the coefficient and factors of the divisor, and prefixing to the quotient the proper sign; as the quotient of $-15a^5bx^4$ divided by $3a^2bx^3$ is $-5a^3x$.

And when the dividend is a compound quantity and the divisor a monomial, the quotient is determined by dividing each term of the dividend by the divisor, and connecting the terms of the quotient by their proper signs; as the quotient of $-10a^3x+8a^2x^2+6ax^3$ divided by $-2ax$ is $+5a^2-4ax-3x^2$.

When the dividend and divisor are both compound quantities. As the process of division is the reverse of multiplication; if the divisor be multiplied by each term of the quotient, and these successive products be subtracted from the dividend, the quotient will be determined.

First.—Let the terms of the dividend and divisor, if necessary, be arranged according to the ascending or descending powers of one of the quantities; then the first term of the quotient is found by dividing the first term of the dividend by the first term of the divisor.

Secondly.—Let the divisor be multiplied by the first term of the quotient, and the product be subtracted from corresponding terms of the dividend.

Thirdly.—To this remainder let one or more terms from the dividend be annexed, and the second term of the quotient is found by dividing the first term of this new dividend by the first term of the divisor.

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duct arranged are + + +, which, taking the first with the second sign, and the second with the third, give + + and + +, two continuations of the same sign when the factors are $x+4$ and $x+3$.

The signs of the second product are + − +, which give + − and − +, when the factors are $x-4$ and $x-3$.

The signs of the third product are + + −, which give + + and + −, when the factors are $x+4$ and $x-3$.

The signs of the fourth product are + − −, which give + − and − −, when the factors are of the form $x-4$ and $x+3$.

Secondly. When the products consist each of three factors.

The signs of the terms of the first product are + + + +, which give + +, + +, + +, when the three factors are of the form $x+a$.

The signs of the terms of the second product are + − + −, which give + −, − +, + −, when the three factors are of the form $x-b$.

The signs of the terms of the third product are + ± − −; (writing ± when the coefficient of any term is zero).

Taking the upper of these two signs, the order of signs is + + − −, which give + +, + −, − −, when two factors are of the form $x+a$, and one of the form $x-b$.

Next taking the lower sign, the order of signs is + − − −, which give + −, − −, − −, where there are two factors of the form $x+a$ and one of the form $x-b$, the same as before.

The signs of the terms of the fourth product are + − + +, which give + −, − +, + +, when two factors are of the form $x-b$, and one of the form $x+a$.

And similarly, the law will be found to be true for the product of four, five, &c., rational binomial factors.

Fourthly.—Let the divisor be multiplied by the second term of the quotient, and the product subtracted as before; and a similar process continued for the third and successive terms of the quotient, until there be no remainder, or a remainder of less dimensions than the divisor.

In the former case, the dividend is equal to the product of the divisor and the quotient; in the latter, the dividend is equal to the sum of the remainder and the product of the divisor and the quotient.

As examples of the reverse process, let $a^2+2ab+b^2$ be divided by $a+b$, and $a^2-2ab+b^2$ by $a-b$.

$$\begin{array}{l} a+b)a^2+2ab+b^2(a+b \\ \quad\; a^2+ab \\ \hline \qquad\quad ab+b^2 \\ \qquad\quad ab+b^2 \\ \hline \end{array} \qquad\qquad \begin{array}{l} a-b)a^2-2ab+b^2(a-b \\ \quad\; a^2-ab \\ \hline \qquad -ab+b^2 \\ \qquad -ab+b \\ \hline \end{array}$$

Thus the quotients are $a+b$ and $a-b$, which with the divisor and dividend may be exhibited by the identical equations:

$$\frac{a^2+2ab+b^2}{a+b}=a+b, \text{ and } \frac{a^2-2ab+b^2}{a+b}=a-b.$$

As another example let a^2-b^2 be divided by $a-b$ and by $a+b$.

$$\begin{array}{l} a-b)a^2-b^2(a+b \\ \quad\; a^2-ab \\ \hline \qquad +ab+b^2 \\ \qquad +ab-b^2 \\ \hline \end{array} \qquad\qquad \begin{array}{l} a+b)a^2-b^2(a-b \\ \quad\; a^2+ab \\ \hline \qquad -ab-b^2 \\ \qquad -ab-b^2 \\ \hline \end{array}$$

These quotients may be exhibited in the same manner:

$$\frac{a^2-b^2}{a-b}=a+b, \text{ and } \frac{a^2-b^2}{a+b}=a-b.$$

The reverse operations of a^2-b^2 divided by $a-b$ and by $a+b$ respectively, give the following theorems:—

If the difference of the squares of two numbers be divided by the difference of the numbers, the quotient is equal to their sum. And if the difference of the squares of two numbers be divided by the sum of the numbers, the quotient is equal to their difference.

It will frequently be found that the divisor is not contained exactly in the dividend, and in such cases there will be a remainder after the quotient has been obtained; as a^2+b^2 is not exactly divisible by $a+b$, but gives the quotient $a-b$ with a remainder $2b^2$; but if divided by $a-b$, gives the quotient $a+b$ with $2b^2$ as remainder.

As a further exemplification of the reverse process, let a^3+b^3 be divided by $a+b$, and a^3-b^3 by $a-b$:

$$\begin{array}{l} a+b)\,a^3+b^3\,(a^2-ab+b^2 \\ \quad\; a^3+a^2b \\ \hline \quad\; -a^2b+b^3 \\ \quad\; -a^2b-ab^2 \\ \hline \qquad\; +ab^2+b^3 \\ \qquad\; +ab^2+b^3 \\ \hline \end{array} \qquad\qquad \begin{array}{l} a-b)\,a^3-b^3\,(a^2+ab+b^2 \\ \quad\; a^3-a^2b \\ \hline \quad\; +a^2b-b^3 \\ \quad\; +a^2b-ab^2 \\ \hline \qquad\; +ab^2-b^3 \\ \qquad\; +ab^2-b^3 \\ \hline \end{array}$$

These also may be similarly denoted:

$$\frac{a^3+b^3}{a+b}=a^2-ab+b^2\text{; and }\frac{a^3-b^3}{a-b}=a^2+ab+b^2.$$

And in the same manner, if $a^3+3a^2b+3ab^2+b^3$ be divided by $a+b$, and $a^3-3a^2b+3ab^2-b^3$ by $a-b$, it will be found that the quotients are $a^2+2ab+b^2$ and $a^2-2ab+b^2$ respectively, and that

$$\frac{a^3+3a^2b+3ab^2+b^3}{a+b}=a^2+2ab+b^2,$$

$$\text{and }\frac{a^3-3a^2b+3ab^2-b^3}{a-b}=a^2-2ab+b^2.$$

The quotients arising from the following dividends and divisors are important.

1. a^n-b^n is exactly divisible by $a-b$ when the n is any positive integer odd or even.
2. a^n+b^n is exactly divisible by $a+b$ when n is an odd integer.
3. a^n-b^n is exactly divisible by $a+b$ when n is an even integer.
4. a^n+b^n is in no case exactly divisible by $a-b$, whether n be odd or even integer.*

---

* To shew that a^n-b^n is divisible by $a-b$, when n is odd or even.

$$\begin{array}{l} a-b)\,a^n-b^n\,(a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots\ldots \\ \qquad\;\; \underline{a^n-b^{n-1}} \\ \qquad\qquad +a^{n-1}b-b^n \\ \qquad\qquad \underline{+a^{n-1}b-a^{n-2}b^2} \\ \qquad\qquad\qquad a^{n-2}b^2-b^n \\ \qquad\qquad\qquad \underline{a^{n-2}b^2-a^{n-3}b^3} \\ \qquad\qquad\qquad\qquad a^{n-3}b^3-b^n,\ \&c. \end{array}$$

It is obvious that at each step of the division, the indices of the powers of a in the remainders are successively diminished by unity, and after n steps there will be no power of a in the remainder.

Let Q denote the quotient and R the remainder,

$$\text{then } a^n-b^n=(a-b)Q+R,$$

and since the remainder R does not contain a, it will not be altered on giving any value whatever to a.

$$\text{Let } a=b,\text{ then } b^n-b^n=(b-b)Q+R,\text{ or } R=0,$$

that is, the remainder is 0 after n terms of the quotient have been obtained, whether n be an odd or even number, and the quotient with the dividend and divisor may be denoted thus :—

$$\frac{a^n-b^n}{a-b}=a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots\ldots\ldots+a^2b^{n-3}+ab^{n-2}+b^{n-1}. \qquad (1)$$

If $n=2$, 3, 4, 5, &c., any integral numbers,

then $\frac{a^2-b^2}{a-b}=a+b$, $\frac{a^3-b^3}{a-b}=a^2+ab+b^2$, $\frac{a^4-b^4}{a-b}=a^3+a^2b+ab^2+b^3$,

$\frac{a^5-b^5}{a-b}=a^4+a^3b+a^2b^2+ab^3+b^4$, and so on.

$$\frac{a^n+b^n}{a+b}=a^{n-1}-a^{n-2}b+a^{n-3}b^2-\ldots\ldots+a^2b^{n-3}-ab^{n-2}+b^{n-1}, \qquad (2)$$

where $n=3$, 5, 7, 9, 11, &c.

Also $\frac{a^n-b^n}{a+b}=a^{n-1}-a^{n-2}b+a^{n-3}b^2-\ldots\ldots\ldots-a^2b^{n-3}+ab^{n-2}-b^{n-1}$, (3)

where $n=2$, 4, 6, 8, 10, &c.

But $\frac{a^n+b^n}{a-b}=a^{n-1}+a^{n-2}b+a^{n-3}b^2+\ldots\ldots+a^2b^{n-3}+ab^{n-2}+b^{n-1}$, (4)

with remainder $2b^n$.

If dividends whose coefficients are rational of the form $x^2 \pm px + q$, $x^3 \pm px^2 \pm qx \pm r$, $x^4 \pm px^3 \pm qx^2 \pm rx \pm s$, $x^5 \pm pa^4 \pm qx^3 \pm rx^2 \pm sx \pm t$, &c., are divided respectively by $x \pm a$; after two, three, four, five, &c., terms of the quotient have been determined, the remainders will be found to be of the same form as the dividends with a in the place of x, in some cases with the same signs as the dividend, in others with different signs.

If $x^3 + px^2 + qx + r$ be divided by $x - a$,*

The coefficients of the dividend are 1, p, q, r.

The coefficients of the quotient are 1, $a+p$, $a^2 + ap + q$,

And the remainder is $a^3 + pa^2 + qa + r$.

This remainder is obviously the same expression as the dividend, having a in the place of x.

Now if a be actually substituted for x in the dividend, this assumes that $x = a$, or $x - a = 0$; hence, when a is substituted for x in the dividend, the remainder will be equal to 0, and $x^3 + px^2 + qx + r$ will be exactly divisible by $x - a$.

When any given expression with numerical coefficients is or is not divisible by $x \pm a$, the quotient and the remainder can be found by substituting the numerical values for the general coefficients of the dividend and divisor.†

---

* The following is the process of the division exhibited, and the proof of its correctness :—

$$
\begin{array}{l}
x-a\,)\,x^3+px^2+qx+r\,(\,x^2+(a+p)x+(a^2+pa+q)\\
\qquad\quad \underline{x^3-ax^2}\\
\qquad\qquad (a+p)x^2+qx\\
\qquad\qquad \underline{(a+p)x^2-(a^2+ap)x}\\
\qquad\qquad\qquad\qquad (a^2+ap+q)x+r\\
\qquad\qquad\qquad\qquad \underline{(a^2+ap+q)x-(a^3+pa^2+qa)}\\
\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{a^3+pa^2+qa+r}
\end{array}
$$

$$
\begin{array}{l}
x^2+(a+p)x+(a^2+pa+q)\\
\underline{x-a}\\
x^3+(a+p)x^2+(a^2+pa+q)x\\
\underline{\qquad\; -ax^2-(a^2+pa)x-(a^3+pa^2+qa)}\\
x^3+px^2+qx-a^3-pa^2-qa\\
\underline{\qquad\qquad\qquad a^3+pa^2+qa+r}\\
\underline{x^3+px^2+qx+r}
\end{array}
$$

† Is $x^3 - 6x^2 + 11x - 6$ divisible by $x - 3$?

The coefficients of divisor are 1, -6, 11, -6.

The coefficients of quotients are 1, -3, 2.

The remainder is $27 - 54 + 38 - 6 = 0$.

The expression $x^3 - 6x^2 + 11x - 6$ is exactly divisible by $x - 3$, and gives $x^2 - 3x + 2$ for the quotient.

Is $x^3 + 7x^2 + 9x + 12$ divisible by $x + 3$?

The coefficients of the dividend are 1, 7, 9, 12.

The coefficients of the quotient are 1, 4, 39.

And the remainder is $3^3 + 7 \cdot 3^2 + 9 \cdot 3 + 12 = 129$.

The expression $x^3 + 7x + 9x + 12$ when divided by $x + 3$ gives the quotient $x^2 + 10x + 39$, with a remainder 129.

Also the conditions may be readily found when any rational expression of these forms is divisible by $x+1$ and by $x-1$.

If x^3+px^2+qx+r is divisible by $x+1$, $x+1$ must be equal to 0, and $x=-1$: by substituting this value for x, the dividend becomes $(-1)^3+p(-1)^2+q(-1)+r$, which must also be equal to 0, or $-1+p-q+r=0$, or $p+r=q+1$, that is: the expression x^3+px^2+qx+r is divisible by $x+1$, if the sums of the alternate coefficients of the dividend are equal.

And if x^3+px^2+qx+r is divisible by $x-1$, then $x-1$ must be equal to 0, and $x=1$, and substituting this value of x, $1+p+q+r$ must be equal to 0, or the sum of coefficients of the terms of the dividend must be equal to 0.

14. The products and quotients of concrete quantities admit of different interpretations, according to the nature of the magnitudes denoted by the symbols. In their application to geometrical magnitudes, they include lines, surfaces, and volumes, as well as angles and curves. The straight line has one dimension, length only; the surface has two dimensions, length and breadth; and the volume has three, length, breadth, and thickness.

The unit of length is arbitrary, and may be any straight line of definite and known length.

The square is the figure assumed as the measure of surfaces, and the unit of area is assumed to be a square, any side of which is an unit in length. The cube is the figure assumed as the measure of volumes, and the unit of volume is a cube, any edge of which is an unit in length.

To find the measure of any line, surface, and volume, is to determine the number of times any length, surface, and volume contains the unit of length, surface, and volume respectively.

If a, b denote the number of lineal units in two adjacent sides of a rectangular parallelogram or rectangle, and A the number of square units in the area, it may be shown that $A=ab$*; that is, the number of square units in the area of a rectangle is denoted by the product of the two numbers which express the number of lineal units in the two adjacent sides of the rectangle. Or, in other words,

If a number representing a line be multiplied by another representing a line, the product is a number representing an area.

And since $A=ab$, it follows that $\frac{A}{a}=b$, a number denoting b lineal units.

Hence. If a number representing an area be divided by another representing a line, the quotient is a number representing a line.

---

* The reader is referred for the proof of the expressions $A=ab$, and $V=abc$, to the notes on the Second and Eleventh Books of the author's edition of Euclid's Elements of Geometry, with geometrical exercises selected from the Cambridge Examination Papers.

Again, if a, b, c denote the number of lineal units in the three adjacent edges of a rectangular parallelopiped, and V denote the number of cubic units in the volume, it may be shown that the product of the numbers a, b, and c is a correct representation of the number of cubic units in the volume of the parallelopiped, and that $V=abc$. Or in words,

The number of cubic units contained in the volume of a rectangular parallelopiped, is expressed by the number of cubic units contained in the product of the numbers of lineal units in any three adjacent edges of the parallelopiped.

And since $V=abc$, and $A=ab$, then $\frac{V}{A}=\frac{abc}{ab}=c$, a line of c units.

Or, if a number representing a volume be divided by a number representing an area, the quotient is a number representing a line.

Next, $V=abc$ and $\frac{V}{c}=ab$, an area of ab square units.

Or, if a number representing a volume be divided by another representing a line, the quotient will be a number representing an area.

It will also be obvious, that if a number representing a line, an area, or a volume be multiplied by any abstract number, the products will each be a multiple or submultiple of the line, the area, or the volume respectively.

And in the same manner, if a line, an area, or a volume be divided by an abstract number, the quotient will denote some part of the line, the area, or the volume respectively.

And if a number representing a line be divided by another representing a line, the quotient will indicate the number of times the one contains the other, and is therefore an abstract number. In the same manner, a number representing an area divided by another representing an area, as also a number representing a volume divided by another representing a volume, will denote abstract numbers in the quotients.

In a similar manner the products and quotients of other concrete quantities may be interpreted; as, for instance, if a body move uniformly over a distance of s units in t units of time, then $\frac{s}{t}$ will denote the distance which the body has moved over in one unit of time, and is defined to be the velocity of the body. If v be assumed to denote this distance or velocity, then $\frac{s}{t}=v$, and $s=tv$, or, the distance moved over with an uniform velocity, is equal to the product of the numbers which denote the velocity and the time.

EXERCISES.

I.

1. What operation bears the same relation to multiplication that multiplication bears to addition?

2. Explain what is meant by a reverse operation in Algebra, and give an example.

3. Explain the meaning of the terms factor and dimension as applied to Algebraical expressions. Give examples.

4. What is meant by the rule of signs in the multiplication of Algebraical quantities? State whether you regard the rule as conventional or demonstrative, giving your reason in the former case, or a demonstration in the latter.

5. Define the meaning of a^m; shew that $a^m.a^n=a^{m+n}$, and $\frac{a^m}{a^n}=a^{m-n}$; and explain the notations a^0, a^{-n}.

6. Explain and interpret the expressions ab, and abc, when a, b, c, represent straight lines.

7. Find the products of $5abx$ and $4cdy$; mab and $-nc$; $-12a^2bc^3$ and $5ab^2d^3$; $-12a^2bc^3$ and $-8ab^2d^3$ respectively.

8. Divide $20abcdxy$ by $4cdy$; $-mnabc$ by mab; $-60a^3b^3c^3d^3$ by $-12a^2bc^3$; $+96a^3b^3c^3d^3$ by $-8ab^2d^3$, respectively.

9. Multiply $9a+2x$ by $3a$; $4a-8b$ by $-3ab$; x^2-ax+a^2 by $-ax$; $3a^2-4ab-5b^2$ by $-6a^2b^2c^2$; $max-nay+paz$ by $axyz$; $-a^2x^2+b^2y^2-c^2z^2$ by $x^2y^2z^2$, respectively.

10. Divide $27a^2+6ax$ by $3a$; $-12a^2b+24ab^2$ by $-3ab$; $-ax^3+a^2x^2-a^3x$ by $-ax$; $18a^4b^2c^2+24a^3b^3c^3+30a^2b^4c^3$ by $-6a^2b^2c^2$; $ma^2x^3yz-na^2xy^2z+pa^2xyz^2$ by $axyz$; $-a^2x^4y^2z^2+b^2x^2y^4z^3-c^2x^2y^2z^4$ by $x^2y^2z^2$, respectively.

11. Shew the truth of, and express in words at length, the identical expressions:

    1. $a(a+b)+b(a+b)$.
    2. $a(a-b)-b(a-b)$.
    3. $(a+b)^2+(a-b)^2=2(a^2+b^2)$.
    4. $(a^2+b)^2-(a-b)^2=4ab$.

12. In the first ten propositions of the second book of Euclid's Elements, if "the rectangle contained by two lines" be exchanged for "the product of two numbers," and "the square on a line" for the square of a number:" shew that the propositions so changed are true for all numbers whatever.

13. In the multiplication and division of Algebraical polynomials, what preliminary steps are necessary in order to avoid confusion in performing these operations?

II.

Find the following products, and prove the correctness of them by reversing the operation in each example:—

1. Of $2x+3y$ and $2x+3y$; $2x-3y$ and $2x+3y$; $2x-3y$ and $2x-3y$.

2. Of x^2+2x+7 and $x+5$; $3x^2-5xy+7y^2$ and $2x-3y$.
3. Of a^2+ax+x^2 and a^2-ax+x^2.
4. Of $8a^3-12a^2b+18ab^2-27b^3$ and $2a+3b$.
5. Of $a^2-2ab+2b^2$ and $a^2+2ab+2b^2$.
6. Of $5x^3-7x^2-3x+8$ and $7x^2+8$.
7. Of $x^3+6x^2+24x+60$ and $x^3-6x^2+12x-60$.
8. Of $x^4-3ax^3+2a^2x$ and $x^2-2ax-2a^2$.
9. Of $81x^4+27x^3y+9x^2y^2+3xy^3+y^4$ and $3x-y$.
10. Of $x^4+x^3+x^2+x+1$ and x^2-x+1.
11. Of $a^4-2a^3+3a^2-2a+1$ and $a^4+2a^3+3a^2+2a+1$.
12. Of $1-5x+10x^2-10x^3+5x^4-x^5$ and $1-2x+x^2$.
13. Of $x^{-3}-x^{-2}+4x^{-1}-8$ and $x^{-3}+x^{-2}+4x^{-1}+8$.
14. Of $x^{-3}-x^{-2}y^{-1}+x^{-1}y^{-2}-y^{-3}$ and $x^{-1}+y^{-1}$.
15. Of $-3x^{-5}+2x^{-4}y^{-1}$ by $-2x^{-5}-3x^{-4}y^{-1}$.
16. Of $5x^3y^{-5}-7xy^{-3}+4x^{-1}y^{-1}-3x^{-3}y+x^{-5}y^3$ by x^3y-xy^3.

III.

Determine the following quotients, and verify their correctness :—

1. $x^4+10x^3+35x^2+50x+24$ by $x+4$.
2. $x^4+x^3-4x^2+5x-3$ by x^2+2x-3.
3. $256x^4+16x^2y^2+y^4$ by $16x^2+4xy+y^2$.
4. a^5-32b^5 by $a-2b$.
5. $x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5$ by x^3-y^3.
6. $21x^5-2x^4-70x^3-23x^2+33x+27$ by $7x^2+4x-9$.
7. $50x^3y^2-6y^5+25xy^4-45x^2y^3-41x^4y+20x^5$ by $5xy^2-3y^3-4x^2y+5x^3$.
8. $21x^3y^2+25x^2y^3+68xy^4-40y^5-56x^5-18x^4y$ by $5y^2-8x^2-6xy$.
9. $40ab^4+49a^3b^2-65a^2b^3-25a^4b+6a^5-25b^5$ by $5ab^2+3a^3-8a^2b-5b^3$.
10. x^6+3x^3+9 by $x+2$.
11. $x^6-2x^5-5x^4+20x^3-25x^2+14x-3$ by x^2+2x-3.
12. $x^7-x^4y^3-x^3y^4+x^7$ by $x^4-x^3y-xy^3+y^4$.
13. $x^2+x^{-2}+1$ by $x+x^{-1}+1$.
14. $a^2x^{-2}-b^{-4}x^2-2b^{-1}-b^2$ by $ax^{-1}+b^{-2}x+b$.
15. $2x^5y^{-3}-5x^4y^{-2}+7x^3y^{-1}-5x^2+2xy$ by $x^3y^{-3}-x^2y^{-2}+xy^{-1}$.

IV.

Find the quotients arising from the division of :—

1. $a^3-3a^2b+3ab^2-b^3$ by $a-b$.
2. $x^3+(4ab-b^2)x-(a-2b)(a^2+3b^2)$ by $x-a+2b$.
3. $(x+y)^3+3xy(1-x-y)-1$ by $x+y-1$.
4. $a^3+b^3+c^3-3abc$ by $a+b+c$.
5. $x^4-(a^2-b-c)x^2-(b-c)ax+bc$ by x^2-ax+c.
6. $(a^3-1)x^3-(a^3+a^2-2)x^2+(4a^2+3a+2)x-3(a+1)$
by $(a-1)x^2-(a-1)x+3$.
7. $(ax+by)^2+(ay-bx)^2+c^2x^2+c^2y^2$ by x^2+y^2.
8. $a^2x^2-a^2y^2+b^2y^2-b^2x^2$ by $ax+by+ay+bx$.

9. $x^4+y^4-z^4+2x^2y^2-2z^2-1$ by $x^2+y^2-z^2-1$.

10. $3(a^4+b^4)-8ab(a^2+b^2)+14a^2b^2$ by $3a^2-2ab+b^2$.

11. $(12b^2-29bc+15c^2)x^3+(23b^3-31b^2c-9bc^2+15c^3)x^2+(10b^4-6b^2c^2)x$ by $(3b-5c)x+2b^2$.

12. $a^6+b^6-ab(a^2+b^2)\{ab-5(a^2-b^2)\}$ by a^2-b^2.

13. $(a-b)x^3-(x-b)a^3+(x-a)b^3$ by $ax-bx+a^2-b^2$.

14. $(a^3+3a^2x+9ax^2+27x^3)\times(a^3-3a^2x+9ax^2-27x^3)$ by $a^4-18a^2x^2+81x^4$.

15. $(x^3-a^3)b+(x^2-a^2)ax+(x-a)a^3$ by $(a+b)(x-a)$.

16. $xy^3+xyz^2-x^3y-xy^2z+x^3z-xz^3+2y^3z-2yz^3$ by $y+z-x$.

17. $xy^3+2xyz^2+y^3z+yz^3-2xy^2z-xz^3-y^2z^2$ by $xy-xz+yz$.

18. $(yz-a)^2-(ca-y^2)(ab-z^2)$ by $(bc-1)(yz-a)-(z-by)(y-cz)$.

19. $(ax+by)^3+(ax-by)^3+(bx-ay)^3+(ay+bx)^3$ by $(a+b)^2x^2-3ab(x^2-y^2)$.

V.

1. From the expression $(a+b)(a-b)=a^2-b^2$, find the difference of the squares of 79 and 81, and of $118\frac{1}{4}$ and $121\frac{3}{4}$.

2. "To multiply two unequal numbers. Take the sum of the two numbers and multiply half of it into itself. From this product subtract the square of half the difference of the two numbers" (*Kholasat-al-Hisab*). Give the Algebraic formula which expresses this rule, and state it in words at length in the simplest form.

3. Subtract the product $(x-7)\times(x-1)$ from $(x-2)\times(x-3)$, and add to the result $(x-1)\times(x-2)$.

4. What value of x will make the difference between $(x+1)(x+2)$ and $(x-1)(x-2)$ equal to 54?

5. Write the Algebraical expressions for the square of the sum of the cubes of a and b, and the cube of the sum of their squares, and find their product.

6. Determine by inspection the common factor in each of the terms of the expression $(x^3+a^3)^2-2ax(x^2-a^2)^2+a^4(x+a)^2$.

7. Multiply $3+\cdot01x$ by $1-x$, and shew that the value of the product is $\cdot0009$ when $x=\cdot1$.

8. Divide $2\cdot091x^3+9\cdot22x^2+3\cdot694x-1\cdot2$ by $1\cdot02x+4$.

9. If a, b, k be positive integers, ascertain whether $(a+k)^3-(b+k)^3$ is divisible by $a-b$.

10. What number must be added to the expression $x^3+x^2-4(x+3)$ that it may be divisible by $x-6$ without remainder?

11. If the dividend be $4a^2b^2+2(3a^4-2b^4)-ab(5a^2-11b^2)$ and the quotient be $2(a+b)a+(a^2-b^2)$, what is the divisor?

12. Write at length the forms of the four products $(mx\pm a)(nx\pm b)$, and exemplify the use of them.

13. Shew that (1) ax^3+bx^2+cx+d is divisible by x^2+b^2, if $ad=ac$.
(2) ax^4+bx^2+c by x^2+b^2, if $ac=b^2$.
(3) $x^5-5qx+4r$ by $(x-c)^2$, if $q^5=r^4$.

14. Multiply $a+x$, $b+y$, $c+z$ together, and deduce $(x+y)^3$ from the result.

VI.

1. Find the continued product of $x+1$, $x+2$, $3-x$, $4-x$.

2. Find what $x^5+4x^3-3x^2+2$ becomes when x is changed into $x+5$, and when x is changed into $x-1$.

3. Multiply $x+a$, $x+b$, $x+c$, $x+d$ together, and deduce from the product the coefficient of x in the product $(x+2)(x+6)(x+10)(x+14)$.

4. Multiply $x^3+(x-y)^3-y^3$ by $x^3-(x+y)^3+y^3$.

5. Show that the difference between $(x^2+2x+3)(3x^2+2x+1)$ and $(x^2-2x+3)(3x^2-2x+1)$ is the same as the difference between $(x^2+4x+1)^2$ and $(x^2-4x+1)^2$.

6. Shew that $(3x^2-4x+2)^2-(2x^2+9x+3)^2$ is divisible by x^2+x+1 without performing the operation of division.

7. By what Algebraical expression must x^3+y^3 be multiplied so that the product may be $x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5$?

8. Multiply $x^3-9x^2y+23xy^2-15y^3$ by $x-7y$ and divide the quotient by $x^2-8xy+7y^2$.

9. Write the coefficients of x^3 and of x^5 in the product of $3x^4-5x^3+7x^2-9x+11$ and $x^5+6x^4+12x^3-12x^2-6x+1$.

10. The product of two factors is $(2x+3y)^3+(2y+3z)^3$ and one of them is $2x+5y+3z$; find the other.

VII.

Form the products of the following sets of factors without performing the process of multiplication.

1. $a+b+c$ and $a+b-c$; $a-b+c$ and $b+c-a$; $a+b+c$ and $a+c-b$; $a+b+c$ and $b+c-a$; $a+b-c$ and $a+c-b$; $a+b-c$ and $b+c-a$.

2. $a+b+c+d$ and $a+b+c-d$; $a+b+d-c$ and $a+d-b+c$;
$a+b-c-d$ and $a-b+c-d$; $a-b+c-d$ and $a+b+c+d$;
$b+c+d-a$ and $a+d-b-c$; $a+b-c-d$ and $b-a+c-d$.

3. a^2+ab+b^2 and a^2-ab+b^2.

4. $1-ax$, $1-bx$, and $1-cx$.

VIII.

Resolve the following expressions into their simple and quadratic factors :—

1. $(a^2+2a)-(c^2-2c)$.
2. $a(b+c)^2+b(a+c)^2+c(a-b)^2$.
3. $ab(a+b)+bc(b+c)+ca(a+c)+2abc$.
4. $a^2-b^2-c^2+d^2-2(ad+bc)$.
5. $4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2$.
6. $(x-5)^2x^2+10(x-5)x+24$.
7. $4(ad-bc)^2-(a^2+d^2-b^2-c^2)^2$.

8. $x^4+bx^2+a^3x+a^3b$.
9. $(x-a)^6-a^6$.
10. $(1+y)^2-2(1+y)^2x^2+(1+y)^2x^4$.
11. $(a+1)x^2-(a+b)(x-y)xy-(b-1)y^2$.
12. $(abc+bcd+cad+abd)^2-(a+b+c+d)^2abcd$.

IX.

Perform the following divisions of—

1. $(a+b)(a+c)-(d+b)(d+c)$ by $a-d$.
2. $a^2(c-b)+b^2(a-c)+c^2(b-a)$ by $a-b$.
3. $(ax+by+cz)^2-(bx+cy+az)^2$ by $(a+b)x+(b+c)y+(c+a)z$.
4. $a^2(b+c)-b^2(a+c)+c^2(a+b)+abc$ by $a-b+c$.
5. $a^8-b^8+a^2b^2(a^4-b^4)$ by $(a^2-ab+b^2)(a^2+ab+b^2)$.
6. $a^3+b^3+c^3-3abc$ by $a+b+c$.
7. $(b+c)^3+(c+a)^3+(a+b)^3-3(b+c)(c+a)(a+b)$ by $a+b+c$.
8. $a^3+b^3+c^3+d^3-3(abc+abd+acd+bcd)$ by $a+b+c+d$.
9. $a^2(b+c)+b^2(a-c)+c^2(a-b)+abc$ by $a+b+c$.
10. $(b-c)a^3+(c-a)b^3+(a-b)c^3$ by $(a-c)(b-c)$.
11. $a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)$ and $(a-b)^5+(b-c)^5+(c-a)^5$ by $(a-b)(b-c)(c-a)$.
12. $a^2(b+c-a)^3+b^2(c+a-b)^3+c^2(a+b-c)^3$ by $(a-b)^2-2(a+b)c+c^2$.

X.

Verify the truth of the following equivalent expressions :—

1. $(a+b)cd+(c+d)ab=(a+d)bc+(b+c)ad$.
2. $(a^2+d^2)bc+(b^2+c^2)ad=(ac+bd)(ab+cd)$.
3. $(a^2+b^2)(c^2+d^2)=(ac+bd)^2+(ad-bc)^2=(ac-bd)^2+(ad+bc)^2$.
4. $(a^2-b^2)(c^2-d^2)=(ac-bd)^2-(ad-bc)^2=(ac+bd)^2-(ad+bc)^2$.
5. $(a^2+b^2+c^2+d^2)(x^2+y^2)=(ax+by)^2+(bx-ay)^2+(xc+dy)^2+(dx-cy)^2$.
6. $(a^2+b^2)(c^2+d^2)(x^2+y^2)$
$=\{(ac-bd)x+(ad+bc)y\}^2+\{(ac-bd)y-(ad+bc)x\}^2$.
7. $(a^2+b^2+c^2+d^2)(v^2+x^2+y^2+z^2)=(av+bx+cy+dz)^2+(ax-bv+cz-dy)^2+(ay-cv-bz+dx)^2+(az-dv+by-cx)^2$.
8. $a^2(b-c)+b^2(c-a)+c^2(a-b)=(a-b)(b-c)(c-a)$.
9. $a^3(a-c)+b^3(c-a)+c^3(a-b)=(a-b)(b-c)(c-a)(a+b+c)$.
10. $a^4(b-c)+b^4(c-a)+c^4(a-b)$
$=(a-b)(b-c)(c-a)(a^2+b^2+c^2+ab+ac+bc)$.
11. $a^5(b-c)+b^5(c-a)+c^5(a-b)$
$=(a-b)(b-c)(c-a)\{a^3+(b+c)a^2+(b^2+bc+c^2)a+(b+c)(b^2+c^2)\}$.
12. $(a+b)^3-(a^3+b^3)=3ab(a+b)$.
$(a+b)^5-(a^5+b^5)=5ab(a+b)(a^2+ab+b^2)$.
$(a+b)^7-(a^7+b^7)=7ab(a+b)(a^2+ab+b^2)^2$.
13. $(a+b+c)^3-(a^3+b^3+c^3)=3(a+b)(b+c)(c+a)$.
$(a+b+c)^5-(a^5+b^5+c^5)$
$=5(a+b)(b+c)(c+a)(a^2+b^2+c^2+ab+ac+bc)$.

$(a+b+c)^7-(a^7+b^7+c^7)$
$=7(a+b)(b+c)(c+a)\{a^2+b^2+c^2+ab+ac+bc)^2+(a+b+c)abc\}.$

XI.

Reduce to their simplest forms—

1. $(a+b)(b+c)-(c+d)(d+a)-(a+c)(b-d).$
2. $a(a+b+c-d)+b(a+b-c+d)+c(a-b+c+d)+d(-a+b+c+d).$
3. $(a+b)^2+(a+c)^2+(b+c)^2+(a+d)^2+(b+d)^2+(c+d)^2.$
4. $(a+b+c+d)^2+(a-b-c+d)^2+(a-b+c-d)^2+(a+b-c-d)^2.$
5. $\{(a-b-c)a-(a+c-b)b+(b+c)c+(a-c)c\}(a-b).$
6. $(a^2-bc)(b-c)+(b^2-ac)(c-a)+(c^2-ab)(a-b).$
7. $(a-2b)(a+2b)+(2b-3c)(2b+3c)+(3c-d)(3c+d).$
8. $(a+b+c)(a+b+d)+(a+c+d)(b+c+d)-(a+b+c+d)^2.$
9. $(ab+cd)(c^2+d^2)+cd(a^2+b^2-c^2-d^2).$
10. $\{(ac+bd)^2+(ad-bc)^2\}.\{(ac+bd)^2-(ad+bc)^2\}.$
11. $4\{(a^2-b^2)cd+(c^2-d^2)ab\}^2+\{(a^2-b^2)(c^2-d^2)-4abcd\}^2.$
12. $(a^2-b^2+c^2-d^2)^2+2\{(ab-cd)^2+(bc+ad)^2-(ac-bd)^2\}.$
13. $(a+b)^3+(b+c)^3+(c+d)^3+(d+a)^3+(a+c)^3+(b+d)^3.$
14. $(a-b)^3+(a+b)^3+3\{(a-b)^2(a+b)+(a+b)^2(a-b)\}.$
15. $a(b+c)^2+b(a+c)^2+c(a+b)^2-c^2(a+b)-b^2(a+c)-a^2(b+c).$
16. $(a+b+c)^3-(a+b-c)^3-(a+c-b)^3-(b+c-a)^3.$
17. $(a+b+c+d)^4-(a^4+b^4+c^4+d^4)$
$-(a+b+d)^4-(a+c+d)^4-(a+b+c)^4-(b+c+d)^4$
$+(a+b)^4+(b+c)^4+(c+d)^4+(a+d)^4+(b+d)^4+(a+c)^4.$
18. $(b^2c^2+a^2d^2)(b-c)(a-d)+(c^2a^2+b^2d^2)(c-a)(b-d)$
$+(a^2b^2+c^2d^2)(a-b)(c-d).$
19. $(a+b+c+d)^5-(b+c+d)^5-(a+c+d)^5-(a+b+d)^5-(a+b+c^5)+$
$(b+c)^5+(a+c)^5+(a+b)^5+(a+d)^5+(b+d)^5+(c+d)^5-(a^5+b^5+c^5+d^5).$

XII.

Verify the following expressions :—

1. $30ab-(9a-8b)(5a+2b)-(4b-3a)(15a+4b)=4ab.$
2. $(ab+ac+bc)^2-(a^2b^2+a^2c^2+b^2c^2)=2(a+b+c)abc.$
3. $a(b+c)(b^2+c^2-a^2)+b(c+a)(c^2+a^2-b^2)+c(a+b)(a^2+b^2-c^2)$
$=2abc(a+b+c).$
4. $(a+b+c)^4-(a+b)^4-(b+c)^4-(c+a)^4+a^4+b^4+c^4$
$=12abc(a+b+c).$
5. $a^3(b+c-a)^2+b^3(c+a-b)^2+c^3(a+b-c)^2+abc(a^2+b^2+c^2)+$
$(a^2+b^2+c^2-bc-ca-ab)(b+c-a)(c+a-b)(a+b-c)=2abc(bc+ca+ab).$
6. $(a+b-2c)^3+(b+c-2a)^3+(c+a-2b)^3$
$=3(a+b-2c)(b+c-2a)(a+c-2b).$
7. $2(a+b+c)^3-(a+b)^3-(b+c)^3-(c+a)^3+3abc$
$=3(ab+bc+ac)(a+b+c).$
8. $8(a+b+c)^3-(a+b)^3-(b+c)^3-(c+a)^3$
$=3(2a+b+c)(a+2b+c)(a+b+2c).$

9. $(a+b+c)^3+a^3+b^3+c^3=(a+b)^3+(b+c)^3+(c+a)^3+6abc$.

10. $(a-b)^3+(b-c)^3+(c-a)^3=3(a-b)(b-c)(c-a)$.

11. $\{(a-b)^2+(b-c)^2+(c-a)^2\}^2=2\{(a-b)^4+(b-c)^4+(c-a)^4\}$.

12. $\{(a-b)^2+(b-c)^2+(c-a)^2\}^3-54(a-b)^2(b-c)^2(c-a)^2$
$=2(a+b-2c)^2(b+c-2a)^2(c+a-2b)^2$.

13. $(a-b)^4+(b-c)^4+(c-a)^4=2\{(a-b)^2(c-a)^2+(a-b)^2(b-c)^2$
$+(b-c)^2(c-a)^2\}=2(a^2+b^2+c^2-ab-bc-ac)^2$.

14. $(b^2+c^2-a^2+ab+bc+ca)^2(c^2-b^2)+(c^2+a^2-b^2+ab+bc+ca)^2(a^2-c^2)$
$+(a^2+b^2-c^2+ab+bc+ca)=(a^2-b^2)(b^2-c^2)(c^2-a^2)$.

15. $a^2(b+c)^2+b^2(c+a)^2+c^2(a+b)^2+abc(a+b+c)$
$+(a^2+b^2+c^2)(bc+ca+ab)=(a+b+c)(b+c)(c+a)(a+b)$.

16. $\{2bc(a-b)-(b^2+c^2-a^2)(a-c)\}^2+(a-c)^2(b+c-a)(c+a-b)$
$(a+b-c)(a+b+c)=4abc\{abc-(b+c-a)(c+a-b)(a+b-c)\}$.

17. $a^4(b+c)^4+b^4(a+c)^4+c^4(a+b)^4=2b^2c^2(a+b)^2(a+c)^2$
$+2c^2a^2(b+c)^2(b+a)^2+2a^2b^2(a+c)^2(c+b)^2-16a^2b^2c^2(bc+ac+ab)$.

18. $(a+b)^2(b+c)^2(c+a)^2+2a^2b^2c^2-a^4(b+c)^2-b^4(c+a)^2-c^4(a+b)^2$
$=2(ab+bc+ca)^3$.

XIII.

Shew the identity in value of the following sets of expressions:—

1. $(a-b)^2+(c-a)(c-b)$, $(b-c)^2+(a-b)(a-c)$ and
$(c-a)^2+(b-a)(b-c)$.

2. $(a+b)^2+(b+c)^2+(c+a)^2-(a^2+b^2+c^2)$,
$3(a^2+b^2+c^2)-\{(a-b)^2+(b-c)^2+(c-a)^2\}$,
and $\frac{1}{2}\{a-b)^2+(b-c)^2+(c-a)^2+6(ab+ac+bc)\}$.

3. $ac(a-c)+ab(b-a)+bc(c-b)$, $a^2(b-c)+b^2(c-a)+c^2(a-b)$, and
$(a-b)(b-c)(c-a)$.

4. $(a+b+c)(ab+ac+bc)-abc$, and $(a+b)(b+c)(c+a)$.

5. $a(b-c)^3+b(c-a)^3+c(a-b)^3$, and $(a-b)(b-c)(c-a)(a+b+c)$.

6. $a(b+c)^2+b(c+a)^2+c(a+b)^2-(a+b)(b+c)(c+a)$, and
$a(b+c-a)^2+b(c+a-b)^2+c(a+b-c)^2+(a+b-c)(b+c-a)(c+a-b)$.

7. $\frac{1}{2}\{(a+b-c)(a+b)+(c+a-b)(a+c)+(b+c-a)(b+c)\}$.
$\frac{1}{4}\{(a+b+c)^2+(a+b-c)^2+(a+c-b)^2+(b+c-a)^2\}$, and
$2(ab+ac+bc)-(a+b-c)(a+c-b)-(b+c-a)(a+b-c)$
$-(a+c-b)(b+c-a)$.

8. $\{a(a-b)+b(b-c)+c(c-a)\}(a+b+c)$, and
$\frac{1}{2}\{(a+b)^3+(b+c)^3+(c+a)^3-3(a+b)(b+c)(c+a)$.

9. $(a^2+b^2+c^2)^3+2(bc+ac+ab)^3-3(a^2+b^2+c^2)(bc+ac+ab)^2$,
$(a^2-bc)^3+(b^2-ac)^3+(c^2-ab)^3-3(a^2-bc)(b^2-ac)(c^2-ab)$, and
$(a^2+2bc)^3+(b^2+2ac)^3+(c^2+2ab)^3-3(a^2+2bc)(b^2+2ac)(c^2+2ab)$.

10. $(a+b+c)(a+b-c)(a+c-b)(b+c-a)$, $4a^2b^2-\{c^2-(a^2+b^2)\}^2$,
$a^2(b^2+c^2-a^2)+b^2(a^2+c^2-b^2)+c^2(a^2+b^2-c^2)$,
$4(a^2b^2+a^2c^2+b^2c^2)-(a^2+b^2+c^2)^2$, $(a^2+b^2)^2-(a^2-b^2)^2-(a^2+b^2-c^2)^2$.
and $2a^2b^2+2a^2c^2+2b^2c^2-a^4-b^4-c^4$.

XIV.

Verify the following equivalent expressions:—

1. $(1+x)^2(1+y^2)-(1+x^2)(1+y)^2=2(x-y)(1-xy)$.

2. $(2x+3y)^3+(3x+2y)^3=5(x+y)(7x^2+11xy+7y^2)$.

3. $(a-b)(x-a)(x-b)+(b-c)(x-b)(x-c)+(c-a)(x-c)(x-a)$
$=(a-b)(b-c)(c-a)$.

4. $(x+b)(x+c)-(a+b+c)(x+b)+a^2+ab+b^2+3ax$
$=(x+a)^2-(a-b)c$.

5. $(xy-a^2)^2+(ay-bx)(ax-cy)=(ca-x^2)(ab-y^2)+(bc-a^2)(xy-a^2)$.

6. $(a+b+c)(x+y+z)+(a+b-c)(x+y-z)+(b+c-a)(y+z-x)$
$+(c+a-b)(z+x-y)=4(ax+by+cz)$.

7. $(x+y+z-xyz)^2+(xy+yz+zx-1)^2=(1+x)^2(1+y^2)(1+z^2)$.

8. $x(y+z)^2+y(x+z)^2+z(x+y)^2-4xyz=(y+z)(z+x)(x+y)$.

9. $(1+xz)^2(1+yz)^2-\{(1-xz)(1-yz)+2xyz\}^2$
$=4(x+y-xy)(xyz^3+xyz^2+z)$.

10. $(xyz+x^2y-y^2z+z^2x)^2+(xyz+xy^2+yz^2-zx^2)^2$
$=(x^2+y^2)(y^2+z^2)(z^2+x^2)$.

11. $(x^2-1)(y^2-1)(z^2-1)+(x+yz)(y+xz)(z+xy)$
$=(xyz+1)(x^2+y^2+z^2+2xyz-1)$.

12. $(y^2-zx)(z^2-xy)+(z^2-xy)(x^2-yz)+(x^2-yz)(y^2-zx)$
$-(x^2-yz)^2-(y^2-zx)^2-(z^2-xy)^2=(x+y+z)(3xyz-x^3-y^3-z^3)$.

13. $(x^2+y^2)(z^2+x^2)(y-z)+(z^2+y^2)(x^2+y^2)(z-x)$
$+(x^2+z^2)(y^2+z^2)(x-y)=x^4(y-z)+y^4(z-x)+z^4(x-y)$.

14. $(y-z)^5+(z-x)^5+(x-y)^5$
$=5(y-z)(z-x)(x-y)\{x^2+y^2+z^2-yz-zx-xy\}$.

15. $(y-z)^6+(z-x)^6+(x-y)^6=3(y-z)^2(z-x)^2(x-y)^2$
$+2(x^2+y^2+z^2-xy-yz-xz)^3$.

XV.

1. If $A=ax+by$, and $B=bx-ay$, shew that
$aA^2+bAB+cB^2=(a+b+c)b^2x^2+(a^2-2ac+b^2)bxy+a^2cy^2$.

2. If $A=a^2-bc$, $B=b^2-ac$, $C=c^2-ab$; prove that
$(A^2-BC)bc=(B^2-AC)ac=(C^2-AB)ab=abc(a+b+c)(A+B+C)$.

3. If $a^2+b^2+c^2=A$, $ab+ac+bc=B$; shew that
$A^3+2B^3-3AB^2=(a^3+b^3+c^3-3abc)^2$.

4. Given $a+b+c+d=A$, $a+b-c-d=B$, $a-b+c-d=C$,
$a-b-c+d=D$;
shew that $AB(A^2+B^2)=CD(C^2+D^2)$, if $ab(a^2+b^2)=cd(c^2+d^2)$.

5. If $a+b+c+d=0, A=bcd$, $B=cda$, $C=dab$, $D=abc$; then shall
$BCD+CDA+DAB+ABC=0$.

6. If $bc-d^2=A$, $ca-e^2=B$, $ab-f^2=C$, $ef-ad=D$, $fd-be=E$,
$de-cf=F$; then shall
$AD^2+BE^2+CF^2=ABC-2DEF-(Aa+eE+fF)^2$.

7. If $X=ax+by+cz$, $Y=cx+by+az$, and $Z=bx+ay+cz$, prove
that $(a^3+b^3+c^3-3abc)(x^3+y^3+z^3-3xyz)=X^3+Y^3+Z^3-3XYZ$.

8. If $X=ax+cy+bz$, $Y=cx+by+az$, $Z=bx+ay+cz$; shew that $X^2+Y^2+Z^2-YZ-ZX-XY$ $=(a^2+b^2+c^2-bc-ac-ab)(x^2+y^2+z^2-yz-zx-xy)$.

9. If the six equivalents, $a=xX$, $b=yY$, $c=zZ$, $2A=yZ+zY$, $2B=zX+xZ$, $2C=xY+yX$, be simultaneously true, then shall $aA^2+bB^2+cC^2=2ABC+abc$.

XVI.

If $2s=a+b+c$, $2\sigma^2=a^2+b^2+c^2$, and $2\sigma^3=a^3+b^3+c^3$; shew the truth of the following equivalents:—

1. $s^2+(s-a)^2+(s-b)^2+(s-c)^2=2\sigma^2$.
2. $(s-a)^3+(s-b)^3+(s-c)^3+3abc=s^3$.
3. $(s-a)(s-b)(s-c)=s^3-s\sigma-abc$.
4. $(b+c)s(s-a)+a(s-b)(s-c)-2bcs$ $=(c+a)s(s-b)+b(s-c)(s-a)-2cas$ $=(a+b)s(s-c)+c(s-a)(s-b)-2abs$.
5. $s(s-b)(s-c)+s(s-c)(s-a)+s(s-a)(s-b)$ $-(s-b)(s-c)(s-a)=abc$.
6. $\{(s-a)(s-b)(s-c)\}^2+\{s(s-b)(s-c)\}^2+\{s(s-c)(s-a)\}^2$ $+\{s(s-a)(s-b)\}^2+2s\sigma^2s(s-a)(s-b)(s-c)=a^2b^2c^2$.
7. $(\sigma^2-a^2)(\sigma^2-b^2)+(\sigma^2-b^2)(\sigma^2-c^2)+(\sigma^2-a^2)(\sigma^2-c^2)$ $=4s(s-a)(s-b)(s-c)$.
8. $(s-a)(s-b)+(s-b)(s-c)+(s-c)(s-a)=s^2-\sigma^2$.
9. $(\sigma^3-a^3)(s-a)+(\sigma^3-b^3)(s-b)+(\sigma^3-c^3)(s-c)=a^4+b^4+c^4-s\sigma^3$.

XVII.

1. If $bz-cy=p$, $cx-az=q$, $ay-bx=r$; then $ap+bq+cr=0$.
2. If $a=y+z-2x$, $b=z+x-2y$, $c=x+y-2z$; find the value of $b^2+c^2+2bc-a^2$ in terms of x, y, z.
3. If $3x=-p+2q+2r$, $3y=2p-q+2r$, $3z=2p+2q-r$; shew that $x^2+y^2+z^2=p^2+q^2+r^2$, and $xy+yz+xz=pq+pr+qr$.
4. If $p=b+c+d-a$, $q=a+c+d-b$, $r=a+b+d-c$, $s=a+b+c-d$; shew that $p(q+pr+ps)+q(a+qs+rs)=4(ab+ac+ad+bc+bd+cd)$.
5. If $(x-y)z^3=c^3$, $(y-z)x^3=a^3$, $(x-z)y^2=b^3$, $(x-y)(y-z)(z-x)=3abc$, then shall $a^3+b^3+c^3-3abc=0$.
6. If $2a=x+z$, $2b=z+x$, $2c=x+y$, find the value of the expression $a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2$ in terms of x, y, z, and express $(x+y+z)(xy+xz+yz)-xyz$ in the form of factors involving a, b, c.
7. $y^3+z^3+m(y+z)=z^3+x^3+m(z+x)=x^3+y^3+m(x+y)$, x, y, z being unequal, then each expression is equal to $2xyz$.
8. If $a^2=y+z$, $b^2=x+z$, $c^2=y+x$, and $2s=a+b+c$, shew that $s(s-a)(s-b)(s-c)=4(xy+xz+yz)$.
9. Shew that $(a+b+c-d)(a+b+d-c)(a+c+d-b)(b+c+d-a)$ $=16(s-a)(s-b)(s-c)(s-d)$; if $2s=a+b+c+d$.
10. If $a+b+c=3s$, shew that $(s-a)^4+(s-b)^4+(s-c)^4$ $=2\{(s-b)^2(s-c)^2+(s-c)^2(s-a)^2+(s-a)^2(s-b)^2\}$.

11. If $x+y=p$ and $xy=q$, find x^2+y^2, x^3+y^3, x^4+y^4, &c., in terms of p and q.

12. If $a^2+b^2=c^2$, the product $(a+b+c)(a+b-c)(a+c-b)(b+c-a)$ is equivalent to $4a^2b^2$.

13. If $x^2-yz=a$, $y^2-xz=b$, and $z^2-xy=c$; then shall $x^3+y^3+z^3-3xyz=ax+by+cz$.

XVIII.

1. If $x(1+y)=1$ and $y(1+z)=z$, then $-z=1+x+2x^2+4x^3+\ldots.$

2. If $ay+bx=a$, $by-ax=b$, then $x^2+y\ =1$.

3. If $x^2y=z(x+y-z)^2$, then shall $(x-z)^2=yz$.

4. If $x+y+z-xyz=2$, then shall $(1-x)^2=(1-xy)(1-xz)$.

5. If $b(bx^2+a^2y)=a(ay^2+b^2z)$, then shall $bx+ay=ab$, and $ay=bx$.

6. If $(a+b-c-d)x=cd-ab$, then $(a+x)(b+x)=(c+x)(d+x)$.

7. If $ax+by=1$, then $ab(x^2+y^2)+(a^2+b^2)xy+(a-b)(x-y)=1$.

8. If $(a^2-bc)x+(b^2-ca)y+(c^2-ab)z=0$ and $x+y+z=0$, prove that $ax+by+cz=0$.

9. If $(a+bc)^2(1-a^2)=(b+ac)^2(1-b^2)$, then $a^2+b^2+c^2+2abc=1$.

10. If $x^3+y^3=z^3$, prove that $(x^3+z^3)^3y^3+(x^3-y^3)^3z^3=(y^3+z^3)^3x^3$.

11. If $a+b+c=0$, then $6(a^5+b^5+c^5)=5(a^3+b^3+c^3)(a^2+b^2+c^2)$, $4(a^7+b^7+c^7)=7(a^4+b^4+c^4)abc$, and $(a^3+b^3+c^3)^3=27a^3b^3c^3$.

12. If $ax^2+b^2y-c^3=0$, $ay^2+b^2x-c^3=0$ and $x+y-c=0$, then $b^2=ac$.

13. If $(2a-3y)y=(z-x)^2$ and $(2a-3z)z=(x-y)^2$, then shall $x+y+z=a$ and $(2a-3x)x=(y-z)^2$.

14. If $a+b+c=0$, and $a(by+cz-ax)=b(cz+ax-by)=c(ax+by-cz)$, then will $x+y+z=0$.

15. If $(by-cx)^2=(b^2-ac)(y^2-cz)$, then shall $(bx-ay)^2=(b^2-ac)(x^2-az)$.

16. If $a+b+c+d=0$, then shall $a^4+b^4+c^4+d^4+4abcd$ $=2\{(ab-cd)^2+(ac-bd)^2+(ad+bc)^2\}$.

17. If $(yz-x)^2+(zx-y)^2+xy-z)^2=(yz-x)(zx-y)(xy-z)+4$, then $x^2+y^2+z^2=xyz+4$; and conversely.

XIX.

1. If a be greater than b, a^n-b^n is greater than $nb^{n-1}(a-b)$ but less than $na^{n-1}(a-b)$.

2. Write the pth term of the quotient of $a^{mn}-b^{mn}$ when divided by a^n-b^n.

3. Shew that a^m-a^n is divisible by $a+1$, when $m-n$ is even; a^m+a^n is divisible by $a+1$ when $m-n$ is odd; and that a^m-a^n is divisible by $a-1$ when $m-n$ is even or odd.

4. Prove that $a^{mn}-1$ is always divisible by a^n-1 and by x^m-1; and write the middle terms of the quotients when m is odd, and when m is even.

5. Determine the number of factors by which $x^{mnp}-1$ is divisible,

when m, n, p are each odd and even numbers; and state the number of terms in each quotient.

6. Shew that $2^{64} = 1+(1+2)(1+2^2)(1+2^4)(1+2^8)(1+2^{16})(1+2^{32})$.

7. Shew that $x^n - na^{n-1}x + (n-1)a^n$ is divisible by $(x-a)^2$, without performing the operation; and write the exact quotient of $x^5-5a^4x+4a^5$ divided by $(x-a)^2$.

8. Prove that $(a^{n-1}-b^{n-1})(a^n-b^n)(a^{n+1}-b^{n+1})$ is divisible by $(a-b)(a^2-b^2)(a^3-b^3)$.

XX.

1. Shew by means of division if $x^2+axy+y^2+bx+cy$ is resolvable into two rational factors, that $abc = b^2+c^2$.

2. Write the coefficient of x^4 in the product of $x^4-ax^3+a^2x^2-a^3x+a^4$, x^2+ax+a^2 and x^2-ax+a^2.

3. How many factors of the form $x+a$, and of the form $x-b$, are contained in the product $x^5-11x^4+33x^3-11x^2-154x+120$?

4. Divide x^3+px^2+px+1 and $x^5+px^4+qx^3+qx^2+px+1$ by $x+1$ respectively, employing any artifice to save the trouble of formal division.

5. What is the criterion by which it is known that an Algebraical polynomial with integral coefficients, and arranged according to descending powers of x, is divisible (1) by $x+1$ and (2) by $x-1$? Shew that $x^5+3x^4-x^3-x^2-2$ is divisible both by $x+1$ and by $x-1$, and write the quotient.

6. Shew that $x^4-px^3+qx^2-rx+s$ is exactly divisible by $x-a$, if $a^4-pa^3+qa^2-ra+s=0$.

7. Divide $x^5-px^4+qx^3-rx^2+sx-t$ by $x-a$ as far as five terms of the quotient, and prove the correctness of the quotient by multiplying the divisor and quotient together and adding the remainder.
Also write the quotient and remainder, (if any)
when $x^5-8x^4+12x^3-18x^2+20x-30$ is divided by $x-4$.

8. Shew when $a_0+a_1x+a_2x^2+a_3x^3+ \; . \; . \; . \; . \; +a_nx^n$, can be divided by $1 \pm x$.

9. If $x^n+p_1x^{n-1}+p_2x^{n-2}+ \; . \; . \; . \; . \; +p_n$, be divisible by $x-a$, without remainder, shew that $a^n+p_1a^{n-1}+p_2a^{n-2}+ \; . \; . \; . \; . \; +p_n$ is equal to zero.

10. If r_1, r_2, r_3 . . . . . r_n be the successive remainders when $x^n+p_1x^{n-1}+p_2x^{n-2}+ \; . \; . \; . \; . \; +p_n$ is divided by $x+a$, n times in succession; then shall $x^n+p_1x^{n-1}+p_2x^{n-2}+ \; . \; . \; . \; . \; +p_n$
$= (x+a)^n+r_n(x+a)^{n-1}+r_{n-1}(x+a)^{n-2}+ \; . \; . \; . \; +r_1$,
Ex. $x^8+5x^6+6x^4+5x^2+1 = (x^2+2)^4-3(x^2+2)^3+9(x^2+2)-9$.

11. If there be a series of factors $x+y$, $x+y-1$, $x+y-2$, &c., and if the second factor be written in the forms $(x-1)+y$ and $x+(y-1)$, then the product of the first two factors will take the form of

$$x(x-1)+2xy+y(y-1).$$

If the third factor be written in the forms $(x-2)+y$, $(x-1)+(y-1)$, and $x+(y-2)$; then the product of the first three factors takes the form of

$$x(x-1)(x-2)+3xy(x-1)+3xy(y-1)+y(y-1)(y-2).$$

Generalise this property.

XXI.

1. If a, b, c be in order of magnitude, ascertain when $ab+ac+bc$ is greater or less than each of the quantities $a(a+b+c)$, $b(a+b+c)$ or $c(a+b+c)$.

2. If $a+b+c=0$ and $x+y+z=0$, then $a^2yz+b^2xz+c^2xy$ will always be a positive quantity.

3. If a, b, c be three unequal positive numbers in order of magnitude, and x, y, z other three; it is impossible that $ay-bx$, $bz-cy$, and $cx-az$ can be all positive.

4. Shew that a^2+b^2 is greater than $2ab$, and $(a+b)^2$ than $4ab$.

5. Is $ab+ac+bc$ greater or less than $a^2+b^2+c^2$?

6. Shew that $(a+b-c)^2+(a+c-b)^2+(b+c-a)^2 > ab+ac+bc$.

7. If a, b, c and $a^2b+ab^2+a^2c+ac^2+b^2c+bc^2-a^3-b^3-c^3-2abc$ be positive, then shall $b+c>a$, $a+b>c$ and $b+c>a$.

8. If $a>b$, then $(a^2+b^2)^2>2ab(x^2-ab+b^2)$.

9. If $a>b$ and $b>c$, then $(a+b+c)(ab+ac+bc)>9abc$ and $(a+b)(b+c)(c+a)>8abc$.

10. Shew that abc is greater than $(a+b-c)(a+c-b)(c+b-a)$.

11. Shew that $a^3+b^3>a^2b+ab^2$ and $a^5+b^5>a^3b^2+a^2b^3$; and generally that $a^{m+n}+b^{m+n}>a^mb^n+a^nb^m$.

12. Is $a^6+a^4b^2+a^2b^4+b^6$ always greater than $(a^3+b^3)^2$?

13. Shew that $a^3+b^3+c^3>3abc$, and $a^4+b^4+c^4>abc(a+b+c)$.

14. Is $(x^2+y^2)(a^2+b^2)>(ax+by)^2$?

15. Which is the greater $(x+a)^2(x^2+b^2)$ or $(x+b)^2(x^2+a^2)$?

16. If $x^2=a^2+b^2$, $y^2=c^2+d^2$, then $xy>ac+bd$ unless $ad=bc$; and $xy>ad+bc$, unless $ac=bd$.

17. Shew that $(a^2+b^2+c^2)(x^2+y^2+z^2)$ is greater than $(ax+by+cz)^2$.

18. Shew that $(x+y+z)^3>27xyz$.

19. Shew that (1) $(x+y-z)^2+(x+z-y)^2+(y+z-x)^2>x^2+y^2+z^2$.
(2) $x^3+y^3+z^3>\frac{1}{2}\{x^2(y+z)+y^2(x+z)+z^2(x+y)\}$.
(3) $(x+y-z)^3+(x+z-y)^3+(y+z-x)^3>x^3+y^3+z^3$.

20. If x, y, z be real quantities in order of magnitude, then shall
(1) $x^2(y-z)+y^2(z-x)+z^2(x-y)$ be positive.
(2) $(y-z)(z-x)+(z-x)(z-y)+(x-y)(y-z)$ be negative.
(3) $x^2(x-y)(x-z)+y^2(y-z)(y-x)+z^2(z-x)(z-y)$ be positive.

RESULTS, HINTS, ETC., FOR THE EXERCISES ON ADDITION AND SUBTRACTION.

I.

1. See Section I., p. 12. 2. Section II., pp. 1, 4, 5. 3. Section II., p. 2, &c. 4. Section 1., pp. 5–8. 5. Section II., p. 16. 6. Section II., p. 30, &c. 7. Section I., pp. 7, 23. Section II., p. 22. 8. Section II., p. 28, &c. Section III., p. 8, &c.

II.

1. See Section IV., Art. 3, p. 2 : Art. 9, p. 10 : and Art. 7, p. 6.

2. Two numbers are equal when each of them consists of the same number of units. When geometrical magnitudes are equal, see School Euclid, Notes on the Axioms, p. 46.

3. The sum is $+12a$; the difference $+7a$.

4. The sum is $-12a$; the difference $-7a$.

5. The sum is $+2a$; the difference $+7a$.

6. The sum is $+2a$; the difference $+2b$.

8. See Section IV., Art. 9, p. 17.

9. 8. 10. $2(m+n+p)$.

11. $2ma$ the sum ; $2na$ the difference.

12. See Section IV., Art. 7, p. 5.

III.

1. $3a+2b+c$. 2. $-a+2b-c$. 3. $a-c$. 4. $-2a+c$. 5. $a-c$. 6. $b+d$. 7. $b-d$. 8. $a+b+c-d$. 9. $5(a+b)-15(c-d)$. 10. $-2(a+c-d)$. 11. $9b$. 12. $10a-2c$.

IV.

1. $2ax$. 2. $5a+b+13c$. 3. $2a-2b+y$. 4. $(c+2b-a)x+2by$. 5. $2bx$. 6. $\{(a+b-c)x+(b+c-a)y+(a+c-b)z\}$.

V.

1. $8x-y$. 2. $-2(x-y-z)$. 3. $-(x+y+z)$. 4. $x-5y$. 5. $53x-2y-4z$. 6. $3x+7y+11z$.

VI.

1. Here $a > b$ and $b > c \therefore a > c$. Hence $a-c > 0$, $b-c > 0$, $a-c > 0 \therefore a-c$, $b-c$, $a-c$ are positive ; and $c-a$, $c-b$, $b-a$ are negative.

2. See Section IV., Art. 6. 3. $+(a-3b+3c)$ and $-(3b-a-3c)$.

4. The sums and differences are $+a-a$, $+a+b$, $+a-b$, $-a+b$, $-a-b$, $+b-b$.

5. The aggregates are, $+a+b+c$, $-a-b-c$, $+a-b-c$, $+b-a-c$, $+c-a-b$, $a+b-c$, $a+c-b$, $b+c-a$.

6. If x, y denote the two quantities, the quotient is $\frac{xy}{(x+y)(x-y)}$.

7. $x^4+32x^2-320x-1024$. 8. The aggregate is 0.

9. $(a^2-ab-b^2)x+(a^2+ab+b^2)y+(b^2-ab-a^2)z$.

10. $(a_1+a_2+a_3-a_4)x+(a_1+a_2+a_4-a_3)y+(a_1+a_3+a_4-a_2)z$.

11. See Section lV., Art. 3, p. 2 ; Art. 9, p. 9

VII.

1. The sum is $2(a+b+c)$, and since $3a=5$, $6b=5$, $12c=5$; $a=\frac{5}{3}$, $b=\frac{5}{6}$, $c=\frac{5}{12}$, the value is $5\frac{5}{6}$.

2. Apply Axiom 1.

3. The simplest form is $9x-14$, and the least integral value of x, which renders the result negative, is unity.

4. Sum $7x+5y+6z$; difference $4x-2y-5z$; sum of results $11x+3y+z$, and the value is 1410.

5. The sum is $5x+13y+6z$; the value is $6{\cdot}306x$.

6. The sum is $9(x+y+z)$, and the value $9{\cdot}99$.

8. The values are ·584192 and ·583808.

9. $-x^3+8x^2y+7xy^2+7y^3$.

10. $px^6-(q+a)x^5+(r-b)x^4-(s+c)x^3+(t-d)x^2-(v+e)x$.

11. See Section IV., Art. 7, p. 6. Note.

RESULTS, HINTS, ETC., FOR THE EXERCISES ON MULTIPLICATION AND DIVISION.

I.

2. See Section IV., Art. 7, p. 7.
3. See Section IV., Art. 3, p. 4; Art. 5, p. 5.
4. See Section IV., Art. 9, p. 9; Art. 10, pp. 14, 15.
5. See Section IV., Art. 12, pp. 15, 16.
6. See Section IV., Art. 14, pp. 24, 25.
7. $20abcdxy$: $-mnxbc$: $-60a^3b^3c^3d^3$: $+96a^3b^3c^3d^3$.
8. $5abx$: $-nc$: $5ab^2d^3$: $-12a^2bc^3$.
9. $27a^2+6ax$: $-12a^2b+24ab^2$: $-ax^3+a^2x^2-a^3x$:
   $-18a^4b^2c^2+24a^3b^3c^2+30a^2b^4c^2$: $ma^2x^2yz-na^2xy^2z+pa^2xyz^2$:
   $-a^2x^4y^2z^2+b^2x^2y^4z^2-c^2x^2y^2z^4$.
10. $9a+2x$: $4a+8b$: x^2-ax+a^2 : $3a^2-4ab-5b^2$: $max-nay+paz$: $-a^2x^2+b^2y^2-c^2z^2$.
11. See Section IV., Art. 12., pp. 17, 18.
12. See Notes on Euclid II., 1–10, of the Editor's Edition of Euclid's Elements.
13. See Section X., Art. 13, p. 20.

II.

The following are the products :—

1. $4x^2+12xy+9y^2$: $4x^2-9y^2$: $4x^2-12xy+9y^2$.
2. $x^3+7x^2+17x+35$: $6x^3-19x^2y+29xy^2-21y^3$.
3. $a^4+a^2x^2+x^4$. 4. $16a^4-81b^4$. 5. a^4+4b^4.
6. $35x^5-49x^4+19x^3-24x+64$. 7. $x^6-12x^4-576x^2-3600$.
8. $x^6-5ax^5+6a^2x^4+6a^3x^3-4a^3x^2-4a^4x$.
9. $243x^5-y^5$. 10. $x^8+x^6+x^4+x^2+1$.
11. $a^8+2a^6+3a^4+2a^2+1$. 12. $1-7x+21x^2-35x^3+40x^4-21x^5+7x^6-x^7$.
13. $x^{-6}+7x^{-4}-64$. 14. $x^{-4}-y^{-4}$. 15. $6x^{-10}+5x^{-9}y^{-1}-6x^{-8}y^{-2}$.
16. $5x^6y^{-4}-12x^4y^{-2}+11x^2-7y^2+4x^{-2}y^4-x^{-4}y^6$.

III.

The quotients are—

1. $x^3+6x^2+11x+6$. 2. x^2-x+1. 3. $16x^2-4xy+y^2$.
4. $a^4+2a^3b+4a^2b^2+8ab^3+16b^4$. 5. x^2-xy+y^2.
6. $3x^3-2x^2-5x-3$. 7. $4x^2-5xy+2y^2$.
8. $7x^3-3x^2y+4xy^2-8y^3$. 9. $2a^2-3ab+5b^2$.
10. $x^5-2x^4+4x^3-5x^2+10x-20$, and remainder $+49$.
11. $x^4-4x^3+6x^2-4x+1$. 12. $x^3+x^2y+xy^2+y^3$.
13. $x-1+x^{-1}$. 14. $ax^{-1}-b-b^{-2}x$. 15. $2x^2-3xy+2y^2$.

IV.

The quotients are—

1. $a^2-2ab+b^2$. 2. $x^2-(a-2b)x+(a^2+3b^2)$. 3. $(x+y)^2+(x+y)-3xy+1$.
4. $a^2+b^2+c^2-ab-ac-bc$. 5. x^2+ax+b. 6. $(a^2+a+1)x-(a+1)$.
7. $a^2+b^2+c^2$. 8. $(a-b)x-(a-b)y$. 9. $x^2+y^2+z^2+1$. 10. $a^2-2ab+3b^2$.
11. $(4b-3c)x^2+(5b^2-3c^2)x$. 12. $a^4-5a^3b+5ab^3-b^4$. 13. $x^2-(a+b)x+ab$.
14. a^2-9x^2. 15. x^2+xa+a^2.

16. Arrange the dividend in ascending powers of x, making $(y+z)-x$ the divisor; perform the division and verify the result. The quotient is—

$$2(y-z)yz+x(y^2-z^2)+x^2(y-z).$$

17. Proceed as in the last example.

18. a. 19. $2(a+b)x$.

V.

1. $81^2-79^2=(81+79)(81-79)=160\times2=320$.
$(121\tfrac{3}{4})^2-(118\tfrac{1}{4})^2=(121\tfrac{3}{4}+\tfrac{1}{4})(121\tfrac{3}{4}-118\tfrac{1}{4})=122\times3\tfrac{1}{2}=427$.

2. $\left\{\tfrac{1}{2}(a+b)\right\}^2-\left\{\tfrac{1}{2}(a-b)\right\}^2=ab$. The product of any two numbers is equal to the square of half their difference subtracted from the square of half their sum.

3. x^2+1. 4. $x=9$.

5. $(a^3+b^3)^2(a^2+b^2)^3=a^{12}+3a^{10}b^2+2a^9b^3+3a^8b^4+6a^7b^5+2a^6b^6+6a^5b^7$ $+3a^4b^8+2a^3b^9+3a^2b^{10}+b^{12}$.

6. $(x+a)^2$. 8. $2{\cdot}05x^2+x-{\cdot}3$. 9. $(a^2+ab+b^2)+3k(a+b)+3k^2$.

10. -216. 11. $2a^2-3ab+4b^2$.

12. $(mx+a)(nx+b)=mnx^2+(mb+na)x+ab$
$(mx-a)(nx-b)=mnx^2-(mb+na)x+ab$
$(mx+a)(nx-b)=mnx^2+(mb-na)x-ab$
$(mx-a)(nx+b)=mnx^2-(mb-na)x-ab$.

As an example, take $(3x\pm5)(5x\pm3)$.

13. (1) The quotient is $ax+b$, the remainder $(c-ab^2)a+(d-b^3)$, of which each part must be equal to zero; or $c-ab^2=0$, and $d-b^3=0$, whence $d=b^3$ and $c=cb^2$ and $ad=bc$.

(2) The quotient is $ax^2-abx+b$, and the remainder $(ab^3-b^2)x+(c-b^3)$; of which each part must vanish; or $ab^3-b^2=0$, $c-b^3=0$, whence $b^3=c$, and $ac-b^2=0$, or $b^2=ac$.

(3) After the quotient $x^3+2cx^2+3c^2x+4c^3$ is found, the remainder is $-5(q-c^4)x+4(r-c^5)$, which must be equal to 0, and consequently $q-c^4=0$ and $r-c^5=0$

$$\therefore c^4=q \text{ and } c^5=r, \therefore c^{20}=q^5 \text{ and } c^{20}=r^2 \therefore q^5=r^4.$$

14. In the product make such substitutions as will make each of the three factors, identical to $x+y$.

VI.

1. $x^4-4x^3-9x^2+13x+18$.

2. $x^5+25x^4+254x^3+1307x^2+3475x+3552$, and $x^5-5x^4+14x^3-25x^2+23x-6$.

3. The product is $x^4+(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2+(abc+abd+acd+bcd)x+abcd$.

The coefficient of x in the required product is $2.6.10+2.6.14+2.10.14+6.10.14=1408$.

4. $-4x^2y(x-y)$.

5. The differences are each $16x(x^2+1)$.

6. By aid of $a^2-b^2=(a+b)(a-b)$; the dividend is equal to $5(x^2+x+1)(x^2-13x-1)$.

7. x^2+xy+y^2. 8. $x^2-8xy-15y^2$.

9. The coefficient of x^3 is $+193$, and of x^5 is $+83$.

10. $4x^2+8xy+4y^2-6xz+3yz+9z^2$.

This may be found by inspection with the aid of the quotient $(a^3+b^3)\div(a+b)$.

VII.

The products under 1, 2, 3 may be inferred from the form $(a+b)(a-b)=a^2-b^2$.

1. $a^2+b^2-c^2+2ab$; $c^2-a^2-b^2+2ab$; $a^2+c^2-b^2+2ac$; $b^2+c^2-a^2+2bc$; $a^2-b^2-c^2+2bc$; $b^2-a^2-c^2+2ac$.

2. $a^2+b^2+c^2-d^2+2ab+2ac+2bc$; $a^2+d^2-b^2-c^2+2ad+2bc$; $a^2+d^2-b^2-c^2-2ad+2bc$; $a^2+c^2-b^2-d^2+2ac-2bd$; $d^2-a^2-b^2-c^2+2ab+2ac-2bc$; $b^2+d^2-a^2-c^2-2bd-2ac$.

3. $a^4+a^2b^2+b^4$. 4. $1-(a+b+c)x+(ab+ac+bc)x^2-abcx^3$.

VIII.

1. $(a-c)(a+c-2)$. 2. $(a+b)(b+c)(c+a)$. 3. $(a+b)(b+c)(c+a)$.

4. $(a+b+c-d)(a-b-c-d)$.

5. $(a+b+c-d)(a+b+d-c)(c+d+a-b)(c+d+b-a)$.

6. $(x+1)(x-2)(x-3)(x-4)$.

7. $(a+b+c+d)(a+d-b-c)(b+c-a-d)(b-c-a+d)$.

8. $(x+a)(x+b)(x^2-ax+a^2)$. 9. $x(x-2a)(x^2-ax+a^2)(x^2-3ax+3a^2)$.

10. $(1+y)^2(1-x)^2(1+x)^2$. 11. $\{(a+1)x-(b-1)y\}(x^2-xy+y^2)$.

12. $(abc+bcd+cad+abd)^2=\{ac(b+d)+bd(a+c)\}^2$
$=a^2c^2(b+d)^2+2abcd(b+d)(a+c)+b^2d^2(a+c)^2$,
and $\{(b+d)+(a+c)\}^2abcd=abcd(b+d)^2+2abcd(b+d)(a+c)+abcd(a+c)^2$,
$\therefore (abc+bcd+cad+abd)^2-(a+b+c+d)^2abcd$
$=ac(b+d)^2(ac-bd)-bd(a+c)^2(ac-bd)=(ac-bd)\{ac(b+d)^2-bd(a+c)^2\}$
$=(ac-bd)(ab-cd)(bc-ad)$.

IX.

1. $(a+b)(a+c)=a^2+(b+c)a+bc$
$(d+b)(d+c)=d^2+(b+c)d+bc$
$\therefore (a+b)(a+c)-(d+b)(d+c)=(a^2-d^2)+(b+c)(a-d)$, which is divisible by $a-d$.

2. The dividend can be arranged thus: $c(a^2-b^2)-ab(a-b)-c^2(a-b)$ which is obviously divisible by $a-b$.

3. The dividend is the difference of two squares, which may be expressed as the product of the sum and difference of the quantities. The quotient is $(a-b)x+(b-c)y+(c-a)z$.

4. By adding $abc - abc$ to the dividend, it becomes by arrangement of the terms $ab(a-b+c)+ac(a-b+c)+bc(a-b+c)$ which is obviously divisible by $a-b+c$.

5. The dividend is $(a^4-b^4)(a^4+a^2b^2+b^4)$ and the divisor is $a^4+c^2b^2+b^4$.

6. The quotient is $a^2+b^2+c^2-ab-ac-bc$.

Deduce from this quotient, the three quotients (1) when $-a$ is written for $+a$, (2) $-b$ for $+b$, and (3) $-c$ for $+c$: and shew that the sum of the four quotients is $4(a^2+b^2+c^2)$.

7. The dividend when reduced is $2(a^3+b^3+c^3-3abc)$.

8. $a^2+b^2+c^2+d^2-ab-ac-ad-bc-bd-cd$.

9. By a like artifice as in 4, it is shewn the quotient is $ab-bc+ac$.

10. The quotient is $(a-b)(a+b+c)$.

11. The first dividend involves the factor $b-c$ which is common to the divisor, which may be removed, and the quotient is $-(ab+ac+bc)$.

The second dividend, when the terms are expanded, will be found to contain the same factor as one of the factors of the divisor. This common factor being removed, the quotient is $5\{a^2-(b+c)a+(b^2-bc+c^2)\}$.

12. $-c^3+(a+b)c^2+(a^2-4ab+b^2)c-(a+b)(a-b)^2$.

X.

3. See Diophantus III. 22.

The four following formulæ are extensions of 3 :—

XI.

1. $(b+a)(b+c)=b^2+(a+c)b+ac$

$(d+c)(d+a)=d^2+(a+c)d+ac$

$\therefore (b+a)(b+c)-(d+c)(d+a)=(b^2-d^2)+(a+c)(b-d)$

$\therefore (b+a)(b+c)-(d+c)(d+a)-(a+c)(b-d)=b^2-d^2$.

2. $(a+b)^2+2(a-b)(c-d)+(c+d)^2$. 3. $(a+b+c+d)^2+2(a^2+b^2+c^2+d^2)$.

4. $4(a^2+b^2+c^2+d^2)$. 5. $(a-b)^3$. 6. 0. 7. a^2-d^2.

8. Bracket $a+b$ and $c+d$, so as to make each factor a binomial. In two lines the aggregate is shewn to be $ab+cd$.

9. $(ac+bd)(bc+ad)$. 10. $(a^4-b^4)(c^4-d^4)$. 11. $(a^2+b^2)^2.(c^2+d^2)^2$.

12. $a^4+b^4+c^4+4abcd$.

13. The six binomials when expanded can be thus exhibited—

$$3a^3+3ab(a+b)=3a(a^2+b^2)+3a^2b$$
$$3b^3+3bc(b+c)=3b(b^2+c^2)+3b^2c$$
$$3c^3+3cd(c+d)=3c(c^2+d^2)+3cd^2$$
$$3d^3+3da(d+a)=3d(d^2+a^2)+3d^2a$$
$$3ac(a+c)+3bd(b+c)$$

Next if $3a^2(a+c)+3bd(b+c)$ be combined with $3a^2b+3b_2c+3cd^2+3d^2a$
they become $3a(c^2+d^2)+3b(a^2+d^2)+3c(b^2+a_2)+3d(b_2+c^2)$
which when added to $3a(a^2+b^2)+3b(b^2+c^2)+3c(c^2+d^2)+3d(d_2+a^2)$
make up $3(a+b+c+d)(a^2+b^2+c^2+d^2)$.

14. $(2a)^3$. 15. $6abc$. 16. $24abc$.

17. There are twelve terms in this expression to be combined. First combine the fifth with the first and the sixth with the eighth, and then the two results.

Next combine the third and the seventh and the fourth and the twelfth.

Thirdly, reduce the results from the first and fifth and sixth and eighth, with that of the ninth, tenth, and eleventh. Then the results from the third and seventh, fourth and twelfth, and the second.

These results being one set positive and the other negative, when combined leave the remainder $24abcd$.

18. $(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$.

19. This reduction can be effected in the same manner as in 12.
The result is $60abcd(a+b+c+d)$.

XII.

The verifications of the expressions under this number may be readily effected by performing the indicated operations.

XIII.

The exercises under this number will occasion no difficulty. The identical expressions may be expanded or resolved into the same factors.

9. The three expressions under this number are each reducible to
$a^6+b^6+c^6-6(a^4bc+b^4ac+c^4ab)+2(a^3b^3+a^3c^3+b^3c^3)+9a^2b^2c^2$.

XIV.

The exercises under number require no directions further than to employ any methods whereby the labour of reducing the expressions may be lessened.

XV.

1. Make the proposed substitutions, which offer no difficulty.

2. Then $A+B+C=a^2+b^2+c^2-ab-ac-bc$
and $B^2-AC=b(b^3+c^3+a^3-3abc)=b(a+b+c)(a^2+b^2+c^2-ab-ac-bc)$
$\therefore \dfrac{B^2-AC}{b}=(a+b+c)(A+B+C)$, similarly $\dfrac{A^2-BC}{a}$ and $\dfrac{C^2-AB}{c}$ are equal to the same quantity;
therefore $\dfrac{B^2-AC}{b}=\dfrac{A^2-BC}{a}=\dfrac{C^2-AB}{c}=(a+b+c)(A+B+C)$
and $(B^2-AC)ac=(A^2-BC)bc=(C^2-AB)ab=abc(a+b+c)(A+B+C)$.

3. Here $A^3+2B^3-3AB^2=(A-B)^2(A+2B)$
$=(a^2+b^2+c^2-ab-ac-bc)^2(a+b+c)^2$.

4. In order that $AB(A^2+B^2)$ may be equal to $CD(C^2+D^2)$, $(a+b)^4-(c+d)^4$ must be equal to $(a-b)^4-(c-d)^4$, whence it may be shewn that $ab(a^2+b^2)=cd(c^2+d^2)$.

XVI.

The exercises under this head offer no difficulties.

XVII.

1. Apply axiom 1.

2. $b^2+c^2+2bc-a^2=(b+c)^2-a^2=(b+c+a)(b+c-a)$; substitute for a, b, c.

5. By adding the three equalities, $(x-y)(y-z)(z-x)=a^3+b^3+c^3$
and $(x-y)(y-z)(x-z)=3abc$. Therefore $a^3+b^3+c^3-3abc=0$.

6. $a^4+b^4+c^4-2b^2c^2-2a^2b^2-2a^2c^2=-\{2a^2b^2+2b^2c^2+2a^2c^2-a^4-b^4-c^4\}$

$$=-(a+b+c)(a+b-c)(a+c-b)(b+c-a)$$

And $2a+2b+2c=2x+2y+2z$, and $a+b+c=x+y+z$,
also $a+c-b=y$, $a+b-c=z$, $b+c-a=x$.

Hence the expression is equal to $-(x+y+z)xyz$ in terms of x, y, z;
and $(x+y+z)(xy+xz+yz)-xyz=(x+y)(x+z)(y+z)=2a.\ 2b.\ 2c=8abc$.

7. From the equality of the first and second terms, $-m=x^2+xy+y^2$, and from the equality of the second and third, $-m=z^2+yz+y^2$.

Hence $z^2+zy+y^2=y^2+xy+x^2$, and $(z^2-x^2)+y(z-x)=0 \therefore x+y+z=0$.

$$\begin{aligned}\therefore y^3+z^3+m(y+z)=(y+z)\{y^2-yz+z^2+m\}&=(y+z)(y^2-yz+z^2-z^2-yz-y^2)\\&=(y+z)(-2yz)\\&=(-x)(-2yz)=+2xyz.\end{aligned}$$

The same value may be found by a similar process for the other two expressions.

If mxy, &c., be substituted for $m(x+y)$, &c., in this exercise, so that the equivalents become $y^3+z^3+myz=z^3+x^3+mzx=x^3+y^3+mxy$, it may be shewn that $m=x+y+z$, and $x^2+y^2+z^2+yz+xz+xy=0$, also that each member of these equations is equal to $xyz-(x+y)(y+z)(z+x)$.

8. $s(s-a)(s-b)(s-c)=2a^2b^2+2b^2c^2+2a^2c^2-a^4-b^4-c^4$. Make the substitutions for a^2, b^2, c^2.

11. $x^2+y^2=p^2-2q$, $x^3+y^3=p^3-3pq$, $x^4+y^4=p^4-4p^2q+2q^2$, $x^5+y^5=p^5-5p^3q+5pq^2$, &c.

12. Multiply the factors and then make the substitution.

13. Multiply the first, second, and third equalities by x, y, z respectively, and add the results.

XVIII.

1. $-z=\dfrac{1-x}{1-2x}$, which when the division is performed will give the series.

2. Square each expression and take their sum.

3. $x^2y=z(\overline{x-z}+y)^2=z(x-z)^2+2yz(x-z)+zy^2$

$\therefore z(x-z)^2=y(x^2-2xz+z^2)+yz(z-y)$,

$\therefore (z-y)(x-z)^2=yz(z-y)$; $\therefore (x-z)^2=yz$.

4. Since $x+y+z-xyz=2$, $\therefore x^2+xy+xz-x^2yz=2x$,

$\therefore -2x+x^2=-xy-xz+x^2yz$,

add unity to each, $\therefore 1-2x+x^2=1-xy-xz+x^2yz$.

5. Since $b(bx^2+a^2y)=a(ay^2+ab^2x)$, then $b^2x^2-a^2y^2=ab^2x-a^2by$,

or $(bx+ay)(bx-ay)=ab(bx-ay)$,

$\therefore (bx-ay)\{bx+ay-ab\}=0$

$\therefore bx-ay=0$ and $bx=ay$

also $bx+ay-ab=0$ and $bx+ay=ab$.

6. $(a+b)x-(c+d)x=cd-ab$; and $(a+b)x+ab=(c+d)x+cd$, add x^2 to each of these equals, and $x^2+(a+b)x+ab=x^2+(c+d)x+cd$.

7. The required expression may be thus arranged—

$ay(ax+by-1)+bx(ax+by-1)+(ax+by-1)=0$

or $(ay+bx+1)(ax+by-1)=0$.

8. $x=-(y+z)$, $\therefore -(a^2-bc)(y+z)+(b^2-ca)y+(c^2-ab)z=0$

and $(b^2-ca-a^2+bc)y=(a^2-bc-c^2+ab)z$

or $(b-a)(b+a+c)y=(a-c)(a+c+b)z \therefore (b-a)y=(a-c)z$ or $by-ay=az-cz$

$\therefore by+cz=a(y+z)=-ax \therefore ax+by+cz=0$.

9. Expand, and remove common quantities, then arrange the terms, and the expression is divisible by a^2-b^2.

11. From $a+b+c=0$, may be found $\frac{1}{2}(a^2+b^2+c^2)=-(a^2+ab+b^2)$, $\frac{1}{3}(a^3+b^3+b^3)=-abc$ and $\frac{1}{5}(a^5+b^5+c^5)=abc(a^2+ab+b^2)$.

12. From the first two equations $a(x+y)=b^2$, and $x+y=c$, $\therefore ac=b^2$.

13. Subtract the second expression from the first and divide the difference by $y-z$, and then $a=x+y+z$.

14. From the first and second members of the given equality, $by-(a-b)z=ax$, from the first and third, $cz-(a-c)y=ax$, and from the second and third, $cz-(b-c)x=by$; these will give $bx=ay$ and $bz=cy$, also $by=by$;

$\therefore b(x+y+z)=(a+b+c)y=0$.

15. $b^2y^2-2bcxy+c^2x^2=b^2y^2-b^2cz-acy^2+ac^2z^2$.

Then $-2bcxy+c^2x^2=-b^2cz-acy^2+ac^2z^2$, by removing b^2y^2.

$-2bxy+cx^2=-b^2z-ay^2+acz^2$ dividing by c.

$-2abxy+acx^2=-b^2az+a^2y^2+a^2cz$, multiplying by a.

$\therefore b^2x^2 - 2abxy + a^2y^2 = b^2x^2 - acx^2 - ab^2z + a^2cz$, adding b^2x^2,

and $(bx - ay)^2 = (b^2 - ac)(x^2 - cz)$.

17. From the given equality may be deduced,

$$(x^2 + y^2 + z^2)(xyz + 1) = (xyz + 1)(xyz + 4).$$

XIX.

1. $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \;.\;.\;.\;.\; + a^2b^{n-3} + ab^{n-2} + b^{n-1})$

$na^{n-1}(a - b) = (a - b)(a^{n-1} + a^{n-1} + a^{n-1} + \;.\;.\;.\;.\; a^{n-1} + a^{n-1} + a^{n-1})$

but $a > b$, $\therefore a^{n-1} > a^{n-2}b$, $a^{n-1} > a^{n-3}b^2$, &c.

$\therefore a^n - b^n$ is less than $na^{n-1}(a - b)$.

In a similar way $a^n - b^n$ may be shewn to be greater than $nb^{n-1}(a - b)$.

3. Here $a^m - a^n = a^n(a^{m-n} - 1)$ supposing $m > n$.

And $a^{m-n} - 1$ is divisible by $a + 1$ when $m - n$ is even.

Similarly for the rest.

5. The product mnp has the divisors m, n, p, mn, np, mp, and m, n, p may be in each case odd or even numbers.

7. $x^n - na^{n-1}x + (n - 1)a^n = (x^n - a^n) - na^{n-1}(x - a)$

$= (x - a)\{x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \;.\;.\;.\;.\;.\; + a^{n-1} - na^{n-1}\}$

$= (x - a)\{(x^{n-1} - a^{n-1}) + (x^{n-2} - a^{n-2})a + (x^{n-3} - a^{n-3})a^2 + \;.\;.\;.\;.\;.$ to n terms$\}$.

The quotient is $x^3 + 2x^2a + 2xa^2 + 3a^3$.

8. If $n - 1$ be an odd number, then $a^{n-1} - b^{n-1}$ is divisible by $a - b$, and $a^n - b^n$ is divisible both by $a - b$, and $a + b$; and it is to be shewn that $a^{n+1} - b^{n+1}$ is divisible by $(a - b)(a^2 + ab + b^2)$.

XX.

1. It is obvious that $bx + cy$ must be one of the factors, and that $x^2 + axy + y^2$ must be divisible by $bx + cy$. After performing the division the remainder will be $\left\{1 - \dfrac{(ab - c)c}{b^2}\right\}y^2$, which must be equal to zero, in order that the division may leave no remainder.

2. The coefficient of x^4 is $3a^4$.

3. Four factors of the form $x + a$ and one of the form $x - b$. See Section IV., Art. 12, pp. 19, 20, and the note.

4. $x^3 + px^2 + px + 1 = (x^3 + 1) + px(x + 1)$; and $x^n + 1$ is always divisible by $x + 1$ when n is an odd number.

5. See Section IV., Art. 13, p. 24.

XXI.

1. The first expression may be shewn greater or less than the second, third, and fourth according as bc is greater or less than a^2, ac than b^2, and ab than c^2.

2. $a = -(b + c)$, $y = -(x + z)$, $z = -(x + y)$; $\therefore a^2yz = +(b + c)^2(x + z)(x + y)$ and similarly, for b^2xz and c^2xy.

3. Suppose each of the three expressions to be positive; then the sum will be positive, and the terms may be arranged thus: $(a - c)y + (c - b)x + (b - a)z$, of which two of the terms are negative and one positive. But if the terms be arranged in this order: $(y - z)a + (z - x)b + (x - y)c$, two of the terms are positive and one negative. Hence the hypothesis of all the three terms being positive is not sustained.

4. If $a > b$, then $a - b > 0$, and $a^2 - 2ab + b^2 > 0 \therefore a^2 + b^2 > 2ab$.

Next $a^2 - 2ab + b^2 > 0$, add $4ab$ to these unequals, and $a^2 + 2ab + b^2 > 4ab$ or $(a + b)^2 > 4ab$.

5. $a^2 + b^2 > 2ab$, $a^2 + c^2 > 2ac$, $b^2 + c^2 > 2bc$. $\therefore 2(a^2 + b^2 + c^2) > 2(ab + ac + bc)$ or $a^2 + b^2 + c^2 > ab + ac + bc$.

6. The aggregate of the three expressions on the left is equal to

$$3(a^2+b^2+c^2)-2(ab+ac+bc).$$

7. The polynomial arises from the product of the three factors $(a+b-c)(a+c-b)(b+c-a)$, and is a positive quantity $\therefore a+b>c$, &c.

8. Since $a>b$, $a^2+b^2>2ab$, and $a^2+b^2>(a^2-ab+b^2)$,

$$\therefore (a^2+b^2)^2>2ab(a^2-ab+b^2).$$

9. $a^2+b^2>2ab$, $b^2+c^2>2bc$, $a^2+c^2>2ac$,
$\therefore (a^2+b^2)(b^2+c^2)(c^2+a^2)>8a^2b^2c^2$, for a^2, b^2, c^2, write a, b, c,

$$\text{and } (a+b)(b+c)(c+a)>8abc.$$

See XIII., 4. $(a+b+c)(ab+bc+ac)-abc=(a+b)(b+c)(a+c)$, and $(a+b)(b+c)(c+a)$ may be shewn greater than $8abc$,
$\therefore (a+b+c)(ab+ac+bc)>9abc$.

10. Let $a>b$ and $a>c$, then $a-b>0$ and $a-c>0$, &c., $\therefore (a-b)(a-c)>0$ or $a^2-ac-ba+bc>0$, and $\therefore bc>ac+ab-a^2$ or $bc>a(c+b-a)$.

Similarly $ac>b(a+c-b)$ and $ab>c(a+b-c)$.

Hence $a^2b^2c^2>abc(c+b-a)(a+c-b)(a+b-c)$
and $\therefore abc>(c+b-a)(a+c-b)(a+b-c)$.

This result illustrates the theorem: "The rectangular parallelopiped formed by the three sides, a, b, c of a triangle for its edges is greater in volume than that formed by the three differences between the sum of every two sides and the third side."

12. Except when $a=b$.

14. From the given expression $(bx-ay)^2>0$; if $bx>ay$ then $(x^2+y^2)(a^2+b^2)>(ax+by)^2$.

15. $(x+a)^2(x^2+b^2)=(x^2+a^2)(x^2+b^2)+2ax(x^2+b^2)$, and $(x+b)^2(x^2+a^2)=(x^2+b^2)(x^2+a^2)+2bx(x^2+a^2)$. The answer depends on whether $2ax(x^2+b^2)$ is greater or less than $2bx(x^2+a^2)$. If $x^2>ab$, then $(x+a)^2(x^2+b^2)$ is greater than $(x+b)^2(x^2+a^2)$.

16. $x^2y^2=(a^2+b^2)(c^2+d^2)=(ac+bd)^2+(ad-bc)^2$, but if $ad=bc$, then $x^2y^2=(ac+bd)^2$ and $xy=ac+bd$.

17. If $(a^2+b^2+c^2)(x^2+y^2+z^2)$ be greater than $(ax+by+cz)^2$, then $(ay-bx)^2+(az-cx)^2+(bz-cy)^2$ is greater than 0, but the squares both of positive and negative quantities are positive, therefore each of these squares is positive, and consequently the first expression is greater than the second.

18. $(a+b+c)^3=a^3+b^3+c^3+3(a+b)(b+c)(c+a)$; first shew $a^3+b^3+c^3>3abc$, next $3(a+b)(b+c)(c+a)>24abc$ from Ex. 9, XXI.

19. (1) $(x+y-z)^2+(x+z-y)^2+(y+z-x)^2=3(x^2+y^2+z^2)-2(xy+xz+yz)$, and $x^2+y^2+z^2$ has been shewn greater than $xy+xz+yz$. (2) and (3) may in like manner be shewn to be true.

20. (1) Let $x>y$ and $y>z$, then $x^2(y-z)$ and $z^2(x-y)$ are positive, and $y^2(z-x)$ is negative; whence $x^2(y-z)+z^2(x-y)>y^2(z-x)$ from which may be deduced, that $x+z>y$, but $x>y$, therefore the property is proved. (2) and (3) may be considered in the same manner.

ELEMENTARY ALGEBRA,
WITH BRIEF NOTICES OF ITS HISTORY.

SECTION V.

MEASURES, MULTIPLES, AND FRACTIONS.

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COMMON MEASURES AND COMMON MULTIPLES.

Art. 1. The subject of algebraic measures and multiples is of extensive use; and it may be remarked that the common measures and common multiples of numbers do not always correspond to the common measures and common multiples of algebraic expressions, when numerical values are assigned to the general symbols.*

A common divisor or common measure of two or more algebraical quantities is any quantity by which each of the given quantities can be divided exactly without any remainder; and *the highest common divisor* is the highest quantity by which each of them can be exactly divided. By "the highest common divisor" must be understood that divisor the terms of which contain greater coefficients and greater exponents of the symbols than any other.

Two or more algebraical quantities are said to be prime to one another when they have no common divisor.

A common multiple of two or more algebraical quantities is that quantity which is the least multiple of each of the given quantities; and *the least common multiple* is the least multiple of each of the given quantities respectively; and is also the least quantity that is exactly divisible by each of the given quantities. Algebraical quantities are said to be prime or composite, in the same manner as numbers in arithmetic.

2. Prop. *To explain the process of finding the highest common divisor of two algebraical expressions.*

The process depends on these two principles.

1. If one quantity be divided by another, and leave a remainder, and if this remainder and the divisor have a common divisor, then this common divisor will also be a common divisor of the dividend and divisor.

Let the quantity A when divided by B give the quotient q and remainder r; and suppose p to be the common divisor of B and r, so that $\frac{B}{p}=m$ and $\frac{r}{p}=n$, and therefore $B=mp$ and $r=np$, or B and r are multiples of p.

But $\frac{A}{B}=q+\frac{r}{B}$ or $A=Bq+r$.

Hence $A=mpq+np=p(mq+n)$.

$\therefore \frac{A}{p}=mq+n$, or p is a common divisor of A as well as of B and r.

2. That a dividend may be multiplied by any factor which is not

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* See the Editor's Elementary Arithmetic, Section viii., on Measures and Multiples.

common to a divisor, and a divisor may be divided by any factor which is not common to a dividend: for in both cases the common divisor of the two expressions is not affected by such operations.

The highest common divisor of Aa and B, of A and Bb, and of Aa and Bb, is the same as that of A and B, if a contain no divisor of Bb, nor b of Aa.

Let A and B denote two polynomials of which the dimensions of B are not higher than those of A.[*]

Let A divided by B give a quotient P, with a remainder Cc. Suppose the factor c in Cc not contained in B, let c be rejected and C taken for the second divisor.

Next, let B divided by C give a quotient Q, with a remainder Dd. Suppose the factor d in Dd not contained in C, let d be rejected and D taken for the third divisor.

Let C divided by D give a quotient R with a remainder O, then D will be the highest common divisor of A and B.

The process may be thus exhibited, with the following equivalents.

$$\begin{array}{lll}
B)A(P & A+BP+Cc, & \text{and } A-BP=Cc, \\
\quad \underline{BP} & & \\
\quad Cc & & \\
\quad C)B(Q & B=CQ+Dd, & B-CQ=Dd, \\
\qquad \underline{CQ} & & \\
\qquad Dd & & \\
\qquad D)C(R & C=DR, & C=DR. \\
\qquad\quad \underline{DR} & & \\
\qquad\quad \underline{O} & &
\end{array}$$

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[*] As an example, find the highest common divisor of $2x^2+9xy+9y^2$ and $3x^2+4xy-15y^2$.

The first term of second quantity cannot be exactly divided by the first term of the first quantity, and in order that it may be divisible, it may be multiplied by the factor 2, which does not enter into every term of the first quantity.

$$\begin{array}{l}
2x^2+9xy+9y^2)3x^2+4xy-15y^2 \\
\qquad\qquad\qquad\qquad \underline{2\qquad\qquad\qquad} \\
\qquad\qquad\qquad\quad 6x^2+8xy-30y^2(3 \\
\qquad\qquad\qquad\quad \underline{6x^2+27xy+27y^2} \\
\qquad\qquad\qquad\qquad -19xy-57y^2=-19y(x+3y)
\end{array}$$

The factor $-19y$ in the remainder may be rejected, as it forms no factor of the preceding divisor.

$$\begin{array}{l}
x+3y)2x^2+9xy+9y^2(2x+3y \\
\qquad\quad \underline{2x^2+6xy} \\
\qquad\qquad\quad 3xy+9y^2 \\
\qquad\qquad\quad \underline{3xy+9y^2}
\end{array}$$

Hence $x+3y$ is highest common divisor of the two expressions.

By this method, the highest common divisor of two algebraical polynomials can always be found, when the expressions can be arranged according to the ascending or descending powers of one of the symbols. In cases, however, where no such arrangement is possible, the common factors may in general be found by inspection.

First, every divisor of A and B divides BP, a multiple of B, and therefore it divides $A-BP$ or Cc, and therefore also C.

And every divisor of A and B divides C, and therefore $B-CQ$ or Dd, and therefore D.

Hence every divisor of A and B divides D.

Secondly. Since $DR=C$, therefore D divides C, and therefore CQ, a multiple of C, also $QC+Dd$ or B, and therefore BP, and consequently $BP+Cc$ or A. Therefore every divisor of A and B divides D, and D divides A and B.

Wherefore D is the highest common divisor of A and B.

The highest common divisor of three quantities A, B, C, may be found by first finding D, the highest common divisor of two of them, A and B; and next finding E the highest common divisor of D and C the third quantity.

Then E will be the highest common divisor of A, B, and C.

The same method may be extended to four and to any number of quantities.

In some cases it will be found more convenient when there is an even number of quantities; to find the highest common divisor of every two of them, and then the highest common divisor of these common divisors.

When two or more given expressions can by inspection be resolved into the product of their prime factors, the factor or the product of factors common to each expression will obviously be the highest common divisor of the given expressions.

3. Prop. *To explain the process for finding the least common multiple of two algebraical expressions.*

Let A, B be the quantities whose least common multiple is required, and let D be their highest common divisor.

Also let A and B when divided by D give respectively the quotients P and Q, which are prime to each other, so that $A=PD$ and $B=QD$; then DPQ is *a common multiple* of A and B; for it contains A, Q times and B, P times.

DPQ is also *the least common multiple* of A and B; for no quantity less than DPQ contains all the factors that enter into A and B.

$$\text{Now } AB=DPQD,$$

$$\therefore DPQ=\frac{AB}{D}, \text{ or } \frac{A}{D}\times B \text{ or } \frac{B}{D}\times A.$$

Hence, the least common multiple of two quantities is found by dividing the product of the two quantities by their highest common divisor. Or,

By multiplying one of the quantities by the quotient arising from the other divided by their highest common divisor.

If the two given quantities be prime to each other, the least common multiple is equal to their product.

If one of the two quantities be a multiple of the other, the former is their least common multiple.

The least common multiple of three quantities is found by first finding the least common multiple of two of them, and then the least common multiple of this and the third quantity. And similarly of four and any number of quantities.

It may also be noted that the highest common divisor of any number of algebraical quantities is equal to the least common multiple of all the common divisors of the quantities. And the least common multiple of any number of algebraical quantities is equal to the highest common divisor of all their common multiples.*

4. Prop. *If one quantity measure the product of two others, but is prime to one of them, it measures the other.*

Let c measure ab, but be prime to a, then c measures b.

For let a be greater than c.

Then let c be contained in a, x times with a remainder r,

$$\therefore \frac{a}{c} = x + \frac{r}{c}, \text{ in which } r \text{ is less than } c.$$

Again, let r be contained y times in c with a remainder s,

$$\therefore \frac{c}{r} = y + \frac{s}{r}.$$

It is manifest that s also is less than r; and as the remainder is continually diminishing at each step, it will at last become unity. For if there be any other remainder, the division may be continued, and if there be no remainder, then the last divisor will measure a and c; and therefore a and c have a common measure which is contrary to the hypothesis.

Suppose then that after the third division the remainder is 1,

$$\text{then } \frac{r}{s} = z + \frac{1}{s}.$$

$$\begin{array}{lll} \therefore a = cx + r & \therefore ab = bcx + br & \text{or } ab - bcx = br \\ \quad c = ry + s & \quad bc = bry + bs & \quad bc - bry = bs \\ \quad r = sz + 1 & \quad br = bsz + b & \quad br - bsz = b. \end{array}$$

Now c measures ab and bcx, it ∴ measures $ab - bcx$ or br, and ∴ it measures $bc - bry$ or bs, and ∴ its measures $br - bsz$ or b.

Hence ∴ if c measure ab, and do not measure a, it measures b. The reasoning is similar if c be supposed greater than a.

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* To find the least common multiple of $2x^2+9xy+9y^2$ and $3x^2+4xy-15y^2$. The highest common divisor of these expressions is $x+3y$.

$$\begin{aligned} \therefore \text{Least common multiple} &= \frac{(3x^2+4xy-15y^2)(2x^2+9xy+9y^2)}{x+3y} \\ &= (3x^2+4xy-15y^2)(2x+3y) \\ &= 6x^3+17x^2y-18xy^2-45y^3. \end{aligned}$$

EXERCISES.

I.

1. Explain the process of finding the highest common divisor of two algebraical expressions, and shew that the result is not affected: (1) by removing any factor which is common to every term of a divisor at any stage of the operation which is not found in every term of the corresponding dividend; (2) by introducing into any dividend a factor which is not contained in every term of the corresponding divisor.

2. Explain in what respects the process of finding the highest common divisor of two algebraical polynomials differs from that of finding the greatest common measure of two numbers.

3. If the quantity D measure A and B; it also measures $mA \pm nB$.

4. Does the common process for finding the highest common divisor of two algebraical expressions enable us to determine all the common divisors whenever they exist?

5. Every other multiple of any two quantities is the same multiple of their least common multiple.

6. If m be the least common multiple of a and b, every other common multiple of a and b is a multiple of m.

7. If m be the least common multiple of a and b, m' the least common multiple of m and c, and m'' the least common multiple of m' and d; shew that m'' will be the least common multiple of a, b, c, d.

8. Find two numbers whose greatest common measure is 2 and whose least common multiple is 36.

9. Determine that algebraic expression which involves the lowest possible powers of x that can be exactly divided by x^2+5x+6 and x^2+7x+8.

10. Shew that the highest common divisor of two compound algebraical quantities is the least common multiple of all the common divisors.

II.

Find the highest common divisor of the following quantities:—

1. $3x^2-2x-1$ and $6x^2-x-1$.
2. x^3-3x+2 and x^3+2x-3.
3. $3x^3-2x^2-2x+1$ and $x^5-x^4-3x^2+4x-1$.
4. $6x^4y+x^3y^2-xy^4$ and $4x^3-6x^2y-4xy^2+3y^3$.
5. $11x^4-9ax^3-a^2x^2-a^4$ and $13x^4-10ax^3-2a^2x^2-a^4$.
6. $2x^5-11x^2-9$ and $4x^5+11x^4+81$.
7. $3x^5-3x^4-53x^3-43x^2+34x+30$
 and $3x^5+3x^4-53x^3+43x^2+34x-30$.
8. $3x^5-10x^3+15x+8$ and $x^5-2x^4-6x^3+4x^2+13x+6$.

9. $x^4+x^3-2x^2+x-1$ and $x^5-2x^4+3x^3-3x^2+2x-1$.

10. $x^3+6x^2+11x+6$, $x^3+10x^2+31x+30$, and $x^3+10x^2+23x+14$.

11. x^6-1 and $x^6-1+x(x^4-1)+x^2(x^2-1)$.

12. $3x^2-(4a+2b)x+2ab+a^2$ and $x^3-(2a+b)x^2+(2ab+a^2)x-a^2b$.

13. $ay(x^3+b^3)+bx(by^2+a^2x)$ and $ax(y^3+b^3)+by(bx^2+a^2y)$.

14. $x^4-px^3+(q-1)x^2+px-q$ and $x^4-qx^3+(p-1)x^2+qx-p$.

15. $1-abx^3+(b-a^2)x^2$ and $1+acx^3-(c-a^2)x^2-2ax$.

16. $x^3+(a+b+c)x^2+(bc+ac+ab)x+a^2b+ab^2$
and $x^3+(a+2b)x^2+(ab+ac+b^2)x+a^2c+abc$.

17. $a^3+b^3+c^3-3abc$ and $a(a+2b)+b(b+2c)+c(c+2a)$.

18. $nx^{n+1}-(n+1)x^n+1$ and $x^n-nx+n-1$.

III.

Find the least common multiple of the following quantities :—

1. $na^{n-1}x$, $(n-1)a^{n-2}x^2$, $(n-2)a^{n-3}x^3$, and $(n-4)a^{n-4}x^4$.

2. $3x^2-5x+2$, and $4x^3-11x^2+x+6$.

3. $x^3-a^2x-ax^2+a^3$, x^4-a^4 and $ax^3+a^3x-a^2x^2-a^4$.

4. $x^5-2x^4-6x^3+4x^2+13x+6$, and $3x^4+4x^3-6x^2-12x-5$.

5. $a^5+b^5+3ab(a^3+b^3)+a^2b^2(a+b)$,
and $2a^5+2b^5+5ab(a^3+b^3)+6a^2b^2(a+b)$.

6. a^6-x^6 and a^2-x^2.

7. x^2-4a^2, $x^3+2ax^2+4a^2x+8a^3$, and $x^3-2ax^2+4a^2x-8a^3$.

8. $21x^2-13x+2$, $28x^2-15x+2$, and $12x^2-7x+1$.

9. x^3-y^3, x^3+y^3, $x^4+x^2y^2+y^4$.

10. $x^3+6x^2+11x+6$, $x^3+7x^2+14x+8$, $x^3+8x^2+19x+12$, and $x^3+9x^2+26x+24$.

11. $6(a^3-b^3)(a-b)^3$, $9(a^4-b^4)(a-b)^2$, and $12(a^2-b^2)^3$.

12. x, $x+1$, x^2+1, x^3+1, and x^4+1.

13. x, $x-1$, x^2-1, x^3-1, and x^4-1.

14. $6x^2-13x+6$, $12x^2-5x-2$, and $15x^2+2x-8$.

15. $x^{mn}-1$, x^m-1 and x^n-1.

IV.

1. Find the value of a for which the following fraction admits of reduction : $\dfrac{x^3-ax^2+19x-a-4}{x^3-(a+1)x^2+23x-a-7}$

2. Find the greatest common divisor of $x^3+y^3+z^3$ and $x^5+y^5+z^5$ when $x+y+z=0$.

3. If $a^m+a^{m-1}b+b^m$ and $a^m+ab^{m-1}+b^m$ have a common divisor, m being an integer, prove that it is of the form $3p-1$, and find the greatest common measure.

4. If the greatest common measure of a, b, c be an odd number m; then the greatest common measure of $a+b$, $b+c$, $c+a$ will be $2m$, if a, b, c be odd numbers, or m if they are not.

5. If p, q, r be the successive quotients in finding the highest

common divisor of the numerator and denominator of the fraction $\frac{a}{b}$; then $\frac{a}{b}=\frac{p+pqr+r}{1+qr}$ or $\frac{1+qr}{p+pqr+r}$, according as a is greater or less than b.

6. Shew that ax^5+bx+c and $a+bx^4+cx^5$ have a common quadratic factor, if $b^2c^2=(c^2-a^2+b^2)(c^2-a^2+ab)$:

7. The expressions ax^2+bx+c and $ax^2+mbx+m^2c$ have a common divisor, if $(m+1)^2ac=mb^2$.

8. If D be the greatest common measure, and M the least common multiple of two decimals which have m and $m+n$ decimal places respectively; prove that $\frac{D}{M}$ is a multiple of 10^n.

9. If the least common multiple of ax^3+cx+d and $a'x^3+cx^2+d'$ be an expression of the fourth degree; shew that $a^2d=aa'c+a'^3$.

10. Shew that if $x+c$ be the highest common divisor of x^2+ax+b and x^2+mx+a; their least common multiple is $x^3+(a+m-c)x^2+(am-c^2)x+(a-c)(m-c)c$.

11. If ax^2+bx+c and mx^2+nx+p have a common factor of the form $x+q$; shew that $(pa-mc)^2=(nc-pb)(na-mb)$.

12. Find the highest common divisor of $x^n-1)(x^{n+1}+1)$ and $(x^n-1)(x^{m+1}+1)$.

13. Find the highest divisor of x^m-y^m and x^n-y^n, (1) when m and n are of the forms $4p$, $4p+2$, and (2) of the forms $4p+1$, $4p+3$.

14. Explain how the highest common divisor of two algebraical expressions, does not always give the greatest common measure of the numbers which result, when particular numerical values are given to the algebraical symbols.

Exemplify in finding the highest common divisor of x^4-x^2-2x-1 and $x^4+2x^3+x^2-1$; and of $3x^2+ax-4a^2$ and $6x^3-7ax^2-20a^2x$; and their greatest common measures, when $x=4$ and $a=1$.

15. If M be the least common multiple of the three numbers a, b, c, shew what are the conditions that $\frac{ab+ac+bc}{M}$ may be integral.

16. m, m_1, m_2, m_3, and d, d_1, d_2, d_3 be the least common multiples and the highest common divisors of A, B, C; of B, C; of C, A; and of A, B, respectively;

Shew that $\frac{m_1m_2m_3}{m}=\frac{ABC}{d}$, and $\frac{d_1d_2d_3}{d}=\frac{ABC}{m}$.

17. If l_1, l_2, l_3 are the lowest common multiples of B and C, of C and A, of A and B, respectively; if g_1, g_2, g_3 are the highest common divisors of the same pairs; and if L, G are the lowest common multiple and highest common divisor of A, B, and C; prove that $\frac{L^2}{G^2}=\frac{l_1l_2l_3}{g_1g_2g_3}$.

FRACTIONS.

Art. 1. Algebraic fractions are subject to the same principles as arithmetical fractions, and every fraction may be considered as representing the quotient arising from the division of the numerator by the denominator.

The fraction $\frac{a}{b}$ cannot be called either a proper or improper fraction, while the values* of the numerator and denominator are not assigned; but it may be considered a proper or improper fraction according as the numerator is less or greater *in form* than that of the denominator, as $\frac{a}{ma+b}$ is a proper and $\frac{ma+b}{a}$ is an improper fraction.

Every quantity having an *integral form* can be expressed as a fraction, by placing unity for the denominator, as a is the same as $\frac{a}{1}$.

2. The general properties of algebraic fractions are the same as those of numerical fractions, with this distinction, that general symbols are employed in the former and definite numbers in the latter.

A fraction is multiplied by any quantity by multiplying the numerator by that quantity; as $\frac{ma}{b}$† is the product of the fraction $\frac{a}{b}$ by the quantity m.

Conversely. A fraction is divided by any quantity m by dividing the numerator by that quantity, as $\frac{a}{b}$ is the quotient of $\frac{ma}{b}$ divided by m.

A fraction is also divided by any quantity by multiplying the denominator by that quantity, as $\frac{a}{mb}$ is the quotient of the fraction $\frac{a}{b}$ divided by the quantity m.

Conversely. A fraction is multiplied by any quantity by dividing the denominator by that quantity, as $\frac{a}{b}$ is the product of the fraction $\frac{a}{mb}$ by the quantity m.

Hence, a fraction is multiplied by any integral quantity either by multiplying the numerator or dividing the denominator by that quantity. And, a fraction is divided by any integral quantity, either by dividing the numerator or multiplying the denominator by that quantity.

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* It must always be remembered that the *value* of a fraction depends on the *relative* values of the numerator and denominator, and not on their absolute values.

† If $m=b$, then $\frac{a}{b}\times b=a$; that is: if a fraction be multiplied by a quantity equal to its denominator, the product is equal to the numerator.

If the numerator and denominator of a fraction $\frac{a}{b}$, be both multiplied by the same quantity m, as $\frac{ma}{mb}$, the value of the fraction is unaltered in value.

By means of this principle two or more fractions can be changed to other fractions which shall have each the same denominator without altering the values of the fractions.

Conversely. If the numerator and denominator of a fraction $\frac{ma}{mb}$, be both divided by the same quantity m, as $\frac{a}{b}$, the value of the fraction is unaltered in value.*

By means of this principle, fractions can be changed to others in a more simple form without altering their value.

The signs of the numerator and denominator of an algebraic fraction may be changed without affecting the value of the fraction, as it is, in fact, multiplying the numerator and denominator by -1. Thus $\frac{+a}{+b}=\frac{-a}{-b}=+\frac{a}{b}$; and $\frac{-a}{+b}=\frac{+a}{-b}=-\frac{a}{b}$.

3. An improper algebraic fraction is reduced to a mixed quantity, and a mixed quantity is reduced to an improper fraction in the same manner as arithmetical fractions; thus the improper fraction $\frac{ma+b}{a}=m+\frac{b}{a}$, and the mixed quantity $m+\frac{b}{a}=\frac{am+b}{a}$.

An algebraic fraction is reduced to its lowest terms by dividing the numerator and denominator by their highest common divisor.

Two or more algebraic fractions are reduced to equivalent fractions having the lowest common denominator, by finding the lowest common multiple of the denominators, next, dividing this quantity by each of the denominators respectively, and lastly, by multiplying the numerator and denominator of each fraction by the corresponding quotient.†

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* These general properties may be shewn to be true for algebraic fractions in the same manner as they have been proved for arithmetical fractions. See Elementary Arithmetic, section ix. pp. 2-4.

† Ex. Reduce the fractions $\frac{x}{a}$, $\frac{y}{b}$, $\frac{z}{c}$ to equivalent fractions having a common denominator.

Here abc is the least common multiple of the denominators.

Then $\frac{x}{a}=\frac{x}{a}\times\frac{bc}{bc}=\frac{bcx}{abc}$; $\frac{y}{b}=\frac{y}{b}\times\frac{ac}{ac}=\frac{acy}{abc}$; $\frac{z}{c}=\frac{z}{c}\times\frac{ab}{ab}=\frac{abz}{abc}$; and $\frac{bcx}{abc}$, $\frac{acy}{abc}$, $\frac{abz}{abc}$ are the equivalent fractions having the common denominator abc.

4. Prop. *To find the sum and difference of two algebraical fractions.*

The sum and difference of two algebraical fractions $\frac{a}{b}$ and $\frac{c}{d}$ are found in the same manner as the sum and difference of two arithmetical fractions.

First reduce, if necessary, the fractions $\frac{a}{b}$ and $\frac{c}{d}$ to the equivalent fractions $\frac{ad}{bd}$ and $\frac{bc}{bd}$ having a common denominator, and then take the sum or difference of the numerators for the sum or difference of the fractions, thus :—

$$\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}=\frac{ad+bc}{bd}\text{; the sum,}$$

$$\frac{a}{b}-\frac{c}{d}=\frac{ad}{bd}-\frac{bc}{bd}=\frac{ad-bc}{bd}\text{, the difference.}$$

In performing the addition and subtraction of algebraic fractions, it will in general be found most convenient, first to combine two of the fractions, and then a third to this result, and so on for the rest.

In the case of mixed quantities, instead of reducing them to improper fractions, it will generally be found more convenient to reduce the fractional parts to a common denominator, and then to annex the sum or difference of the fractions to the sum or difference of the other quantities.

5. Prop. *To find the product and quotient of two algebraical fractions.*

The product and quotient of two algebraic fractions is subject to the same rules as the product and quotient of two arithmetical fractions.

To find the product of two fractions $\frac{a}{b}$ and $\frac{c}{d}$; take the product ac of the numerators a and c, and the product bd of the denominators b and d; these will be the numerator and denominator of the product of the fractions, thus :—

$$\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}\text{, the product.*}$$

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* To multiply any fraction $\frac{a}{b}$ by another $\frac{c}{d}$, means that the fraction $\frac{a}{b}$ is to be multiplied by c, and this product divided by d.

By an extension of the meaning of words, multiplication includes *taking parts* of a time ; and every multiplication may be converted into a division, and every division into a multiplication.

Thus $a\times\frac{1}{b}=\frac{a}{b}$ or $a\div b$, that is, the product a by $\frac{1}{b}$ is equivalent to a divided b ; and conversely, a divided by b is equivalent to a multiplied by $\frac{1}{b}$.

The inverse or reciprocal of any quantity is unity divided by that quantity, as $\frac{1}{a}$ is the inverse or reciprocal of a, and $\frac{b}{a}$ is the inverse or reciprocal of $\frac{a}{b}$.

In finding the product of two or more algebraical fractions, when equal factors are found in the numerator and denominator of the product, these factors may be struck out, by the principle, that the value of a fraction is not altered by dividing the numerator and denominator by the same quantity, as the value of a fraction depends not on the absolute, but on the relative values of the numerator and denominator.

The quotient of two algebraic fractions is also found in the same manner as the quotient of two arithmetical fractions.

To find the quotient of the fraction $\frac{a}{b}$ divided by $\frac{c}{d}$, is the reverse process of finding the product, which is effected by multiplying the dividend $\frac{a}{b}$ by the reciprocal $\frac{d}{c}$ of the divisor, thus:—

$$\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc},\text{ the quotient.*}$$

Hence the Rule. Multiply the dividend by the reciprocal of the divisor, and the product will be the quotient of the two fractions.

When the numerator and denominator of the dividend can be divided by the numerator and denominator of the divisor respectively; the quotient may be found directly, but this is very seldom possible.

It may be remarked that algebraic fractions are in general considered as representing parts of abstract number. But algebraic fractions may also be regarded as denoting concrete quantity. With respect to the interpretation of the algebraic addition and subtraction of concrete fractions, it is obvious that the sums and differences of such fractions are only possible when the concrete quantities are of the same kind.

In the interpretation of the results of the multiplication and division of concrete algebraic fractions, the same method applies as in the interpretation of the algebraic products and quotients of integral concrete quantities.†

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* The rule for division of one fraction by another may be readily shewn by the principle of the dividend being equal to the product of the divisor and quotient. Let $\frac{a}{b}$ be the dividend, $\frac{c}{d}$ the divisor, and let x denote the quotient.

Then $\frac{c}{d}\times x=\frac{a}{b}$. Multiply these equals by $\frac{d}{c}$. $\therefore\frac{d}{c}\times\frac{c}{d}\times x=\frac{a}{b}\times\frac{d}{c}$, but $\frac{d}{c}\times\frac{c}{d}=\frac{cd}{cd}=1$. Hence x or $\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}$.

† The student is referred to the Elementary Arithmetic, section ix. pp. 7-9, where the reasons of the processes of arithmetical fractions are fully given, and which may readily be adapted to general expressions of number and quantity.

EXERCISES.

I.

1. In what respects do algebraical fractions differ from arithmetical fractions?

2. If the numerator and denominator are prime to each other, the fraction is irreducible, and cannot be expressed in a more simple form.

3. When a fraction $\frac{a}{b}$ is equal to another irreducible fraction $\frac{c}{d}$, the terms of the former are respectively equimultiples of the terms of the latter.

4. What alterations are made in the value of a fraction $\frac{a}{b}$, (1) when a quantity x is added to, (2) when subtracted from, the numerator and denominator?

5. If $\frac{a}{b}$ and $\frac{c}{d}$ be two unequal fractions, the fraction $\frac{a+c}{b+d}$ is intermediate in value between $\frac{a}{b}$ and $\frac{c}{d}$.

6. Neither the sum nor the difference of two fractions which are in their lowest terms, and of which the denominator of one contains a factor not contained in the other, can be an integer.

7. Explain how a meaning is given to the process of multiplying two fractions together.

8. Reduce the fractions $\frac{ab}{c^2}$, $\frac{bc}{a^2}$, $\frac{ca}{b^2}$ to equivalent fractions having (1) the same common denominator, (2) the same common numerator.

9. Shew that the cube of the difference of a fraction and its reciprocal is less than the difference of their cubes by three times the difference of the fraction and its reciprocal.

10. Shew that $\frac{(ac+bd)^2}{(a^2+b^2)(c^2+d^2)}$ is always a proper fraction.

II.

Determine the value of—

1. $\frac{a-3}{2a+2}+\frac{a-2}{3a+9}+\frac{a+3}{a^2-1}$ when $a=5$.

2. Find the value of $\frac{a-b}{b-c}+\frac{b-c}{c-a}+\frac{c-a}{a-b}$, when $a=10$, $b=5$, $c=1$.

3. Find the value of $a-\frac{a^2}{2}+\frac{a^3}{3}-\frac{a^4}{4}+\frac{a^5}{5}$ to five places of decimals when $a=\cdot1$.

4. Find the value of $\frac{a^2+b^2-c^2}{ab+ac-bc}$, when $a=1$, $b=2$, $c=3$.

5. What is the value of $\frac{x-a}{b}+\frac{x-b}{a}$ when $x=\frac{a^2}{a+b}$?

6. Find the value of $\frac{5(x^2+xy)}{5y^2+2xy+5x^2}$ when $x=\frac{2y}{3}$.

7. Shew that $\frac{x^3+y^3+z^3-3xyz}{a^3+b^3+c^3-3abc}=2$, when $x=b+c$, $y=c+a$, $z=a+b$.

8. When $a=4$, $b=6$, $c=8$, find the numerical value of

$$\left\{\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right\}\cdot\left\{\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right\}\cdot\left\{\frac{1}{a}+\frac{1}{c}-\frac{1}{b}\right\}.$$

9. Find in terms of a, the value of $\frac{x^2+ax+a^2}{x^3-a^3}-\frac{x^2-ax+a^2}{x^3+a^3}$, when $x=+3a$, and when $x=-\frac{1}{3}a$.

10. If $\mu=1{\cdot}5$, $f=20$, $y=2$, $r=-10$, $s=-15$, find the value of $-\frac{\mu-1}{\mu^2}\left\{\frac{1}{r}+\left(\frac{1}{s}-\frac{1}{f}\right)\cdot\left(\frac{1}{s}-\frac{\mu+1}{f}\right)\right\}\cdot\frac{f^2y^2}{2}$, to two places of decimals.

11. Shew that $\frac{x^3+(x+1)^3+(x+2)^3}{x+(x+1)+(x+2)}$ is integral for all integral values of x. Can $\frac{x^3+y^3+z^3}{x+y+z}$ be integral for all integral values of x, y, z?

12. Shew that the sum s shillings, d pence, and q farthings is equivalent to $\frac{3.4^2.s+2^2.d+q}{1.2^6.3.5}$ of one pound.

III.

Determine the highest common divisor of the numerator and denominator of the following fractions respectively, and reduce them to their lowest terms :—

1. $\frac{x^2-4x+3}{x^2-2x-3}$. 2. $\frac{3a^3x^3+6a^2x^4+6ax^5+3x^6}{7a^6+14a^5x+7a^4x^2}$. 3. $\frac{2x^2-xy-6y^2}{3x^2-8xy+4y^2}$.

4. $\frac{x^5+3x^4y-4x^3y^2}{x^4-x^3y-xy^3+y^4}$. 5. $\frac{10x^4-7x^3+x^2}{4x^4-2x^3-2x+1}$. 6. $\frac{30a^2x^4-5a^3x^3+5a^5x}{9ax^3-a^3x+2a^4}$.

7. $\frac{1-2x+2x^2+2x^3-3x^4}{1-x+2x^2+x^3+3x^4}$. 8. $\frac{x^5-2x^4-x^3+5x^2-4x+1}{x^4-x^3-3x^2+5x-2}$.

9. $\frac{x^3+2x^2+2x}{x^5+4x}$. 10. $\frac{10x^3-54x^2+87x-45}{5x^4-36x^3+87x^2-90x+54}$.

11. $\frac{(2x^2+5x+2)(x^3+3x^2-x+3)}{(x^3+6x^2+11x+6)(2x^2-x-1)}$. 12. $\frac{5x^4-4x^5-1}{x^5-x^4-x+1}$.

13. $\frac{3(x^3-y^3)-5xy(x-y)+y^3}{3(x^3+y^3)+xy(x+y)-5y^3}$. 14. $\frac{8x^7-29x^3+21}{21x^7-29x^4+8}$.

15. $\frac{ab^2+(ab-b^2)x+(a-b)x^2+x^3}{ab^2+(ab+b^2)x+(a-b)x^2+x^3}$.

16. $\frac{a^2(b^2-c^2)-ab(2b^2+bc-c^2)+b^3(b+c)}{a^3(b^2+2bc+c^2)-a^2b(2b^2-3bc+c^2)+ab^3(b+c)}$.

17. $\frac{3x^2-(4a+2b)x+2ab+a^2}{x^3-(2a+b)x^2+(2ab+a^2)-a^2b}$.

18. $\frac{(a^2-1)x^3+(2a^2+a)x^2+2ax+1}{(a-1)^2x^3-2(a-1)x^2-2ax-1}$.

IV.

Find by inspection the highest common divisor of the numerator and denominator of each of the following fractions, and reduce them to their simplest form:—

1. $\frac{ma^3b^2x}{na^2b^3y}$, $\frac{ab-b^2}{b^2+ab}$, $\frac{a^2bc-ab^2c}{a^2bd-ab^2d}$, and $\frac{x^2-y^2}{(x-y)^2}$.

2. $\frac{x^3+x^2y-xy^2-y^3}{x^4-y^4}$. 3. $\frac{2a^2-ab-b^2}{a^3+a^2b-a-b}$.

4. $\frac{a^2+b^2+c^2+2ab+2ac+2bc}{a^2-b^2-c^2-2bc}$. 5. $\frac{a^2+b^2+c^2-ab-ac-bc}{a^3+b^3+c^3-3abc}$.

6. $\frac{a^4+a^3b-ab^3-b^4}{a^4-a^3b-ab^3+b^4}$. 7. $\frac{x^6+x^2y-x^4y^2-y^3}{x^4-x^2y-x^2y^2+y^3}$.

8. $\frac{(a+c)(a-c)+b(2a+b)}{(a+c)(a-c)-b(2c+b)}$. 9. $\frac{3bcz+6mxz+36mx+18bc}{4adz+7rsz+24ad+42rs}$.

10. $\frac{ab(x^2+y^2)+xy(a^2+b^2)}{ab(x^2-y^2)+xy(a^2-b^2)}$. 11. $\frac{(1-xy)(1+xy)-(x-y)(x+y)}{(1+xy)^2+(x-y)^2}$.

12. $\frac{a^2b+c^2d+a^2d+bc^2}{ab^2+cd^2+abd+bcd+acd+abc+b^2d+bd^2}$.

13. $\frac{9x^2-3xy-6x-2y}{6x^4-4x^3-3xy^2-2y^2}$. 14. $\frac{(a+b)^2-(c+d)^2}{(a+d)^2-(b+c)^2}$.

15. $\frac{(ax+by)^2-(a-b)(x+z)(ax+by)+(a-b)^2xz}{(ax-by)^2-(a+b)(x+z)(ax-by)+(a+b)^2xz}$.

V.

Find the respective sums of the following examples:—

1. $\frac{2x}{3}+\frac{5x}{7}$. 2. $5x+\frac{3x-2}{7}$. 3. $\frac{x}{7}+\frac{3x}{8}+\frac{5x}{9}$. 4. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

5. $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}$. 6. $\frac{bc}{a}+\frac{ac}{b}+\frac{ab}{c}$. 7. $\frac{a}{2a-2b}+\frac{b}{2b-2a}$.

8. $\frac{x}{x+a}+\frac{a}{x-a}$. 9. $\frac{1}{x+7}+\frac{1}{x-3}$. 10. $\frac{3a+2x}{3a-2x}+\frac{3a-2x}{3a+2x}$.

11. $\frac{b^2-ab}{ba-a^2}+\frac{a^2+ab}{ba+b^2}$. 12. $\frac{ba}{a-b}+\frac{ab}{b-a}$. 13. $\frac{a+b}{x-a}+\frac{a-b}{x-b}$.

14. $\frac{x}{x+2y}+\frac{y}{x-2y}$. 15. $\frac{b}{d}+\frac{ad-bc}{d(c+dx)}$. 16. $\frac{x}{ab}+\frac{y}{ac}+\frac{z}{bc}$.

17. $\frac{a-b}{ab}+\frac{c-a}{ac}+\frac{b-c}{bc}$. 18. $\frac{x-y}{x^2-xy+y^2}+\frac{x+y}{x^2+xy+y^2}$.

19. $\frac{x^2+xy+y^2}{x^2-xy+y^2}+\frac{x^2-xy+y^2}{x^2+xy+y^2}$.

VI.

Find the respective differences of the following examples :—

1. $\frac{x}{5}-\frac{x}{7}$. 2. $\frac{3x}{4}-\frac{4x}{5}$. 3. $\frac{3}{2x}-\frac{2x+1}{5x}$. 4. $\frac{2x+1}{3x}-\frac{4x+2}{12x}$.

5. $a-\frac{a^2}{a-b}$. 6. $\frac{a}{b}-\frac{ac}{b(b+c)}$. 7. $\frac{1}{x-3}-\frac{1}{x-4}$. 8. $\frac{a+b}{a-b}-\frac{a-b}{a+b}$.

9. $\frac{ma}{bx+c}-\frac{na}{bx-c}$. 10. $\frac{a-c}{a-b}-\frac{b+c}{a+b}$. 11. $\frac{a^2}{x-a}-\frac{b^2}{x-b}$.

12. $\frac{a^2}{(x-a)(a-b)}-\frac{b^2}{(x-b)(b-a)}$. 13. $\frac{3x}{2}-\left\{\frac{2x}{13}+\frac{5x}{11}+\frac{11x}{12}\right\}$.

14. $\frac{x^2-xy+y^2}{x-y}-\frac{y^2}{x+y}$.

VII.

Verify the correctness of the results expressed by the following equivalents :—

1. $\frac{a+b}{2}-\frac{c+d}{2}-\left\{\frac{3(b-d}{2}-\frac{a-c}{2}\right\}=a-b-c+d.$

2. $\frac{2x}{3}+\frac{4}{5}\cdot\frac{5x-4}{4}-\frac{1}{4}\left\{\frac{10}{3}-\frac{4-4x}{5}\right\}=\frac{44x-43}{30}.$

3. $\frac{3a-4b}{7}-\frac{2a-b-c}{3}+\frac{15a-4c}{12}-\frac{a+b4b}{84}=a-b.$

4. $b\left\{1-\frac{(b+c-a)(c+a-b)}{2ab}\right\}+c\left\{1-\frac{(b+c-a)(a+b-c)}{2ac}\right\}$
$=a+b+c.$

5. $\frac{(a+b+c)(a^2+b^2+c^2)}{abc}-\left\{\frac{b+c}{a}+\frac{c+a}{b}+\frac{a+b}{c}\right\}=\frac{a^3+b^3+c^3}{abc}.$

6. $\frac{(a+b)(a^2+b^2-c^2)}{ab}+\frac{(b+c)(b^2+c^2-a^2)}{bc}+\frac{(c+a)(c^2+a^2-b^2)}{ca}$
$=2(a+b+c).$

7. $\frac{(b^2+c^2)-a^2}{2bc}+\frac{(c+a)^2-b^2}{2ac}+\frac{(a+b)^2-c^2}{2ab}$
$=\frac{(a+b+c)(2ab+2ac+2bc-a^2-b^2-c^2)}{2abc}.$

8. $\frac{b^2+c^2-a^2}{bc}+\frac{c^2+a^2-b^2}{ac}+\frac{a^2+b^2-c^2}{ab}$
$=2+\frac{(b+c-a)(c+a-b)(a+b-c)}{abc}.$

9. $\frac{(a+b)(a+c)}{a^2}+\frac{(b+a)(b+c)}{b^2}+\frac{(c+a)(c+b)}{c^2}-\frac{(a+b)(b+c)(c+a)}{abc}$
$=1+\frac{a^3b^3+a^3c^3+b^3c^3}{a^2b^2c^2}.$

10. (1) $\frac{1}{a}\frac{1}{+3}\frac{1}{+c}=\frac{3c+1}{3ac+a+c}.$ (2) $\frac{1}{a}\frac{1}{+b}\frac{1}{+c}=\frac{bc+1}{abc+a+c}.$

(3) $\frac{a}{b}\,\frac{b}{+c}\,\frac{c}{+d}=\frac{ac(d+c)}{b(cd+c+d)}$. (4) $\frac{a}{x}\,\frac{b}{+y}\,\frac{c}{+z}=\frac{ayz+ac}{xyz+ax+dz}$.

VIII.

Find the following products :—

1. $\frac{3x}{2x-3y}\times\frac{2x+3y}{2y}$. 2. $\left\{\frac{a}{x}+\frac{b}{y}\right\}\times\left\{\frac{x}{a}+\frac{y}{b}\right\}$.

3. $\left\{\frac{x}{4}-\frac{x}{5}+\frac{x}{6}\right\}\times\left\{\frac{2}{3x}-\frac{3}{4x}\right\}$. 4. $\frac{x(a-x)}{a^2+2ax+x^2}\times\frac{a(a+x)}{a^2-2ax+x^2}$.

5. $\frac{x^2-xy}{x^2-y^2}\times\frac{x^4-y^4}{xy(x-y)^2}$. 6. $\frac{x^9-y^9}{x^6-y^6}\times\frac{x^4+x^2y^2+y^4}{x^6+x^3y^3+y^6}$.

7. $\left\{1-x+x^2-\frac{x^3}{1+x}\right\}\times(1+x)$.

8. $\left\{x+3y+\frac{x(x-y)}{3x+2y}\right\}\times\frac{3x+2y}{2x+3y}$.

9. $\left\{3+4x-\frac{11+25x+12x^2}{4+3x}\right\}\times\left\{2-3x+\frac{13x-6-5x^2}{3-2x}\right\}$.

10. $\left\{a^2+\frac{2(a+b)x}{3}-\frac{x^2}{2}\right\}\times\left\{a^2-\frac{5(a-b)x}{2}+\frac{4x^2}{9}\right\}$.

11. $\left\{\frac{5}{7}x^3-\frac{7}{5}x^2+\frac{3}{7}x-\frac{7}{3}\right\}\times\left\{\frac{7}{5}x^2-\frac{5}{7}x\right\}$.

12. $\frac{a^6-b^6}{a^3+b^3}\times\frac{a+b}{a-b}\times\frac{(a^2-ab+b^2)^2}{(a^2+ab+b^2)^2}$.

IX.

Resolve the following expressions into their constituent simple and quadratic factors :—

1. $\frac{(a^2+ab+b^2)^2-(a^2-ab+b^2)^2}{4(a^2+b^2)}$. 2. $1-\frac{a^2+b^2-c^2}{2ab}$.

3. $\frac{a^2+b^2-c}{2ab}-1$. 4. $1-\frac{a^2-(b+c)^2}{(a+b+c)^2}$. 5. $\frac{(a^2-(b+c)^2}{(a-b-c)^2}-1$.

6. $\frac{(a+b)\{(a+b)^2-c^2\}}{4b^2c^2-(a^2-b^2-c^2)^2}$. 7. $\frac{(ac-bd)^2+(ad+bc)^2}{a^2+b^2}-c^2$.

8. $\frac{a^2-b^2-c^2+d^2-2(ad-bc)}{a-b+c-d}$. 9. $1-\left\{\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}\right\}^2$.

10. $1-\left\{\frac{a^2+b^2-x^2}{2ab}\right\}^2$, when $x^2=\frac{ab(c^2+d^2)-cd(a^2+b^2)}{ab-cd}$.

X.

Determine the following quotients :—

1. $\left\{\frac{x}{2}-\frac{x}{3}+\frac{x}{4}-\frac{x}{5}\right\}\div\left\{\frac{x}{2}+\frac{x}{3}\right\}$. 2. $\left\{\frac{a}{x}-\frac{y}{b}\right\}\div\left\{\frac{y}{a}-\frac{b}{x}\right\}$.

3. $\frac{y^2-9}{5}\div\frac{y+3}{4}$. 4. $\frac{2x^2}{a^3+x^3}\div\frac{x}{x+a}$.

5. $(a^3+b^3)\div\left\{\frac{1}{a}+\frac{1}{b}\right\}$. 6. $\left\{\frac{a}{b}+\frac{b}{a}\right\}\div\left\{\frac{a^2}{b^2}-\frac{b^2}{a^2}\right\}$.

7. $\frac{a^2-2ab}{a+2b}\div\frac{a-2b}{2b^2+ab}$. 8. $\frac{x^4-b^4}{x^2-2bx+b^2}\div\frac{x^2+bx}{x-b}$.

9. $\frac{4a(a^2-x^2)}{3b(c^2-x^2)}\div\frac{a^2-ax}{bc-bx}$. 10. $\frac{ax-x^2}{(a+x)^2}\div\frac{x^2}{a^2-x^2}$.

11. $\frac{(x^3+y^3)(x^2+y^2)}{x^4-y^4}\div\frac{x^2-xy+y^2}{x+y}$.

12 $\left\{1+\frac{2xy}{x^2+y^2}\right\}\div\left\{\frac{x^3-y^3}{x-y}-3xy\right\}$.

13. $\left\{\frac{x+2y}{x+y}+\frac{x}{y}\right\}\div\left\{\frac{y}{x+y}+\frac{x+y}{y}\right\}$.

14. $(a^2-b^2-c^2-2bc)\div\frac{a+b+c}{a-b-c}$.

XI.

Perform the following divisions and verify the results:—

1. $x^3+\frac{2x^2}{3}+\frac{x}{6}+\frac{1}{9}$ by $\frac{x}{2}+\frac{1}{3}$. 2. $\frac{x^2}{a}-8a^3+\frac{12a^5}{x^2}$ by $x-\frac{6a^2}{x}$.

3. $\frac{x^4}{2}-\frac{x^3}{12}-\frac{11x^2}{72}+\frac{61x}{144}-\frac{1}{12}$ by $\frac{3x^2}{2}+\frac{5x}{4}-\frac{1}{3}$.

4. $\frac{3x^5}{4}-4x^4+\frac{77x^3}{8}-\frac{43x^2}{4}-\frac{33x}{4}+27$ by $\frac{x^2}{2}-x+3$.

5. $\frac{9a^{10}}{16b^2}-\frac{4a^8}{25b^4}+\frac{7a^4}{20b^5}-\frac{49a^2}{64b^6}$ by $\frac{3a^5}{4b}+\frac{2a^3}{5b^2}-\frac{9a}{8b^3}$.

6. $\frac{x^7y^2}{5}-\frac{47x^6y^3}{40}+\frac{9x^5y^4}{2}-12x^4y^5$ by $\frac{2x^3y^2}{5}-\frac{3x^2y^3}{4}+6xy^4$.

7. $\frac{x^3}{a^3}+\frac{a^3}{x^3}-3\left\{\frac{x^2}{a^2}-\frac{a^2}{x^2}\right\}+4\left\{\frac{x}{a}+\frac{a}{x}\right\}$ by $\frac{x}{a}+\frac{a}{x}$.

8. $a^6+\frac{1}{a^6}+a^4+\frac{1}{a^4}+a^2+\frac{1}{a^2}+2$ by $a^3+\frac{1}{a^3}+a+\frac{1}{a}$.

XII.

1. Divide $1+2x$ by $1-x$ and $1+x$ by $1-2x$, each to five terms, and write the 20th terms of the quotient.

2. Divide $1+5x$ by $2+3x$ to five terms and write the 16th term.

3. Divide $a-x$ by $a+x$ to four terms of the quotient, and write the general term. Shew also that if the sum of any two consecutive terms of the quotient be divided by their difference, the quotient will be the same as $a-x$ divided by $a+x$.

4. Shew that $\frac{a+x}{b+x}=\frac{a}{b}-\frac{(a-b)x}{b^2}+\frac{(a-b)x^2}{b^3}-\frac{(a-b)x^3}{b^4}+$ &c., and $\frac{x+a}{x+b}=1-\frac{b-a}{x}+\frac{(b-a)b}{x^2}-\frac{(b-a)b^2}{x^3}+$, &c.

5. Divide $a+bx$ by $c-dx$ to five terms of the quotient and write the nth term.

6. Verify the truth of the following divisions:—

$$\frac{a+bx}{b+ax}=\frac{a}{b}-\frac{(a^2-b^2)x}{b^2}+\frac{(a^2-b^2)ax^2}{b^3}-\frac{(a^2-b^2)a^2x^3}{b^4}+\ \&c.$$

$$\frac{a+bx}{ax+b}=\frac{1}{x}-\frac{1-x^2}{x^2}\cdot\frac{b}{a}+\frac{1-x^2}{x^3}\cdot\frac{b^2}{a^2}-\frac{1-x^2}{x^4}\cdot\frac{b^3}{a^3}+\ \&c.$$

$$\frac{bx+a}{ax+b}=\frac{b}{a}-\frac{(b^2-a^2)}{a^2x}+\frac{(b^2-a^2)b}{a^3x^2}-\frac{(b^2-a^2)b^2}{a^4x^3}+\ \&c.$$

And shew that the sum divided by the difference of any two consecutive terms of the quotients gives $\frac{b-ax}{b+ax}$ in the first, and $\frac{ax-b}{ax+b}$ in the last two quotients.

7. Divide 1 by $1-x+x^2$ and by x^2-x+1, proceeding to six terms in the quotient and obtaining two different results, in one of which the powers of x continually increase and in the other decrease. Which of these will give the true value when $x=3$?

XIII.

Verify the correctness of the following identities:—

1. $\frac{x+a}{(x-a)(x-b)}+\frac{x-a}{(x+a)(x+b)}=\frac{2(x^2+a^2+2ab)x}{(x^2-a^2)(x^2-b^2}.$

2. $\frac{5+11x+16x^2}{(2-5x-3x^2)^5}-\frac{3}{(2-5x-3x^2)^4}=\frac{25x^2+26x-1}{(2-5x-3x^2)^5}.$

3. $\frac{8+6x^2-7x^4}{(5-3x-7x^2)^4}-\frac{3+4x-5x^2}{(5-3x-7x^2)^3}=-\frac{42x^4-13x^3-64x^2+11x+7}{(5-3x-7x^2)^4}.$

4. $\frac{x^7-a^7+ax(x^5-a^5)}{x^2-a^2}-\frac{x^7+a^7-ax(x^5+a^5)}{x^2-a^2}=2a(x^4+a^2x^2+a^4).$

5. $\frac{3a+2x}{3a-2x}-\frac{3a-2x}{3a+2x}-\frac{16x^2}{9a^2-4x^2}=\frac{8x}{3a+2x}.$

6. $\frac{1}{2x-1}-\frac{1}{2x+1}-\frac{1}{1-2x}=\frac{2x+3}{4x^2-1}.$

7. $\frac{1}{x-1}-\frac{1}{2(x+1)}-\frac{x+3}{2(x^2+1)}=\frac{x+3}{x^4-1}.$

8. $\frac{5}{2}\cdot\frac{1}{x+1}-\frac{1}{10}\cdot\frac{1}{x-1}-\frac{24}{5}\cdot\frac{1}{2x+3}=\frac{2x-3}{2x+3}\cdot\frac{1}{1-x^2}.$

9. $\frac{a+b}{ax+by}+\frac{a-b}{ax-by}+\frac{2(a^2x+b^2y)}{a^2x^2+b^2y^2}=\frac{4(a^4x^3-b^4y^3)}{a^4x^4-b^4y^4}.$

10. $\frac{3}{2x-3}-\frac{2x+15}{4x^2+9}-\frac{2}{2x+3}=\frac{18(2x+15)}{4x^4-81}.$

11. $\frac{x+y}{y}-\frac{2x}{x+y}+\frac{x^3-xy^2}{y^3-x^2y}=\frac{y-x}{y+x}.$

12. $\frac{a^2+b^2}{a^2-b^2}-\frac{a^2-b^2}{a^2+b^2}-\frac{a+b}{a-b}=-\frac{a^4+2a^3b-2a^2b^2+2ab^3+b^4}{a^4-b^4}.$

13. $\frac{1}{2(x+1)}-\frac{4}{x+2}+\frac{9}{2(x+3)}=\frac{x^2}{+1)(x+2)(x+3)}.$

14. $\dfrac{1}{x^2+3x+2}+\dfrac{2x}{x^2+4x+3}+\dfrac{1}{x^2+5x+6}=\dfrac{2}{x+3}.$

15. $\dfrac{1}{3}\cdot\dfrac{1-2x}{x^2-2x+1}+\dfrac{1}{2}\cdot\dfrac{1+x}{x^2+1}+\dfrac{1}{6}\cdot\dfrac{1}{x+1}=\dfrac{1}{(x^3+1)(x^2+1)}.$

16. $\dfrac{1}{(x-2)(x-3)}+\dfrac{2}{(x-1)(2-x)}+\dfrac{1}{(x-1)(x-2)}=0.$

17. $\dfrac{x-5}{(x-4)(x-3)}-\dfrac{2(x-4)}{(x-3)(x-5)}+\dfrac{x-3}{(x-5)(x-4)}$
$=\dfrac{2}{(x-3)(x-4)(x-5)}.$

18. $\dfrac{1}{x}+\dfrac{1}{x-2}+\dfrac{1}{(x-1)^2}+\dfrac{1}{x(x-2)}-\dfrac{3}{x(x-1)(x-3)}$
$=\dfrac{2x^4-10x^3+11x^2+2x-3}{x(x-)^2(x-2)(x-3)}.$

19. $\dfrac{1}{2(x-3)}-\dfrac{1}{6(x-2)}-\dfrac{1}{2(x-4)}+\dfrac{1}{6(x-5)}$
$=\dfrac{1}{(x-2)(x-3)(x-4)(x-5)}.$

XIV.

Verify the correctness of the following expressions:—

1. $\dfrac{4a^2+6ab+9b^2}{4a^2-6ab+9b^2}\times(64ab^6-729b^6)\times\dfrac{2a-3b}{2a+3b}=\dfrac{(2a+3b\{(2a)^6-(3b)^6\}}{2a-3b}.$

2. $\left\{\dfrac{x}{x+y}+\dfrac{x}{x-y}-\dfrac{2xy}{x^2-y^2}\right\}\times\left\{\dfrac{x^2+y^2}{2xy}+1\right\}=\dfrac{x+y}{y}.$

3. $\left\{\dfrac{x^2}{x^2-y^2}-\dfrac{y^2}{x^2+y^2}\right\}\times\dfrac{(x^2-y^2)^2}{(x^2-y^2)^2-(x^2+y^2)^2}=-\dfrac{(x^4+y^4)(x^2-y^2)}{4x^2y^2(x^2+y^2)}.$

4. $\left\{\dfrac{x}{y}-\dfrac{y}{x}\right\}\cdot\left\{\dfrac{1}{x-y}+\dfrac{1}{x+y}\right\}\div\dfrac{x}{y}=\dfrac{2}{x}.$

5. $\left\{\dfrac{1+x^3}{1-x+x^2}+\dfrac{1-x^3}{1+x+x^2}\right\}\div\left\{\dfrac{1}{1-x}+\dfrac{1}{1+x}\right\}=1-x^2.$

6. $\left\{\dfrac{x+2y}{x+y}+\dfrac{x}{y}\right\}\div\left\{\dfrac{x+2y}{y}-\dfrac{x}{x+y}\right\}=1.$

7. $\left\{\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}\right\}\div\left\{\dfrac{a+x}{a-x}+\dfrac{a-x}{a+x}\right\}=\dfrac{2ax}{a^2+x^2}.$

8. $\left\{a+\dfrac{ax}{a-x}\right\}\times\left\{a-\dfrac{ax}{a+x}\right\}\times\left\{\dfrac{a^2-x^2}{a^2+x^2}\right\}=\dfrac{a^4}{a^2+x^2}.$

9. $\left\{\dfrac{1}{x+y}+\dfrac{2}{x-y}-\dfrac{9}{3x-y}\right\}\div\dfrac{8y}{y^2-9x^2}=-\dfrac{5(3x+y)y}{4(3x-y)(x^2-y^2)}.$

10. $\left\{\dfrac{1}{x}\;\dfrac{1}{+x}\;\dfrac{1}{+y}\right\}\div\left\{\dfrac{x}{1}\;\dfrac{1}{+y}\right\}-\dfrac{1}{y(xyz+x+z)}=1.$

XV.

Shew the correctness of the following aggregates:—

1. $\dfrac{1}{6}\left\{x(x+1)(x+2)+x(x-1)(x-2)\right\}+\dfrac{2}{3}(x-1)(x+1)x=x^3.$

2. $\frac{1}{6}\left\{\frac{1}{x-1}-\frac{1}{x+1}+\frac{x-2}{x^2-x+1}-\frac{x+2}{x^2+x+1}\right\}=\frac{1}{x^6-1}.$

3. $\frac{1}{x^3}+\frac{1}{x^2}-\frac{1}{x}-\frac{1}{(x^2+1)^2}+\frac{x-1}{x^2+1}=\frac{x^2+x+1}{x^3(x^2+1)^2}.$

4. $\frac{13}{10}\cdot\frac{1}{(x-3)^3}-\frac{2}{15}\cdot\frac{1}{(x-3)^2}+\frac{9}{500}\cdot\frac{1}{x-3}+\frac{1}{500}\cdot\frac{9x-13}{x^2+1}$
$=\frac{x^2+x+1}{(x-3)^3(x^2+1)}.$

5. $\left\{\frac{x}{a}-\frac{y}{b}\right\}\frac{z}{c}+\left\{\frac{x}{a}-\frac{z}{c}\right\}\frac{y}{b}+\left\{\frac{y}{b}-\frac{z}{c}\right\}\frac{x}{a}=\frac{2y}{b}\left\{\frac{x}{a}-\frac{z}{c}\right\}.$

6. $\left\{\frac{a}{b}+\frac{b}{a}\right\}\cdot\left\{\frac{c}{d}+\frac{d}{c}\right\}-\left\{\frac{a}{b}-\frac{b}{a}\right\}\cdot\left\{\frac{c}{d}-\frac{d}{c}\right\}=2\left\{\frac{ad}{bc}+\frac{bc}{ad}\right\}.$

7. $\left\{\frac{b}{c}+\frac{c}{b}\right\}^2+\left\{\frac{c}{a}+\frac{a}{c}\right\}^2+\left\{\frac{a}{b}+\frac{b}{a}\right\}^2$
$=4+\left\{\frac{b}{c}+\frac{c}{b}\right\}\cdot\left\{\frac{c}{a}+\frac{a}{c}\right\}\cdot\left\{\frac{a}{b}+\frac{b}{a}\right\}.$

8. $\left\{\frac{a}{b}+\frac{b}{a}-\frac{c}{d}-\frac{d}{c}\right\}\cdot\left\{\frac{a}{b}+\frac{b}{a}+\frac{c}{d}+\frac{d}{c}\right\}=\frac{a^2}{b^2}-\frac{c^2}{d^2}+\frac{b^2}{a^2}-\frac{d^2}{c^2}.$

9. $\frac{a}{b}+\frac{b}{a}+\frac{a}{c}+\frac{c}{a}+\frac{b}{c}+\frac{c}{b}=\frac{(a+b+c)(ab+ac+bc)-3abc}{abc}.$

XVI.

Verify the accuracy of the following equivalent expressions:—

1. $\left\{\frac{a+b}{2(a-b)}-\frac{a-b}{2(a+b)}+\frac{b^2}{a^2-b^2}\right\}\cdot\frac{a-b}{b}=1.$

2. $abc+(b+c)(c+a)(a+b)=abc(a+b+c)\left\{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right\}.$

3. $\frac{2ay+bx}{2bx}-\frac{1}{2}\left\{\frac{a}{x}+\frac{x}{a}\right\}\cdot\left\{\frac{b}{y}+\frac{y}{b}\right\}+\frac{2bx-ay}{2ay}$
$+\frac{1}{2}\left\{\frac{a}{x}-\frac{x}{a}\right\}\cdot\left\{\frac{b}{y}+\frac{y}{b}\right\}=0.$

4. $\left\{\frac{a-b}{a+b}-\frac{a^2-b^2}{a^2+b^2}\right\}\div\left\{\frac{a+b}{a-b}+\frac{a^2+b^2}{a^2-b^2}\right\}=-\frac{ab(a-b)^2}{a^4+a^2b^2+b^4}.$

5. $\left\{\frac{a-1}{a}+\frac{b-1}{b}+\frac{c-1}{c}\right\}\div\left\{\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right\}-\frac{abc}{bc+ac-ab}$
$=\frac{2abc-(ab+ac+bd)}{bc+ac-ab}.$

6. $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{1}{a+b+c}=\frac{(a+b)(b+c)(c+a)}{abc(a+b+c)}.$

7. $\frac{a^2-acx+(ac-b^2+bc)x^2-bcx^3}{a^2+abx+(ac-c^2+bc)x^2+c^2x^3}=\frac{a+bx}{a+cx}.$

8. $\left\{a+\frac{b^2}{b+a}\right\}\cdot\left\{a-\frac{b}{b-a}\right\}\cdot(a^6-b^6)=(a^4+a^2b^2+b^4)^2.$

9. $$\frac{a}{b-c}+\frac{b}{c-a}+\frac{c}{a-b}+\frac{a^3+b^3+c^3+3abc}{(b-c)(c-a)(a-b)}=\frac{(b+c)(c+a)(a+b)}{(b-c)(c-a)(a-b)}.$$

10. $$\frac{a^2b+b^2c+c^2a}{(a+b)(b+c)(c+a)}-\frac{a-c}{a+b}-\frac{b-a}{b+c}-\frac{c-b}{c+a}=-\frac{a^3+b^3+c^3}{(a+b)(b+c)(c+a)}.$$

11. $$\frac{a^4+a^2b^2+b^4}{a^6-b^6}+\frac{a+b}{a^3-b^3}-\frac{a-b}{a^3+b^3}-\frac{1}{a^2-b^2}=\frac{2ab(a^2+b^2)}{a^6-b^6}.$$

12. $$\left\{\frac{1}{b-c}+\frac{1}{c-a}+\frac{1}{a-b}\right\}^2=\frac{1}{(b-c)^2}+\frac{1}{(c-a)^2}+\frac{1}{(a-b)^2}.$$

13. $$\frac{(a-b)^2+(b-c)^2+(c-a)^2}{(a-b)(b-c)(c-a)}+\frac{2}{a-b}+\frac{2}{b-c}+\frac{2}{c-a}=0.$$

14. $$\frac{a(b+c-a)}{bc}+\frac{b(c+a-b)}{ac}+\frac{c(a+b-c)}{ab}$$
$$=\frac{(a+b)(b+c)(c+a)-(a^3+b^3+c^3+2abc)}{abc}.$$

15. $$\frac{a^2(b+c)}{a(b+c)-(a^2+bc)}+\frac{b^2(c+a)}{b(c+a)-(b^2+ac)}+\frac{c^2(a+b)}{c(a+b)-(c^2+ab)}=0.$$

16. $$\frac{(a+d)bc}{(a-b)(a-c)}+\frac{(b+d)ac}{(b-a)(b-c)}+\frac{(c+d)ab}{(c-a)(c-b)}=d.$$

17. $$\frac{(a-b)^2-(b-c)^2}{a^2+ab-bc-c^2}+\frac{(b-c)^2-(c-a)^2}{b^2+bc-ca-a^2}+\frac{(c-a)^2-(a-b)^2}{c^2+ca-ab-b^2}=0.$$

18. $$\frac{a^2+b^2-c^2}{ab}+\frac{b^2+c^2-a^2}{bc}+\frac{c^2+a^2-b^2}{ca}+6$$
$$=\left\{4ab-(a+b-c)^2\right\}\cdot\left\{\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right\}.$$

19. $$\frac{abcd-(ab+ad+cd-1)}{bcd-(b+d)}-\frac{abc-(a+c)}{bc-1}$$
$$=a+c-\frac{(ad+bc)(bc-1)-b(c+a)}{(bcd-b-d)(bc-1)}.$$

20. $$\left\{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right\}^2-\frac{(a+b+c+d)^2}{abcd}$$
$$=\left\{\frac{1}{ad}-\frac{1}{bc}\right\}\cdot\left\{\frac{1}{bd}-\frac{1}{ac}\right\}\cdot\left\{\frac{1}{cd}-\frac{1}{ab}\right\}.$$

XVII.

Reduce to their simplest forms :—

1. $$\frac{1}{(a-b)(a-c)}+\frac{1}{(b-a)(b-c)}+\frac{1}{(c-b)(c-a)}.$$

2. $$\frac{a}{(a-b)(a-c)}+\frac{b}{(b-a)(b-c)}+\frac{c}{(c-a)(c-b)}.$$

3. $$\frac{a-b}{(b+c)(c+a)}+\frac{b-c}{(c+a)(a+b)}+\frac{c-a}{(a+b)(b+c)}.$$

4. $$\frac{a+b}{(b-c)(c-a)}+\frac{b+c}{(c-a)(a-b)}+\frac{c+a}{(a-b)(b-c)}.$$

5. $\dfrac{ab}{(b-c)(c-a)}+\dfrac{bc}{(c-a)(a-b)}+\dfrac{ca}{(a-b)(b-c)}.$

6. $\dfrac{a^2}{(a-b)(a-c)}+\dfrac{b^2}{(b-a)(b-c)}+\dfrac{c^2}{(c-a)(c-b)}.$

7. $\dfrac{(b-c)^2}{(c-a)(a-b)}+\dfrac{(c-a)^2}{(a-b)(b-c)}+\dfrac{(a-b)^2}{(b-c)(c-a)}.$

8. $\dfrac{a^3}{(a-b)(a-c)}+\dfrac{b^3}{(b-c)(b-a)}+\dfrac{c^3}{(c-a)(c-b)}.$

9. $\dfrac{a^2(c-b)}{(a+b)(a+c)}+\dfrac{b^2(a-c)}{(b+c)(b+a)}+\dfrac{c^2(b-a)}{(c+a)(c+b)}.$

10. $\dfrac{a^2-bc}{(a+b)(a+c)}+\dfrac{b^2-ca}{(b+c)(b+a)}+\dfrac{c^2-ab}{(c+a)(c+b)}.$

11. $\dfrac{a+b-c}{(b-c)(c-a)}+\dfrac{b+c-a}{(c-a)(a-b)}+\dfrac{c+a-b}{(a-b)(b-c)}.$

12. $\dfrac{a+b}{(a^2-bc)(b^2-ac)}+\dfrac{b+c}{(b^2-ac)(c^2-ab)}+\dfrac{c+a}{(c^2-ab)(a^2-bc)}.$

13. $\dfrac{(1+ab)(1+ac)}{(a-b)(c-a)}+\dfrac{(1+bc)(1+ab)}{(a-b)(b-c)}+\dfrac{(1+ac)(1+bc)}{(b-c)(c-a)}.$

14. $\dfrac{a^2(a+b)(a+c)}{(a-b)(a-c)}+\dfrac{b^2(b+c)(b+a)}{(b-c)(b-a)}+\dfrac{c^2(c+a)(c+b)}{(c-a)(c-b)}.$

15. $\dfrac{a^3(a+b)(a+c)}{(a-b)(a-c)}+\dfrac{b^3(b+c)(b+a)}{(b-c)(b-a)}+\dfrac{c^3(c+a)(c+b)}{(c-a)(c-b)}.$

XVIII.

Reduce to their simplest forms the expressions:—

1. $\dfrac{a-b}{(x-a)(x-b)}+\dfrac{b-c}{(x-b)(x-c)}+\dfrac{c-a}{(x-a)(x-c)}.$

2. $\dfrac{x+c}{(x-a)(x-b)}+\dfrac{x+b}{(x-a)(x-c)}+\dfrac{x+a}{(x-b)(x-c)}.$

3. $\dfrac{x-a}{(a-b)(a-c)}+\dfrac{x-b}{(b-c)(b-a)}+\dfrac{x-c}{(c-a)(c-b)}.$

4. $\dfrac{(x+b)(x+c)}{(a-b)(a-c)}+\dfrac{(x+a)(x+c)}{(b-a)(b-c)}+\dfrac{(x+a)(x+b)}{(c-a)(c-b)}.$

5. $\dfrac{(x-b)(x-c)}{(a-b)(a-c)}+\dfrac{(x-c)(x-a)}{(b-c)(b-a)}+\dfrac{(x-a)(x-b)}{(c-a)(c-b)}.$

6. $\dfrac{xy}{ab}+\dfrac{(x-a)(y-a)}{a(a-b)}+\dfrac{(x-b)(y-b)}{b(b-a)}.$

7. $\dfrac{bc(x-a)^2}{(a-b)(a-c)}+\dfrac{ca(x-b)^2}{(b-c)(b-a)}+\dfrac{ab(x-c)^2}{(c-a)(c-b)}.$

8. $\dfrac{(x-a)(y-a)(z-a)}{(a-b)(a-c)}+\dfrac{(x-b)(y-b)(z-b)}{(b-a)(b-c)}+\dfrac{(x-c)(y-c)(z-c)}{(c-b)(c-a)}.$

9. $\dfrac{abx}{(ax-by)(ax-cz)}+\dfrac{acy}{(by-ax)(by-cz)}+\dfrac{bcz}{(cz-ax)(cz-by)}.$

XIX.

Determine the aggregates of—

1. $\dfrac{1}{a(a-b)(a-c)}+\dfrac{1}{b(b-c)(b-a)}+\dfrac{1}{c(c-a)(c-b)}$.

2. $\dfrac{bc}{a(a^2-b^2)(a^2-c^2)}+\dfrac{ac}{b(b^2-a^2)(b^2-c^2)}+\dfrac{ab}{c(c^2-b^2)(c^2-a^2)}$.

3. $\dfrac{1}{(b-a)(c-a)(x+a)}+\dfrac{1}{(a-b)(c-b)(x+b)}+\dfrac{1}{(a-c)(b-c)(x+c)}$.

4. $\dfrac{a^2}{(a-b)(a-c)(x-a)}+\dfrac{b^2}{(b-a)(b-c)(x-b)}+\dfrac{c^2}{(c-a)(c-b)(x-c)}$.

5. $\dfrac{a^2}{(a-b)(a-c)(a+x)}-\dfrac{b^2}{(a-b)(b-c)(b+x)}+\dfrac{c^2}{(a-c)(b-c)(c+x)}$.

6. $\dfrac{a^2}{(a-b)(a-c)(b+c)}+\dfrac{b^2}{(b-c)(b-a)(a+c)}+\dfrac{c^2}{(c-a)(c-b)(b+a)}$.

7. $\dfrac{b+c-a}{(b+c)(c-a)(a-b)}+\dfrac{c+a-b}{(c+a)(a-b)(b-c)}+\dfrac{a+b-c}{(a+b)(b-c)(c-a)}$.

8. $\dfrac{1}{(a-b)(a-c)(a-d)}+\dfrac{1}{(b-a)(b-c)(b-d)}+\dfrac{1}{(c-a)(c-b)(c-d)}$
$+\dfrac{1}{(d-a)(d-b)(d-c)}$.

XX.

Find the simplest forms of the following expressions:—

1. $\dfrac{a}{b+c-a}-\dfrac{b}{c+a-b}+\dfrac{c}{a+b-c}$.

2. $\dfrac{1}{a+b+c}+\dfrac{1}{a-b-c}+\dfrac{1}{b-c-a}+\dfrac{1}{c-a-b}$.

3. $\dfrac{a}{(c+a-b)(a+b-c)}-\dfrac{b}{(a+b-c)(b+c-a)}+\dfrac{c}{(b+c-a)(c+a-b)}$.

4. $\dfrac{(a+c-b)(a+b-c)}{(a-b)(a-c)}+\dfrac{(a+b-c)(b+c-a)}{(b-c)(b-a)}+\dfrac{(b+c-a)(a+c-b)}{(c-a)(c-b)}$.

5. $\dfrac{a(b+c)}{b+c-a}+\dfrac{b(c+a)}{c+a-b}+\dfrac{c(a+b)}{a+b-c}$.

6. $\dfrac{a+b-c}{a-b+c}-\dfrac{a-b+c}{a+b-c}-\dfrac{(b-c)^2}{a^2-(b-c)^2}$.

7. $\left\{\dfrac{a^2+b^2-c^2}{ab}-\dfrac{b^2+c^2-a^2}{bc}\right\}\cdot\left\{\dfrac{a+b+c}{a+c-b}-\dfrac{a+c-b}{a+b+c}\right\}$.

8. $\dfrac{a^2-(b-c)^2}{(b+c)^2-a^2}+\dfrac{b^2-(c-a)^2}{(c+a)^2-b^2}+\dfrac{c^2-(a-b)^2}{(a+b)^2-c^2}$.

9. $\dfrac{a^2-(b-c)^2}{(b+c)^2-a^2}\times\dfrac{b^2-(c-a)^2}{(c+a)^2-b^2}\times\dfrac{c^2-(a-b)^2}{(a+b)^2-c^2}$.

10. $\left\{\dfrac{b+c-2a}{b-c}+\dfrac{c+a-2b}{c-a}+\dfrac{a+b-2c}{a-b}\right\}\cdot\left\{\dfrac{b-c}{b+c-2a}+\dfrac{c-a}{c+a-2c}\right.$
$\left.+\dfrac{a-b}{a+b-2c}\right\}$.

XXI.

Verify the equivalent expressions—

1. $\dfrac{a}{a+b}+\dfrac{b}{a-b}=\dfrac{a}{a-b}-\dfrac{b}{a+b}.$

2. $\dfrac{a^2+b^2}{a^2-b^2}+2+\dfrac{a^2-b^2}{a^2+b^4}=\dfrac{4a^4}{a^4-b^4}.$

3. $\dfrac{a^2(b^2-c^2)-ab(2b^2+bc-c^2)+b^3(b+c)}{a^2(b+c)^2-ab(2b^2+3bc+c^2)+b^3(b+c)}=\dfrac{a(b-c)-b^2}{a(b+c)-b^2}.$

4. $\dfrac{(a+b+d)(a+b)-c(c+d)}{(a+b+d)(a+d)-c(b+c)}=\dfrac{a+b-c}{a-c+d}.$

5. $\dfrac{d(a-b)(b-c)+b(a-d)(c-d)}{c(a-b)(a-d)+a(b-c)(c-d)}=\dfrac{b-d}{a-c}.$

6. $\dfrac{(a-b)(b-d)+(b-c)(c-a)+(d-a)(a-c)}{(a-c)(c-d)+(b-d)(d-a)+(d-b)(b-c)}=\dfrac{a-c}{b-d}.$

7. $\dfrac{(a^2+yz)(b^2+xz)-(ab+cz)^2}{(a^2+yz)(c^2-xy)-(ac-by)^2}=-\dfrac{z}{y}.$

8. $\dfrac{(a+bc)(b+ca)(c+ab)+(a-bc)(b-ca)(c-ab)}{(1+a+b+c)^2+(1+a-b-c)^2+(1-a+b-c)^2+(1-a-b+c)^2}$
$$=\frac{abc}{2}.$$

9. $\dfrac{(2b-c-a)^3-(2c-a-b)^3}{(c-a)^3-(a-b)^3}=\dfrac{9(b-c)}{b+c-2a}.$

10. $\dfrac{a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)}{a^3(b-c)+b^3(c-a)+c^3(a-b)}=\dfrac{bc+ca+ab}{a+b+c}.$

11. $\dfrac{(ca-b^2)(ab-c^2)+(ab-c^2)(bc-a^2)+(bc-a^2)(ca-b^2)}{(bc-a^2)^2+(ca-b^2)^2+(ab-c^2)^2}$
$$=\frac{a^2+b^2+c^2+ab+ac+bc}{ab+ac+bc}.$$

12. $\dfrac{(ab+cd)(a^2+b^2-c^2-d^2)-(ac+bd)(a^2-b^2+c^2-d^2)}{(a^2+b^2-c^2-d^2)(a^2-b^2+c^2-d^2)+4(ab+cd)(ac+bd)}$
$$=\frac{(a-d)(b-c)}{(a+d)^2-(b-c)^2}.$$

13. $\dfrac{\{(a-1)x-(b-1)y\}(ax-by)-(a-b)^2xy}{\{(a-1)x-(b-1)y\}(ay-bx)-(a-b)^2xy}=-\dfrac{a(a-1)x-b(b-1)y}{b(a-1)x+a(b-1)y}.$

14. $\dfrac{(2x-y-z)^2+(2y-z-x)^2+(2z-x-y)^2}{(y-z)^2+(z-x)^2+(x-y)^2}$
$$=\frac{(x+y+z)^2+(z-y)^2+(x-z)^2+(y-x)^2}{x^2+y^2+z^2}.$$

15. $\dfrac{b^2c^2(c^2-b^2)+c^2a^2(a^2-c^2)+a^2b^2(b^2-a^2)}{bc(c-b)+ca(a-c)+ab(b-a)}$
$$=\frac{(a^2-b^2)^3+(b^2-c^2)^3+(c^2-a^2)^3}{(a-b)^3+(b-c)^3+(c-a)^3}.$$

16. $$(x-a)^2+(y-b)^2+z^2-\frac{\{z+m(x-a)+n(y-b)\}^2}{1+m^2+n^2}$$
$$=\frac{(x-mz-a)^2+(y-nz-b)^2+\{n(x-a)-m(y-b)\}^2}{1+m^2+n^2}.$$

XXII.

Reduce to their simplest forms—

1. $$\left\{1+\frac{20}{x+1}-\frac{180}{x+2}+\frac{420}{x+3}-\frac{280}{x+4}\right\}\cdot\left\{1-\frac{20}{x-1}+\frac{180}{x-2}-\frac{420}{x-3}\right.$$
$$\left.+\frac{280}{x-4}\right\}.$$

2. $$1+\frac{1}{4}\left\{\frac{(x+a)}{2(x-a)}+\frac{x-a}{2(x+a)}-\frac{x^2-a^2}{x^2+a)}\right\}^2.$$

3. $$\frac{(x+y)(1-xy)}{(1-xy)^2-(x+y)^2}-\frac{x(1-y^2)+y(1-x^2)}{(1-x^2)(1-y^2)-4xy}.$$

4. $$1+\frac{1-x}{1+x}+\frac{(1-x)(1-x^2)}{(1+x)(1+x^2)}+\frac{(1-x)(1-x^2)(1-x^3)}{(1+x)(1+x^2)(1+x^3)}.$$

5. $$\left\{\frac{ax^2-ay^2+2bxy}{x^2+y^2}\right\}^2+\left\{\frac{by^2-bx^2+2axy}{x^2+y^2}\right\}^2.$$

6. $$\frac{(ay-bx)^3-(ax-by)^3}{(a+b)(y-x)}+\frac{(ay-bx)^3+(ax-by)_3}{(a-b)(y+x)}.$$

7. $$\frac{x^2+xy+y^2}{x^2y^2}\cdot\left\{\frac{1}{x}+\frac{1}{y}\right\}+\frac{y^2+yz+z^2}{y^2z^2}\cdot\left\{\frac{1}{y}+\frac{1}{z}\right\}$$
$$-\frac{x^2+xz+z^2}{x^2z^2}\cdot\left\{\frac{1}{x}+\frac{1}{z}\right\}.$$

8. $$\left\{\frac{2x}{x+y}-\frac{x^2}{x^2-y^2}+\frac{2y}{x-y}\right\}\times\left\{\frac{1}{x}+\frac{1}{y}\right\}\div\left\{\frac{3}{x-y}+\frac{1}{y}+\frac{2}{x}\right\}.$$

9. $$\left\{\frac{a+bx}{a-bx}+\frac{b+ax}{b-ax}\right\}\div\left\{\frac{a+bx}{a-bx}-\frac{b+ax}{b-ax}\right\}.$$

10. $$\frac{\{ab-1-(a-b)x\}^2}{\{ab-1-(a-b)x\}^2-\{a-b-(ab-1)x\}^2}$$
$$\div\frac{\{ab-1-(a-b)x\}^2}{(1-ab)\{1+ab+(a-b)x\}-(a-b)\{a+b+(ab+1)x\}}.$$

11. $$\frac{(a+b)(a^2+b^2-c^2)}{ab}+\frac{(b+c)(b^2+c^2-a^2)}{bc}+\frac{(a+c)(a^2+c^2-b^2)}{ac}.$$

12. $$\left\{\frac{1+x}{1-x}+\frac{4x}{1+x^2}+\frac{8x}{1-x^4}-\frac{1-x}{1+x}\right\}\div\left\{\frac{1+x^2}{1-x^2}+\frac{4x^2}{1+x^4}-\frac{1-x^2}{1+x^2}\right\}.$$

13. $$\left\{\frac{(x+y)^2}{(x-y)^2}+1\right\}\cdot\left\{\frac{(y+z)^2}{(y-z)^2}+1\right\}\cdot\left\{\frac{(z+x)^2}{(z-x)^2}+1\right\}$$
$$\times\left\{\frac{(x-y)^2}{x^2+y^2}-1\right\}\cdot\left\{\frac{(y-z)^2}{y^2+z^2}-1\right\}\cdot\left\{\frac{(z-x)}{z^2+x^2}-1\right\}.$$

14. $$\left\{\frac{(x+2y)^2x}{(x-y)^2}-\frac{(2x+y)^2y}{(x-y)^2}\right\}\times\left\{\frac{(x-2y)^2x}{(x+y)^2}+\frac{(2x-y)^2y}{(x+y)^2}\right\}.$$

15. $\left\{\frac{(a-2b)^2a^2}{a^2+b^2}-\frac{(2a+b)^2b^2}{a^2+b^2}\right\}\div\left\{\frac{(a+2b)^2a^2}{a^2-b^2}-\frac{(2a+b)^2b^2}{a^2-b^2}\right\}.$

16. $\left\{\frac{(x-2y)^3x}{(x+y)^3}+\frac{(2x-y)^3y}{(x+y)^3}\right\}\div\left\{\frac{(x+2y)^3x}{(x-y)^3}-\frac{(2x+y)^3y}{(x-y)^3}\right\}.$

XXIII.

1. If $s=a+b+c+d$;

then $\frac{s-a}{a}+\frac{s-b}{b}+\frac{s-c}{c}+\frac{s-d}{d}=\left\{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right\}\cdot s-4.$

2. If $s=a+b+c+d+\ \ldots\ldots$ to n terms,

then shall $\frac{s-a}{s}+\frac{s-b}{s}+\frac{s-c}{s}+\ \ldots\ldots\ =n-1.$

3. If $x^2-yz=a^2$, $y^2-zx=b^2$, $z^2-xy=c^2$,

find the value of $\frac{a^2x+b^2y+c^2z}{x+y+z}$ in terms of a, b, c.

4. If $x=\frac{b-c}{a}$, $y=\frac{c-a}{b}$, $z=\frac{a-b}{c}$; then $xyz+x+y+z=0$.

5. If $y+z+u=ax$, $z+u+x=by$, $u+x+y=cz$, $x+y+z=du$;

then shall $\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}+\frac{1}{d+1}=1.$

6. If $a=\frac{x-y}{x+y}$, $b=\frac{y-z}{y+z}$, $c=\frac{z-x}{z+x}$;

prove that $\frac{1+a}{1-a}\cdot\frac{1+b}{1-b}\cdot\frac{1+c}{1-c}=1.$

7. If $\frac{a}{x}(b-c)+\frac{b}{y}(c-a)+\frac{c}{z}(a-b)=0$;

then $\frac{x}{a}(z-y)+\frac{y}{b}(x-z)+\frac{z}{c}(y-x)=0.$

8. If $2s=a+b+c$,

then shall $\frac{1}{s-a}+\frac{1}{s-b}+\frac{1}{s-c}+\frac{1}{s}=\frac{abc}{s(s-a)(s-b)(s-c)}.$

9. If $\frac{b^2+c^2-a^2}{2bc}+\frac{c^2+a^2-b^2}{2ac}+\frac{a^2+b^2-c^2}{2ab}=1$;

then shall $(a+b-c)(a+c-b)(b+c-a)=0$.

10. If $\frac{a^2+bc}{a^2-bc}+\frac{b^2+ca}{b^2-ca}+\frac{c^2+ab}{c^2-ab}=1$, a, b, c being each greater than 0;

then shall $\frac{a^2}{b^2+c^2-a^2}+\frac{b^2}{a^2+c^2-b^2}+\frac{c^2}{a^2+b^2-c^2}=-\frac{3}{2}.$

11. If $(a^2-bc)(b^2-ac)(c^2-ab)=0$,

then shall $\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=\frac{a^3+b^3+c^3}{a^2b^2c^2}.$

XXIV.

1. Find the difference of $\frac{x}{x-a}+\frac{x}{x-b}+\frac{x}{x-c}$ and $\frac{a}{x-a}+\frac{b}{x-b}+\frac{c}{x-c}$,

and ascertain if the value of the difference is dependent on the value of x.

2. If $xyz=1$, then $(1+x+y^{-1})^{-1}+(1+z+x^{-1})^{-1}+(1+y+z^{-1})^{-1}=1$.

3. If $a=(b+c)x$, $b=(c+a)y$, $c=(a+b)z$;
then $1-xy-yz-zx-2xyz=0$.

4. If $x=\dfrac{a+b-c}{a+b+c}$, then $\dfrac{a+bx^2}{b+cx^2}=\dfrac{(a-b+c)^2+4ab}{(b-a+c)^2+4ab}$.

5. If $x=\dfrac{2ab+b^2}{a^2+ab+b^2}$ and $y=\dfrac{a^2-b^2}{a^2+ab+b^2}$; then $x^3+y=y^3+x$.

6. If $x=a+b+\dfrac{(a-b)^2}{4(a+b)}$ and $y=\dfrac{a+b}{4}+\dfrac{ab}{a+b}$;
shew that $(x-a)^2-(y-b)^2=b^2$.

7. If $x+y+z=0$; then $\dfrac{x(y^3-z^3)}{y-z}+\dfrac{y(z^3-x^3)}{z-x}+\dfrac{z(x^3-y^3)}{x-y}=0$.

8. If $\dfrac{a}{b+x}-\dfrac{c}{d+y}=\dfrac{a}{b}-\dfrac{c}{d}$; shew that either number is equal to $\dfrac{bc}{dx}-\dfrac{ad}{by}$.

9. Of the fractions $\dfrac{(a+b)(c+d)}{ab+cd}$, $\dfrac{(a+c)(b+d)}{ac+bd}$, $\dfrac{(a+d)(b+c)}{ad+bc}$,
if any two be equal, the third is equal to either of them, and each to -1 : a, b, c, being unequal quantities.

10. If $\dfrac{(ab+ac-bc)(bc+ac-ab)}{ab+bc-ac}=\dfrac{(ab+ac-bc)(ab+bc-ac)}{bc+ac-ab}$
$=\dfrac{(ab+bc-ac)(bc+ac-ab)}{ab+ac-bc}$; then shall $a=b=c$.

11. Shew that for all values of x, the sum of these two expressions
$\dfrac{(x-a)^2}{(a-b)(a-c)}+\dfrac{(x-b)^2}{(b-c)(b-a)}+\dfrac{(x-c)^2}{(c-a)(c-b)}$, and
$\left\{\dfrac{1}{a-b}+\dfrac{1}{a-c}\right\}(x-a)+\left\{\dfrac{1}{b-c}+\dfrac{1}{b-a}\right\}(x-b)$
$+\left\{\dfrac{1}{c-a}+\dfrac{1}{c-b}\right\}(x-c)$, is equal to a *constant* negative quantity; supposing a, b, c constant and x variable.

12. Determine the value of the fraction $\dfrac{x^3-2x^2+3}{x^3+3x^2-2}$ when $x=-1$.

13. What value of x will cause both the numerator and the denominator of the fraction $\dfrac{12x^3-23x^2a+19xa^2-6a^3}{9x^3-21x^2a+22xa^2-8a^3}$ to vanish, and what is the true value of the fraction in that case?

14. Reduce $\dfrac{x^3+6x^2+13x+10}{x^4+6x^3+14x^2+15x+7}$ to the form of a continued fraction, and verify the truth of the result by reversing the process.

XXV.

1. Which is the greater fraction, $\frac{1+a^2}{1-a^2}$ or $\frac{1+a}{1-a}$?

2. Shew that $\frac{a}{b^2}+\frac{b}{a^2}$ is always greater than $\frac{1}{b}+\frac{1}{a}$.

3. If m be less than n, then shall $\frac{a+mb}{b+ma}$ be greater than, equal to, or less than $\frac{a+nb}{b+na}$, according as a is greater than, equal to, or less than b.

4. If $\frac{a}{b}$ and $\frac{c}{d}$ be any two fractions, and m, n be any integral numbers; shew that $\frac{ma+nc}{mb+nd}$ is intermediate in value between $\frac{a}{b}$ and $\frac{c}{d}$.

5. Which is the greater fraction $\frac{ax-by}{ay+bx}$ or $\frac{ay-bx}{ax+by}$?

6. Prove that $\frac{1}{a+c}$ is always less than $\frac{2b}{(a+b)(b+c)}$, if b be intermediate in value to a and c.

7. What are the integral values of x, if $\frac{1}{4}(x+2)+\frac{x}{3}$ be less than $\frac{1}{2}(x+4)+3$, but greater than $\frac{1}{2}(x+1)+\frac{1}{3}$?

8. If $x+\frac{2}{x}$ be greater than 3: then shall $x^2+\frac{2}{x^2}>3$.

9. Shew that the sum of every fraction and its reciprocal is equal to, or greater than 2, and that $\frac{a}{b}+\frac{b}{c}+\frac{c}{d}+\frac{c}{b}+\frac{b}{a}+\frac{d}{c}>6$.

10. If a, b, c be unequal, then $a^2(b+c)+b^2(a+c)+c^2(a+b)>6abc$.

11. If a, b, c be three quantities in order of magnitude,

$$\text{then shall } \frac{a}{c}+\frac{b}{a}+\frac{c}{b}>\frac{b}{c}+\frac{a}{b}+\frac{c}{a}.$$

12. Shew that $\frac{1}{b+c-a}+\frac{1}{c+a-b}+\frac{1}{a+b-c}>\frac{9}{a+b+c}$.

13. If a, b, c be such that the sum of any two of them is greater than the third; shew that $\frac{a}{b+c-a}+\frac{b}{c+a-b}+\frac{c}{a+b-c}>3$.

XXVI.

1. If $a+\frac{1}{b}=1$ and $c+\frac{1}{a}=1$, then $b+\frac{1}{c}=1$.

2. If $a-\frac{bc}{a}=b-\frac{ac}{b}$, then each is equal to $c-\frac{ab}{c}$.

3. If $a-ab=1$ and $b-bc=2$, then $c=\frac{1-a}{1-2a}$.

4. If $ax+a'=bx+b'=cx+c'$; then $ab'+a'c+bc'=a'b+b'c+c'a$.

5. If $\frac{a-b}{1+ab}+\frac{c-d}{1+cd}=0$, prove that $\frac{b-c}{1+bc}+\frac{d-a}{1+ad}=0$.

6. If $\frac{y}{b}+\frac{c}{z}=1$ and $\frac{z}{c}+\frac{a}{x}=1$, then $\frac{xyz}{abc}=-1$.

7. If $x=\frac{e+z}{1+ez}$, then shall $\frac{1-x}{1+x}=\frac{1-e}{1+e}\cdot\frac{1-z}{1+z}$.

8. If $y=\frac{a+bx}{c+dx}$ and $z=\frac{a+by}{c+dy}$; then shall $x=\frac{a+bz}{c+dz}$, provided that $b^2+c^2+ad+bc=0$.

9. If $\frac{a^2(b-c)}{a-d}=\frac{b^2(a-c)}{b-d}$, then $\frac{1}{a}+\frac{1}{b}=\frac{1}{c}+\frac{1}{d}$; and conversely.

10. If $(a^2+b^2)x^2=1$, then shall $\frac{a^3}{1-bx}+\frac{b^3}{1+ax}=(a+b)(a^2+b^2)$.

11. If $xy=ab(a+b)$ and $x^2-xy+y^2=a^3+b^3$, shew that $\left\{\frac{x}{a}-\frac{y}{b}\right\}\left\{\frac{x}{b}-\frac{y}{a}\right\}=0$.

12. If $\frac{x^2-y^2}{a-b}=\frac{xy}{z}$, and $\frac{y^2-z^2}{b-c}=\frac{yz}{x}$, then shall $\frac{z^2-x^2}{c-a}=\frac{zx}{y}$.

13. If $\frac{ad-bc}{a-b-c+d}=\frac{ac-bd}{a-b+c-d}$, then $a+b=c+d$, and each fraction is equal to $\frac{1}{4}(a+b+c+d)$.

14. If $\frac{1}{x^3}\left\{\frac{y}{z}+\frac{z}{y}\right\}=\frac{1}{y^3}\left\{\frac{z}{x}+\frac{x}{z}\right\}=\frac{1}{z^3}\left\{\frac{x}{y}+\frac{y}{x}\right\}$, then shall each $=\frac{2}{xyz}$.

15. If $\frac{x}{a-b}+\frac{y}{b-c}+\frac{z}{c-a}=0$, then $x+y+z=0$.

16. If $\frac{1}{x-y}+\frac{1}{y-z}+\frac{1}{z-x}=0$; shew that $x=y=z$.

17. If $\frac{x}{b-c}=\frac{y}{c-a}=\frac{z}{a-b}$; then shall $ax+by+cz=0$.

18. If $ab+ac+bc=0$, then $\frac{1}{a^2-bc}+\frac{1}{b^2-ac}+\frac{1}{c^2-ab}=0$.

19. If $\frac{x}{y+z}=a$, $\frac{y}{x+z}=b$, $\frac{z}{x+y}=c$,

then shall $\frac{x^2}{a(1-bc)}=\frac{y^2}{b(1-ac)}=\frac{z^2}{c(1-ab)}$.

20. If $\frac{x+y-z}{ax+by-cz}=\frac{y+z-x}{by+cz-ax}=\frac{x+z-y}{ax+cz-by}$, then $a=b=c$.

21. Shew that if $\left\{\frac{y+z}{x}+\frac{z+x}{y}+\frac{x+y}{z}\right\}=a$, and $\left\{\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right\}=b$, then $ab=9$, if $x+y+z=0$.

RESULTS, HINTS, ETC., FOR THE EXERCISES ON COMMON MEASURES AND COMMON MULTIPLES.

I.

1. See Section V., Art. 2, pp. 1, 2.
8. The numbers 12 and 6 ; 4 and 18 ; 2 and 36, respectively fulfil the conditions.
9. $x^4+12x^3+49x^2+82x+48$.

II.

The following are the highest common divisors :—

1. $3x+1$. 2. $x-1$. 3. $x-1$. 4. $2x-y$. 5. $x-a$. 6. x^2+2x. 7. $3x^2-2$.
8. x^3+3x^2+3x+1. 9. $x-1$. 10. $x+2$. 11. x^2-1. 12. $x-a$.
13. $xy+ab$. 14. x^2-1. 15. $1-ax$. 16. $x+a+b$. 17. $a+b+c$. 18. $(x-1)^2$.

NOTE.—In the examples numbered 2, 3, 5, 9, 11, 14, 15, 18, the highest common divisors may be found by inspection. See Section IV., Art. 13, p. 24.

III.

The following are the least common multiples :—

1. $n(n-1)(n-2)(n-4)a^{n-1}x^4$. 2. $12x^4-41x^3+25x^2+16x-12$.
3. $a(x+a)(x-a)^2(x^2+a^2)$. 4. $3x^6-11x^5-8x^4+42x^3+19x^2-47x-30$.
5. $(2a^4+3a^3b+3a^2b^2+3ab^3+2b^4)(a^5+3a^4b+a^3b^2+a^2b^3+3ab^4+b^5)$.
6. $(a^2+ax+x^2)(a^2-x^2)$. 7. x^4-16a^4. 8. $(3x-1)(4x-1)(7x-2)$.
9. $(x^2-y^2)(x^4+x^2y^2+y^4)$. 10. $x^4+10x^3+35x^2+50x+24$.
11. $36(a^2+b^2)(a^3-b^3)(a-b)^3.(a+b)$. 12. $x(x+1)(x^2+1)(x^2-x+1)$.
13. $x(x+1)(x^2+x+1)(x^4-1)$. 14. $(2x-3)(3x-2)(4x+1)(5x+4)$.
15. See Exercises XIX., No. 4, p. 35.

IV.

1. $a=8$, the highest common divisor is x^2-4x+3, and the fraction in its lowest terms is $\frac{x-4}{x-5}$.

2. It may be shewn that $x^3+y^3+z^3=-3xyz$, and $x^5+y^5+z^5=7xyz(yx-x^2)$. The truth of this may be verified if $x=2$, $y=3$, $z=-5$.

3. At the third step of the process for finding the highest common divisor of the two quantities, it appears that the question is reduced to ascertaining if $a^{m-2}-b^{m-2}$ is divisible by a^2+ab+b^2.

4. It is sufficient to remark that the quotient of an odd number divided by an odd number is an odd number, and the sum of two odd numbers is an even number.

5. It is possible, that when numerical values are assigned to the symbols of two algebraical expressions, and the greatest common measure of these numbers be found, the result may not be the same as that found by assigning the same numerical values to the symbols of the highest common divisor. This may arise either from the relation of the numbers, or from the introduction or removal of factors in the process of finding the highest common divisors.

In the first example, the highest common divisor is x^2+x+1. When 4 is substituted for x in the two given expressions, their values become 231 and 399, and the greatest common measure of these two numbers is 21. And when 4 is substituted for x in x^2+x+1, the highest common divisor, the result is 21, which is the same as the greatest common measure of the two numbers found by the same substitution.

In the second example, the highest common divisor is $3x+4a$. When 4 is substituted for x and 1 for a, the two expressions become 48 and 192, and their greatest common measure is 48. But when these substitutions are made for x and a in $3x+4a$, the highest common divisor, the result is 16 for the greatest common measure which is different from 48. The reason of this discrepancy arises from the fact that in the algebraical process, factors have been rejected which are common measures of the corresponding arithmetical expressions.

6. As the common divisor is a quadratic factor, the remainder will be of the form $px+q$, and must be equal to zero. And as x is not zero, there must be $p=o$ and $q=o$, from which the required condition will be determined.

NOTE.—The second remainder in the process is $bcx^2+(c^2+b^2-a^2)x+bc$, which is equal to $bc(x^2+\frac{c^2+b^2-a^2}{bc}.x+1)$. If m be put for $\frac{c^2+b^2-a^2}{bc}$, the rest of the process may be considerably simplified.

7. The common divisor is of the form $x+b$, and the remainder does not contain x. This remainder made equal to zero, when reduced, will give the required condition.

8. Let $\frac{p}{10^m}$ and $\frac{q}{10^{m+n}}$ denote the two decimals, and suppose r to be the greatest common measure of p and q.

Then $D=\frac{r}{10^m}$, and $M=\frac{pq}{r10^{m+n}}$, $\therefore$ $\frac{D}{M}=\frac{r}{10^m}\times\frac{r.10^{m+n}}{pq}=\frac{r^2}{pq}.10^n$.

9. The highest common divisor is a quadratic factor of the form x^2+px+q if a and a' are prime to each other.

10. If x^2+ax+b be divided by $x+c$, the quotient is $x+(a-c)$ and the remainder $b-(a-c)c=0$, and $b=c(a-c)$.

If x^2+mx+a be also divided by $x+c$, the quotient is $x+(m-c)$ and the remainder $a-c(m-c)=0$.

The product of x^2+ax+b and $x+(m-c)$ will be the least common multiple. If $c(a-c)$ be substituted for b, the required form will be determined.

11. If each expression be divided by $x+q$, the remainders will not contain x. If each be made equal to zero, a value of q may be found for each expression, and by equating these values of q, the required condition will be determined.

RESULTS, HINTS, ETC., FOR THE EXERCISES ON THE ALGEBRAIC FRACTIONS.

I.

1. See Section V., Art. 2, p. 8.

2. Let $\frac{a}{b}$ be a fraction in which a is prime to b. If possible let the fraction $\frac{a}{b}$ be equivalent to the fraction $\frac{c}{d}$, having its terms respectively less than a and b. Then, since $\frac{a}{b}=\frac{c}{d}$, $\therefore$ $\frac{ad}{b}=c$, and ad is divisible by b. Since b is prime to a, d must be divisible by b.

This is impossible, for d is less than b.

3. For since $\frac{a}{b}=\frac{c}{d}$; $\therefore$ $a=\frac{bc}{d}$; hence bc is divisible by d. But as d is prime to c, must be divisible by d. Let $\frac{b}{d}=m$, then $b=md$, and $a=\frac{cmd}{d}=mc$; or a and b are respectively equimultiples of c and d.

4. The fraction $\frac{a+x}{b+x} > \frac{a}{b}$, if $b > a$; and $\frac{a+x}{b+x} < \frac{a}{b}$, if $b < a$.

The fraction $\frac{a-x}{b-x} < \frac{a}{b}$, if $b > a$; and $\frac{a-x}{b-x} > \frac{a}{b}$, if $b < a$.

5. Let $\frac{a}{b} > \frac{c}{d}$, then $ad > bc$, and $ab = ba$, $\therefore a(b+d) > b(a+c)$,

and $\therefore \frac{a}{b} > \frac{a+c}{b+d}$; similarly, $\frac{c}{d} < \frac{a+c}{b+d}$.

Hence $\frac{a+c}{b+d}$ is greater than $\frac{c}{d}$, but less than $\frac{a}{b}$, consequently $\frac{a+c}{b+d}$ lies between $\frac{a}{b}$ and $\frac{c}{d}$.

It may be added here, that the fraction formed of the sums of the numerators and the sums of the denominators of three, four, or more unequal fractions is always intermediate in value between the greatest and least of these fractions.

6. For let $\frac{a}{b}$ and $\frac{c}{dx}$ be two fractions in their lowest terms, of which x a factor in the denominator of one is not contained in b the denominator of the other.

Now $\frac{a}{b} \pm \frac{c}{dx} = \frac{adx \pm cb}{bdx}$.

And if $\frac{a}{b} \pm \frac{c}{dx}$ be an integer,

then $\frac{adx \pm cb}{bdx}$ must also be an integer, which cannot be the case, unless cb be divisible by bdx, that is, unless cb be divisible by each of the factors of bdx.

But b and c are each prime to x.

Hence cb cannot be divisible by x;

$\therefore \frac{a}{b} \pm \frac{c}{dx}$ cannot be an integer.

8. See Art. 3, p. 9.

9. Let $\frac{a}{b}$ be the fraction, then $\left\{\frac{a}{b} - \frac{b}{a}\right\}^3 = \left\{\frac{a}{b}\right\}^3 - \left\{\frac{b}{a}\right\}^3 - 3\left\{\frac{a}{b} - \frac{b}{a}\right\}$,

and obviously $\left\{\frac{a}{b} - \frac{b}{a}\right\}^3$ is less than $\left\{\frac{a}{b}\right\}^3 - \left\{\frac{b}{a}\right\}^3$ by $3\left\{\frac{a}{b} - \frac{b}{a}\right\}$.

10. $(a^2+b^2)(c^2+d^2) = (ac+bd)^2 + (ad-bc)^2$.

11. The expression is equivalent to x^2+2x+3, which will be integral for all integral values of x. If $x^3+y^3+z^3$ be divided by $x+y+z$, there is a remainder $-3yz(y+z)$.

II.

1. $\frac{5}{8}$. 2. $-\frac{179}{180}$. 3. $\cdot 095310\dot{3}$. 4. $+4$. 5. $-\frac{b}{a}$. 6. $\frac{50}{77}$. 8. $\frac{35}{(24)^2}$.

9. $\frac{1}{6a}$ and $\frac{-9}{4a}$. 10. $\cdot 36913 \ldots .$

III.

1. The H. C. D. is $x+3$ Lowest terms $\frac{x-1}{x+1}$.

2 ... ... $x+a$ ... ... $\frac{3x^3(x^2+ax+a^2)}{7a^4(x+a)}$.

3 ... ... $x-2y$ ... ... $\frac{2x+3y}{3x-2y}$.

4 H. C. D. is	$x-y$	Lowest terms	$\frac{x^2(x^2+4xy+4y^2)}{x^3-y^3}$.
5	$2x-1$	... ...	$\frac{(5x-1)x^2}{2x^3-1}$.
6	$(3x^2-2ax+a^2)a$	... ...	$\frac{5ax(2x+a)}{3x+2a}$.
7	$1-2x+3x^2$	... ...	$\frac{1-x^2}{1+x+x^2}$.
8	x^3-3x^2+3x-1	... ...	$\frac{x+2}{x^2+x-1}$.
9	$x(x^2+2x+2)$	... ...	$\frac{1}{x^2-2x+2}$.
10	$x-3$	... ...	$\frac{10x^2-24x+15}{5x^3-21x^2+24x-18}$.
11	$(x+2)(x+3)(2x+1)$	... ...	$\frac{x^2+1}{x^2-1}$.
12	$x-1$	... ...	$\frac{5x^3+x^2+x+1}{x^4-1}$.
13	$3x-2y$	... ...	$\frac{x^2-xy+y^2}{x^2+xy+y^2}$.
14	$x-1$	... ...	$\frac{8(x^3+x^2+x+1)x^3-21(x^2+x+1)}{21(x^2+x+1)x^4-8(x^3+x^2+x+1)}$.
15	$x+a$	... ...	$\frac{x^2-bx+b^2}{x^2+bx+b^2}$.
16	$(b+c)(a-b)$	... ...	$\frac{ab-ac-b^2}{a(ab+ac-b^2)}$.
17	$x-a$	... ...	$\frac{3x-(a+2b)}{x^2-(a+b)x+ab}$.
18	$(a-1)x+1$	... ...	$\frac{(a^2+a+1)x^2+(a+1)x-1}{(a-1)^2x^2-(a+1)x-1}$.

Note to 14. If the numerator be subtracted from the denominator, then the remainder is $21(x^2-1)-21(x-1)x^3-8(x^2-1)$ of which each term is divisible by $x-1$.

Note to 16. If the factor $b+c$ be removed from the numerator and the denominator, the highest common divisor of the resulting numerator and denominator will be found to be $a-b$. This will be found to be the shorter process.

IV.

1 H. C. D.	a^2b^2, b, $ab(a-b)$, $x-y$	Simplest form	$\frac{max}{nby}$, $\frac{a-b}{a+b}$, $\frac{c}{d}$, $\frac{x^2+xy+y^2}{x-y}$.
2	x^2-y^2	... ...	$\frac{x-y}{x^2+y^2}$.
3	$a+b$	... ...	$\frac{2a-b}{a^2-1}$.
4	$a+b+c$	... ...	$\frac{a+b+c}{a-b-c}$.
5	$a^2+b^2+c^2-ab-ac-bc$	... ...	$\frac{1}{a+b+c}$.
6	a^3-b^3	... ...	$\frac{a+b}{a-b}$.
7	x^2-y^2	... ...	$\frac{x^4-y}{x^2-y}$.

8	H. C. D.	$a+b+c$	Simplest form	$\frac{a+b-c}{a-b-c}$.
9	... ...	$z+6$	... ...	$\frac{3bc+6mn}{4ad-7rs}$.
10	... ...	$ay+bx$	... ...	$\frac{ax+by}{ax-by}$.
11	... ...	$1+yz$	. . ...	$\frac{1-x^2}{1+x^2}$.
12	... ...	$b+d$	... ...	$\frac{a^2+c^2}{(a+d)(b+c)}$.
13	... ...	$3x-2$	... ...	$\frac{3x-y}{2x^2-y^2}$.
14	... ...	$a+b+c+d$	... ...	$\frac{a+b-c-d}{a-b-c+d}$.
15	... ...	$x+y$	... ...	$\frac{b^2(y+z)+ab(x-z)}{b^2(y+z)-ab(x-z)}$.

V.

1. $\frac{29x}{21}$ or $x+\frac{8x}{21}$. 2. $\frac{38x-2}{7}$. 3. $\frac{541x}{504}$. 4. $\frac{ab+ac+bc}{abc}$.
5. $\frac{ayz+bxz+cxy}{xyz}$. 6. $\frac{a^2b^2+a^2c^2+b^2c^2}{abc}$. 7. $\frac{1}{2}$. 8. $\frac{x^2+a^2}{x^2-a^2}$.
9. $\frac{2x+4}{x^2+4x-21}$. 10. $\frac{2(9a^2+4x^2)}{9x^2-4a^2}$. 11. $\frac{a^2+b^2}{ab}$. 12. 0.
13. $-\frac{a^2-2ax+b^2}{(x-a)(x-b)}$. 14. $\frac{x^2-xy+2y^2}{x^2-4y^2}$. 15. $\frac{a+bx}{c+dx}$.
16. $\frac{cx+by+az}{abc}$. 17. 0. 18. $\frac{2x^2}{x^4+x^2y^2+y^4}$. 19. $\frac{4xy(x^2+y^2)}{x^4+x^2y^2+y^4}$.

VI.

1. $\frac{2x}{35}$. 2. $-\frac{x}{20}$. 3. $-\frac{4x-13}{10x}$. 4. $\frac{2x+1}{6x}$. 5. $-\frac{ab}{a-b}$. 6. $\frac{a}{b+c}$.
7. $\frac{1}{(x-3)(x-4)}$. 8 $\frac{4ab}{a^2-b^2}$. 9. $\frac{(m-n)abx-(m+n)ac}{b^2x^2-c^2}$. 10. $\frac{a^2-2ac+b^2}{a^2-b^2}$.
11. $\frac{(a-b)\{(a+b)x-ab\}}{(x-a)(x-b)}$. 12. $\frac{(a^2+b^2)x-(a+b)ab}{(x-a)(x-b)(a-b)}$. 13. $-\frac{11x}{156}$.
14. $\frac{x^3-xy^2+2y^3}{x^2-y^2}$.

VII.

The examples under this number offer no difficulties. For the reduction of the continued fractions in Ex. 10 reference may be made to the note, pp. 5, 6 of Sect. IX. of the Elementary Arithmetic, where the reduction of a continued numerical fraction is fully explained.

VIII.

1. $\frac{6x^2+9y^2}{4xy-6y^2}$. 2. $\frac{ay}{bx}+2+\frac{bx}{ay}$ 3. $-\frac{13}{720}$. 4. $\frac{ax}{a^2-x^2}$. 5. $\frac{x^2+y^2}{xy-y^2}$.
6. $\frac{x^2-xy+y^2}{x-y}$. 7. 1. 8. $2(x+y)$. 9. $\frac{x^2}{(3-2x)(4+3x)}$.

10. $a^4-3(a-b)a^2x-\frac{(31a^2-30b^2)x^2}{18}+\frac{(167a-103b)x^3}{108}-\frac{2x^4}{9}$.

11. $x^5-\frac{3026}{1225}x^4+\frac{8}{5}x^3-\frac{2626}{735}x^2+\frac{5}{3}x$. 12. $\frac{a^2-ab+b^2}{a^2+ab+b^2}$.

IX.

1. ab.

2. $1-\frac{a^2+b^2-c^2}{2ab}=\frac{2ab-a^2-b^2+c^2}{2ab}=\frac{c^2-(a-b)^2}{2ab}=\frac{(c+a-b)(c-a+b)}{2ab}$.

3. $\frac{(a-b+c)(a-b-c)}{2ab}$.

4. $1-\frac{a^2-(b+c)^2}{(a+b+c)^2}=1-\frac{(a+b+c)(a-b-c)}{(a+b+c)^2}=1-\frac{a-b-c}{a+b+c}=\frac{2(b+c)}{a+b+c}$.

5. $\frac{2(b+c)}{a-b-c}$.

6. $$\frac{(a+b)\{(a+b)^2-c^2\}}{4b^2c^2-(a^2-b^2-c^2)^2}=\frac{(a+b)\{(a+b)^2-c^2\}}{(2bc+a^2-b^2-c^2)(2bc-a^2+b^2+c^2)}$$

$$=\frac{(a+b)\cdot\{(a+b)^2-c^2\}}{\{a^2-(b-c)^2\}\cdot\{(b+c)^2-a^2\}}=\frac{(a+b)(a+b+c)(a+b-c)}{(a+b-c)(a-b+c)(b+c+a)(b+c-a)}$$

$$=\frac{a+b}{(a-b+c)(b+c-a)}.$$

7. d^2. 8. $a+b-c-d$.

9. $\frac{(a+b+c-d)(a+b+d-c)(a+c+d-b)(b+c+d-a)}{4(ab+cd)^2}$

10. $1-\left\{\frac{a^2+b^2-x^2}{2ab}\right\}^2=\left\{1+\frac{a^2+b^2-x^2}{2ab}\right\}.\left\{1-\frac{a^2+b^2-x^2}{2ab}\right\}$.

Substitute for x^2, and the reduced expression is—

$$\frac{(a+b+c+d)(a+b-c-d)(a+c-b-d)(b+c-a-d)}{4(ab-cd)^2}$$

X.

1. $\frac{45}{32}$. 2. $-\frac{a}{b}$. 3. $\frac{4y-12}{5}$. 4. $\frac{2x}{x^2-ax+a^2}$. 5. $(a^2-ab+b^2)ab$.

6. $\frac{ab}{a^2-b^2}$. 7. ab. 8. $\frac{x^2+b^2}{x}$. 9. $\frac{4(a+x)}{3(c+x)}$. 10. $\frac{(a-x)^2}{(a+x)x}$. 11. $\frac{(x+y)^2}{x^2-y^2}$.

12. $\frac{(x^2-y^2)^2}{x^2+y^2}$. 13. 1. 14. $(a-b-c)^2$.

XI.

The following are the respective quotients :—

1. $2x^2+\frac{1}{3}$. 2. $\frac{x}{a}-\frac{2a^2}{x}$. 3. $\frac{x^2}{3}-\frac{x}{3}+\frac{1}{4}$. 4. $\frac{3x^3}{2}-5x^2+\frac{x}{4}+9$.

5. $\frac{3a^5}{4b}-\frac{2a^3}{5b^2}+\frac{7a}{8b^3}$. 6. $\frac{x^4}{2}-2x^3y$. 7. $\frac{x^2}{a^2}-\frac{3x}{a}-\frac{3a}{x}+\frac{a^2}{x^2}+3$. 8. $a^3-\frac{1}{a^3}$.

XII.

1. $1+3x+3x^2+3x^3+3x^4+\ldots\ldots+3x^{19}+\ldots\ldots$
$1+3x+6x^2+12x^3+24x^4+\ldots\ldots+3.2^{19}.x^{19}+\ldots\ldots$

2. $\frac{1}{2}+\frac{7x}{4}-\frac{21x^2}{8}+\frac{63x^3}{16}-\frac{189x^4}{32}+\ldots\ldots\pm\frac{7.3^{14}.x^{15}}{2^{16}}\ldots\ldots$

3. $1-\frac{2x}{a^2}+\frac{2x^2}{a^2}-\frac{2x^3}{a^3}+\,.\,.\,.\,.\,.$

5. $\frac{a}{c}+\frac{(ad+bc)x}{c^2}+\frac{(ad+bc)ax^2}{c^3}+\frac{(ac+bc)d^2x^3}{c^4}+\,.\,.\,.+\frac{(ad+bc)d^{n-2}x^{n-1}}{c^n}+\,.\,.\,.\,.$

7. $1+x-x^3-x^4+x^6+x^7-x^9-x^{10}+\,.\,.\,.\,.\,.$

$\frac{1}{x^2}+\frac{1}{x^3}-\frac{1}{x^5}-\frac{1}{x^6}+\frac{1}{x^8}+\frac{1}{x^9}-\,.\,.\,.\,.\,.$

XIII.

The operations under this number involve no difficulties.

XIV.

The multiplications and divisions contain no anomalies which require remark.

XV.

The exercises under this number require no remarks.

XVI.

The first eleven examples under this number offer no difficulties.

12. In this example it may be shewn that

$$\frac{2}{(b-c)(c-a)}+\frac{2}{(c-a)(a-b)}+\frac{2}{(a-b)(b-c)}=0.$$

17. Each fraction may be simplified as the first,

$$\frac{(a-b)^2-(b-c)^2}{a^2+ab-bc-c^2}=\frac{(a-b+b-c)(a-b-b+c)}{(a^2-c^2)+b(a-c)}=\frac{(a-c)(a+c-2b)}{(a-c)(a+c+b)}=\frac{a+c-2b}{a+b+c}.$$

18. The shortest method here is to reduce each expression to the same identical form.

Thus $\frac{a^2+b^2-c^2}{ab}+2+\frac{b^2+c^2-a^2}{bc}+2+\frac{c^2+a^2-b^2}{ac}+2$

$$=\frac{(a+b)^2-c^2}{ab}+\frac{(b+c)^2-a^2}{bc}+\frac{(c+a)^2-b^2}{ac}$$

$$=\frac{(a+b+c)(a+b-c)}{ab}+\frac{(b+c+a)(b+c-a)}{bc}+\frac{(c+a+b)(c+a-b)}{ac},$$

$$=(a+b+c)\cdot\left\{\frac{a+b-c}{ab}+\frac{b+c-a}{bc}+\frac{c+a-b}{ac}\right\}$$

$$=\frac{(a+b+c)(2ac+2bc+2ab-a^2-b^2-c^2)}{abc}.$$

Next $\left\{4ab-(a+b-c)^2\right\}.\left\{\frac{1}{ab}+\frac{1}{bc}+\frac{1}{ca}\right\}$

$$=\left\{4ab-(a+b)^2+2(a+b)c-c^2\right\}.\left\{\frac{c+a+b}{abc}\right\}$$

$$=\frac{(2ac+2ab+2bc-a^2-b^2-c^2)(a+b+c)}{abc}.$$

XVII.

1. 0. 2. 0. 3. 0. 4. 0. 5. -1. 6. 1. 7. $+3$. 8. $a+b+c$. 9. 0.

10. 0. 11. $\frac{2c}{(b-c)(a-b)}$. 12. 0. 13. 1. 14. $a^2+b^2+c^2+ac+ab+bc$.

15. $\frac{a^3(a+b)(a+c)}{(a-b)(a-c)}=\frac{a^3(a^2+(b+c)a+bc)}{a^2-(b+c)+bc}=a^3\left\{1+\frac{2(b+c)a}{(a-b)(a-c)}\right\}.$

Similarly $\frac{b^3(b+c)(b+a)}{(b-c)(b-a)}=b^3\left\{1+\frac{2(a+c)b}{(b-c)(b-a)}\right\},$

and $\frac{c^3(c+a)(c+b)}{(c-a)(c-b)}=c^3\left\{1+\frac{2(a+b)c}{(c-a)(c-b)}\right\}$.

Then $\frac{2(b+c)a^4}{(a-b)(a-c)}-\frac{2(a+c)b^4}{(a-b)(b-c)}=\frac{2}{a-b}\left\{\frac{(b+c)a^4}{a-c}-\frac{(a+c)b^4}{b-c}\right\}$

$$=\frac{2}{a-b}\left\{\frac{(b^2-c^2)a^4-(a^2-c^2)b^4}{(a-c)(b-c)}\right\},$$

$$=\frac{2}{a-b}\cdot\left\{\frac{a^2b^2(a^2-b^2)-c^2(a^4-b^4)}{(a-c)(b-c)}\right\},$$

$$=\frac{2\{a^2b^2(a+b)-c^2(a^2+b^2)(a+b)\}}{(a-c)(b-c)},$$

And $\frac{2\{a^2b^2(a+b)-c^2(a^2+b^2)(a+b)\}}{(a-c)(b-c)}+\frac{2(a+b)c^4}{(c-a)(c-b)}$

$$=\frac{2\{(a+b)c^4-(a+b)(a^2+b^2)c^2+(a+b)a^2b^2\}}{(c-a)(c-b)},$$

$$=\frac{2(a+b)\{c^4-(a^2+b^2)c^2+a^2b^2\}}{(c-a)(c-b)},$$

$$=\frac{2(a+b)(c^2-a^2)(c^2-b^2)}{(c-a)(c-b)},$$

$$=2(a+b)(b+c)(c+a).$$

And the sum of the three fractions is $a^3+b^3+c^3+2(a+b)(b+c)(c+a)$.

XVIII.

1. 0. 2. $\frac{3x^2-(a^2+b^2+c^2)}{(x-a)(x-b)(x-c)}$. 3. 0. 4. 1.

5. First $\frac{x-c}{a-b}\left\{\frac{x-b}{a-c}-\frac{x-a}{b-c}\right\}=\frac{x-c}{a-b}\left\{\frac{-x(a-b)+(a^2-b^2)-c(a-b)}{(a-c)(b-c)}\right\}$

$$=\frac{-(x-c)(x-a-b+c)}{(a-c)(b-c)}.$$

Next $\frac{(x-a)(x-b)}{(c-a)(c-b)}-\frac{(x-c)(x-a-b+c)}{(a-c)(b-c)}$

$$=\frac{x^2-(a+b)x+ab-\{x^2-(a+b)x+(a+b)c-c^2\}}{(a-c)(b-c)},$$

$$=\frac{c^2-(a+b)c+ab}{(a-c)(b-c)}=1.$$

6. 1. 7. x^2. 8. $(x-a)+(y-b)+(z-c)$.

10. $\frac{(b^2-ac)y}{(ax-by)(by-cz)}$.

XIX.

1. $\frac{1}{abc}$. 2. $\frac{1}{abc}$. 3. $\frac{1}{(x+a)(x+b)(x+c)}$. 4. $\frac{1}{(x-a)(x-b)(x-c)}$.

5. $\frac{x^2}{(a+x)(b+x)(c+x)}$. 6. $\frac{(a+b+c)^2}{(a+b)(b+c)(c+a)}$.

7. Sometimes by separating each fraction into two parts, the process may be simplified.

$$\frac{b+c-a}{(b+c)(c-a)(a-b)}=\frac{b+c}{(b+c)(c-a)(a-b)}-\frac{a}{(b+c)(c-a)(a-b)}$$

$$=\frac{1}{(c-a)(a-b)}-\frac{a}{(b+c)(c-a)(a-b)}.$$

Similarly $\frac{c+a-b}{(c+a)(a-b)(b-c)}=\frac{1}{(a-b)(b-c)}-\frac{b}{(c+a)(a-b)(b-c)}$,

and $\frac{a+b-c}{(a+b)(b-c)(c-a)}=\frac{1}{(b-c)(c-a)}-\frac{c}{(a+b)(b-c)(c-a)}$.

Then, $\frac{1}{(c-a)(a-b)}+\frac{1}{(a-b)(b-c)}+\frac{1}{(b-c)(c-a)}=0$. Sec. XVII. Ex. I.

And the aggregate of

$$-\left\{\frac{a}{(b+c)(c-a)(a-b)}+\frac{b}{(c+a)(a-b)(b-c)}+\frac{c}{(a+b)(b-c)(c-a)}\right\}$$

will be found to be $\frac{c(a^2+ab+b^2-c^2)}{(a+b)(b^2-c^2)(c^2-a^2)}$.

8. Add the first two fractions, next the other two, and the sum of the results is zero.

XX.

1. $-\frac{b^3+(a+c)b^2-(a+c)^2b-(a-c)^2(a+c)}{(b+c-a)(a+b-c)(c+a-b)}$.

2. $\frac{8abc}{(a^2+b^2+c^2)^2-4(a^2b^2+a^2c^2+b^2c^2)}$. 3. $\frac{1}{c+a-b}$. 4. 4.

5. $\frac{2(a+b+c)abc}{(b+c-a)(c+a-b)(a+b-c)}$. 6. $\frac{4(b-c)}{a+b-c}$. 7. $\frac{4(a^2-c^2)}{ac}$. 8. 1.

9. $\frac{(b+a-c)(a+c-b)(b+c-a)}{(a+b+c)^3}$.

10. Let $y=b+c-2a=(c-a)-(a-b)$

$z=c+a-2b=(a-b)-(b-c)$

$x=a+b-2c=(b-c)-(c-c)$

$\therefore x+y+z=0$,

also $c-a=\frac{x+y-z}{2}$, $b-c=\frac{x+z-y}{2}$, $a-b=\frac{y+z-x}{2}$.

Hence $\frac{a+b-2c}{a-b}+\frac{b+c-2a}{b-c}+\frac{c+a-2b}{c-a}=\frac{2x}{y+z-x}+\frac{2y}{x+z-y}+\frac{2z}{x+y-z}$,

$$=\frac{2x}{-2x}+\frac{2y}{-2y}+\frac{2z}{-2z}=-3.$$

Similarly $\frac{a-b}{a+b-2c}+\frac{b-c}{b+c-2a}+\frac{c-a}{c+a-2b}=\frac{-2x}{2x}+\frac{-2y}{2y}+\frac{-2z}{2z}=-3$.

Hence the product of the quantities $=-3\times-3=+9$.

XXI.

4. $\frac{(a+b+d)(a+b)-c(c+d)}{(a+b+d)(a+d)-c(b+c)}=\frac{(a+b)^2-c^2+(a+b-c)d}{(a+d)^2-c^2+(a+d-c)b}$

$$=\frac{(a+b+c)(a+b-c)+(a+b-c)d}{(a+d+c)(a+d-c)+(a+d-c)b}$$

$$=\frac{(a+b-c)(a+b+c+d)}{(a+d-c)(a+d+c+b)}=\frac{a+b-c}{a-c+d}.$$

8. The fraction when reduced is $\frac{2abc(1+a^2+b^2+c^2)}{4(1+a^2+b^2+c^2)}$.

9. By means of the formula $x^3-y^3=(x-y)(x^2+xy+y^2)$, the expression is reduced to $\frac{9(b-c)(a^2+b^2+c^2-ab-bc-ca)}{(b+c-2a)(a^2+b^2+c^2-ab-ac-bc)}$.

10. By inspection it may be seen that $b-c$ is a common factor of the numerator and denominator. When this factor is removed, the fraction is $\frac{(b+c)a^3-(b^2+bc+c^2)a^2+b^2c^2}{a^3-(b^2+bc+c^2)a+bc(b+c)}$, and its highest common divisor is $a^2-(b+c)a+ab$.

11. After reducing and arranging the quantities according to the descending powers of a, the highest common divisor will be found to be

$$a^2-(b+c)a+(b^2-bc+c^2).$$

12. Here $\dfrac{(ab+cd)(a^2+b^2-c^2-d^2)-(ac+bd)(a^2-b^2+c^2-d^2)}{(a^2+b^2-c^2-d^2)(a^2-b^2+c^2-d^2)+4(ab+cd)(ac+bd)}$

$$=\frac{(ac+bd+ac+bd)(b^2-c^2)+(ab+cd-ac-bd)(a^2-d^2)}{\{(a^2-d^2)+(b^2-c^2)\}.\{(a^2-d^2)-(b^2-c^2)\}+4(a^2bc+b^2ad+c^2ad+d^2bc)}$$

$$=\frac{(a+d)(b+c)(b^2-c^2)+(a-d)(b-c)(a^2-d^2)}{(a^2-d^2)^2-(b^2-c^2)^2+4ad(b^2+c^2)+4bc(a^2+d^2)}$$

$$=\frac{(a+d)(b-c)\{(b+c)(b+c)+(a-d)(a-d)\}}{(a^2-d^2)^2-(b^2-c^2)^2+(a+d)^2(b+c)^2-(b-c)^2(a-d)^2}$$

$$=\frac{(a+d)(b-c).\{(b+c)^2+(a-d)^2\}}{\{(a+d)^2-(b-c)^2\}.\{(b+c)^2+(a-d)^2\}}$$

$$=\frac{(a+d)(b-c)}{(a+d)^2-(b-c)^2}.$$

15. The first expression may be arranged in the form

$$\frac{(c^2-b^2)\{a^4-(c^2+b^2)a^2+b^2c^2\}}{(c-b)\{a^2-(c+b)a+bc\}}=\frac{(c+b)(c-b)(a^2-b^2)(a^2-c^2)}{(a-b)(a-c)}.$$

$$=(c+b)(a+b)(a+c).$$

The second in the same manner.

$$\frac{-3(b^2-c^2)a^4+3(b^4-c^4)a^2-3b^2c^2(b^2-c^2)}{-3(b-c)a^2+3(b^2-c^2)a-3bc(b-c)}$$

$$=\frac{(b+c)\{a^4-(b^2+c^2)a^2+b^2c_2\}}{a^2-(b+c)a+bc}=\frac{(b+c)(a^2-b^2)(a^2-c^2)}{(a-b)(a-c)}=(b+c)(a+b)(a+c).$$

16. The given expression when reduced to an improper fraction becomes

$$\frac{(1+m_2+n^2)(x-a)^2+(1+m^2+n^2)(y-b)^2+(1+m^2+n^2)z^2-\{z+m(x-a)+n(y-b)\}^2}{1+m^2+n2}$$

and when the terms of the numerator are expanded, they are

$$(1+m^2+n^2)(x-a)^2=(x-a)^2+m^2(x-a)_2+n^2(x-a)_2,$$
$$(1+m^2+n^2)(y-b)_2=(y-b)^2+m^2(y-b)^2+n^2(y-b)^2,$$
$$(1+m^2+n^2)z^2=z^2+m^2z^2+n^2z^2,$$
$$-\{z+m(x-a)+n(y-b)\}^2=-z^2-2mz(x-a)-2nz(y-b)-m^2(x-a)^2$$
$$-2mn(x-a)(y-b)-n^2(y-b)^2.$$

Here z^2-z^2, $m^2(x-a)^2-m^2(x-a)^2$, $n^2(y-b)^2-n^2(y-b^2)$ disappear, and the remaining nine terms may be arranged as follows :—

$$(x-a)^2-2mz(x-a)+m^2x^2=(x-a--mz)^2,$$
$$(y-b)^2-2nz(y-b)+n^2z^2=(y-b-nz)^2,$$
$$n^2(x-n)^2-2mn(x-a)(y-b)+m^2(y-b)^2=\{n(x-a)-m(y-b)\}^2.$$

Hence the given expression becomes by substitution

$$\frac{(x-mz-a)^2+(y-nz-b)^2+\{n(x-a)-m(y-b)\}^2}{1+m^2+n^2}.$$

XXII.

1. $\dfrac{(x-1)(x-2)(x-3)(x-4)}{(x+1)(x+2)(x+3)(x+4)}$. 2. $\left\{\dfrac{x^4+a^4}{x^4-a^4}\right\}^2$. 3. 0.

4. $1+\dfrac{3-4x+x^2-2x^3-x^4-x^5}{(1+x)(1+x^2)(1+x^3)}$. 5. a^2+b^2. 6. $2(ay-bx)^2+2(ax-by)^2$.

7. $\frac{1}{5}\{cx+y)^5+(y+z)^5+(x+z)^5-2(x^5+y^5+z^5)\}$. See Ex. 12, p. 30, Sec. IV.

8. $\dfrac{x(x+2y)}{x^2+4xy-2y^2}$. 9. $\dfrac{ab(1-x^2)}{2(b^2-a^2)x}$. 10. $\dfrac{(1-a^2)(1+b^2)+2ab(a-b)x}{(1-a^2)(1-b^2)(1-x^2)}$.

11. $2(a+b+c)$. 12. $\frac{2}{x^2}$. 13. $\frac{8x^2y^2z^2}{(x+y)^2(y+z)^2(z+x)^2}$. 14. $\frac{x^4+x^2y^2+y^4}{x^2-y^2}$.

15. $\frac{a^2+4ab+b^2}{c^2-4ab-b^2}$. 16. $\frac{x-y}{x+y}$.

XXIII.

1 and 2 offer no difficulties.

3. $a^2x=x^3-xyz,\ b^2y=y^3-xyz,\ c^2z=z^3-3xy$;

$\therefore\ a^2x+b^2y+c^2z=x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-xz-yz)$

$\frac{a^2x+b^2y+c^2z}{x+y+z}=x^2+y^2+z^2-xy-xz-yz=a^2+b^2+c^2$.

4. Shew that xyz and $x+y+z$ are equal with contrary signs.

5. Add x to each side of $y+z+u=ax$, then $x+y+z+u=(a+1)x$,

and $\frac{1}{a+1}=\frac{x}{x+y+z+u}$, Similarly for each of the other three expressions.

6. Since $a=\frac{x-y}{x+y}$, $\therefore ax+ay=x-y$, and $(1+a)y=(1-a)x$

$\therefore \frac{x}{y}=\frac{1+a}{1-a}$; similarly $\frac{y}{z}=\frac{1+b}{1-b}$, and $\frac{z}{x}=\frac{1+c}{1-c}$.

Whence $\frac{1+a}{1-a}.\frac{1+b}{1-b}.\frac{1+c}{1-c}=\frac{x}{y}.\frac{y}{z}.\frac{z}{x}=1$.

7. Multiply by xyz, and arrnge the terms so that x, y, z may be included in brackets in the same manner as a, b, c in the given expression.

8. The sum of the numerators of the four fractions when reduced to a common denominator is $4s^3-3(a+b+c)s^2+2(ab+ac+bc)s-abc$, which when reduced and the substitution for s made, the result is abc.

9. The given expression can be put under the form

$$\frac{b^2+c^2-a^2}{2bc}-1+\frac{c^2+a^2-b^2}{2ac}+1+\frac{a^2+b^2-c^2}{2ab}-1=0,$$

$$\text{or } \frac{(b-c)^2-a^2}{2bc}+\frac{(c+a)^2-b^2}{2ac}+\frac{(a-b)^2-c^2}{2ab}=0.$$

Whence $a(b-c+a).(b-c-a)+b(c+a+b)(c+a-b)+c(a-b+c)(a-b-c)$

$$=(a+c-b).\{c(a-b-c)+b(c+a+b)-a(b-c+a)\},$$

$$=(a+c-b).\{b^2-(a-c)^2\}=(a+c-b)(b+a-c)(b+c-a)=0.$$

10. First add the three fractions, then as the sum is equal to unity, the numerator of the sum is equal to the common denominator.

Whence there results $a^3+b^3+c^3-3abc=0$, and $a^3+b^3+c^3=3abc$,

$$\text{or } (a+b+c)(a^2+b^2+c^2-ab-ac-bc)=0.$$

Taking $a+b+c=0$, then $a^2+b^2-c^2=-2ab$, $a^2+c^2-b^2=-2ac$,

$$\text{and } b^2+c^2-a^2=-2bc.$$

Whence $\frac{a^2}{b^2+c^2-a^2}+\frac{b^2}{a^2+c^2-b^2}+\frac{c^2}{a^2+b^2-c^2}=-\frac{1}{2}\left\{\frac{a^2}{bc}+\frac{b^2}{ac}+\frac{c^2}{ab}\right\}$

$$=-\frac{1}{2}\left\{\frac{a^3+b^3+c^3}{abc}\right\},$$

$$=-\frac{1}{2}.\frac{3abc}{abc}=-\frac{3}{2}.$$

11. Here $(a^2-bc)(b^2-ac)(c^2-ab)=0$,

$$\text{or } a^2b^2c^2-b^3c^3-a^3c^3+abc^4-a^3b^3+b^4ac+a^4bc-a^2b^2c^2=0,$$

$$\therefore\ a^3b^3+a^3c^3+b^3c^3=abc^4+acb^4+bca^4,$$

and dividing these equals by $a^3b^3c^3$, there results $\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=\frac{a^3+b^3+c^3}{a^2b^2c^2}$.

The student may try to resolve $a^3b^3+a^3c^3+b^3c^3-abc(a^3+b^3+c^3)$
and $a^3b^3+b^3c^3+a^3c^3+abc(a^3+b^3+c^3+2abc)$, each into three factors.

XXIV.

1. The difference is $\dfrac{3(x-a)(x-b)(x-c)}{(x-a)(x-b)(x-c)}$ or 3.

2. Here $xyz=1$ and $y=\dfrac{1}{xz}$ and $xz=\dfrac{1}{y}$,

$$\text{then } \frac{1}{1+x+\frac{1}{y}}+\frac{1}{1+z+\frac{1}{x}}+\frac{1}{1+y+\frac{1}{z}}=\frac{1}{1+x+xz}+\frac{x}{1+x+xz}+\frac{xz}{1+x+xz}=\frac{1+x+xz}{1+x+xz}=1.$$

3. The substitutions for x, y, z being made, the expression becomes

$$1-\frac{ab}{(a+c)(b+c)}-\frac{bc}{(a+c)(a+b)}-\frac{ac}{(b+c)(a+b)}-\frac{2abc}{(a+b)(b+c)(c+a)},$$

which when reduced gives the required result.

4. Only requires the substitutions to be made for x and y.

5. The truth of this may be directly shewn by simply substituting for x and y their values in terms of a and b.

Or indirectly, by assuming the result as true, namely, $x^3+y=y^3+x$, then $x^3-y^3=x-y$ and $x^2+xy+y^2=1$, and if for x and y their values be substituted, the sum be equal to unity, and the truth is proved.

6. $x-a=\dfrac{a^2+2ab+5b^2}{4(a+b)}$, $y-b=\dfrac{a^2+2ab-3b^2}{4(a-b)}$,

$$\text{then } (x-a)^2+(y-b)^2=\frac{16(a+b)^2b^2}{16(a+b)^2}=b^2.$$

7. The given expression is reducible to $(x+y+z)(xy+xz+yz)=0$.

8. First, $\dfrac{a}{b+x}-\dfrac{a}{b}=\dfrac{c}{d+y}-\dfrac{c}{a}$ or $\dfrac{ax}{b(b+x)}=\dfrac{cy}{d(d+y)}$, whence $adx(d+y)=bcy(b+x)$

and $b^2cy-ad^2x=xy(ad-bc)$, then dividing these equals by $bdxy$

$$\therefore \frac{bc}{dx}-\frac{ad}{by}=\frac{xy(ad-bc)}{bdxy}=\frac{a}{b}-\frac{c}{d}.$$

9. If $ab+cd$, $ac+bd$, $ad+bc$ be denoted by p, q, r respectively, it may be easily shewn from the equality of the first and second fractions, that $p+q+r=0$. The same condition may be found from the equality of the second and third fractions, whence $\dfrac{p+q}{r}=\dfrac{p+r}{q}=\dfrac{q+r}{q}=-1$.

10. Let $ab+ac-bc=p$, $bc+ac-ab=q$, $ab+bc-ac=r$,

then $\dfrac{pq}{r}=\dfrac{pr}{q}=\dfrac{qr}{p}$, whence $p=q$ and $q=r$,

and restoring the values of p, q, r, $ab+ac-bc=bc+ac-ab$, $\therefore a=c$,
$bc+ac-ab=ab+bc-ac$, $\therefore c=b$, hence $a=b=c$.

11. The sum of these six fractions may be put under the form of

$$\frac{(x-a)(x+a-b-c)}{(a-b)(a-c)}+\frac{(x-b)(x+b-a-c)}{(b-c)(b-a)}+\frac{(x-c)(x+c-a-b)}{(c-a)(c-b)},$$

and their sum is $\dfrac{-2ab+2(a+b)c-2c^2}{(c-a)(c-b)}$ or -2 a constant negative quantity.

XXV.

1. $\dfrac{1+a^2}{1-a^2}$ is greater or less than $\dfrac{1+a}{1-a}$ according as $1+a^2$ is greater or less than $(1+a)^2$.

2. Shew that $(a-b)^2>0$.

3. See Elementary Arithmetic, p. 13, Sect. IX., Exercises VIII. 9–12.

5. The former is greater or less than the latter, according as x^2 is greater or less than y^2.

6. $\frac{1}{a+c}$ is less than $\frac{2b}{(a+b)(b+c)}$ if $(a+b)(b+c)$ be less than $2b(a+c)$ or if $ac+b^2$ be less than $ab+bc$, or if $(a-b)c$ be less than $(a-b)b$, that is if c be less than b, and $a-b$ be positive or a greater than b.

7. x may have any integral value between 1 and 7.

8. If $x+\frac{2}{x}>3$, then $x^2+2>3x$ and $1+\frac{2}{x^2}>\frac{3}{x}$, $\therefore\ \frac{2}{x^2}>\frac{3}{x}-1$,

and $x^2>3x-2$, $\therefore\ x^2+\frac{2}{x^2}>3(x+\frac{1}{x}-1)$, but $x+\frac{1}{x}>2$ $\therefore\ x^2+\frac{2}{x^2}>3$.

9. If $a>b$ then $a-b>0$ and $a^2-2ab+b^2>0$, $\therefore\ a^2+b^2>2ab$, and $\frac{a}{b}+\frac{b}{a}>2$.

10. Let the expression $a^2(b+c)+b^2(a+c)+c^2(a+b)$ or its equivalent

$ab(a+b)+ac(a+c)+bc(b+c)$ be divided by abc,

then $\frac{a}{c}+\frac{b}{c}+\frac{a}{b}+\frac{c}{b}+\frac{b}{a}+\frac{c}{a}$ is the quotient, and $\frac{a}{b}+\frac{b}{a}>2$, $\frac{a}{c}+\frac{c}{a}>2$, $\frac{b}{c}+\frac{c}{a}>2$,

$$\therefore\ \frac{a}{c}+\frac{b}{c}+\frac{a}{b}+\frac{c}{b}+\frac{b}{a}+\frac{c}{a}>6, \text{ or } ab(a+b)+ac(a+c)+bc(b+c)>6abc.$$

11. Since $a>b$, $b>c$, and $a>c$ $\therefore\ a-b>0$, $b-c>0$, and $a-c>0$,

$$\therefore\ (a-b)(b-c)(a-c)>0, \text{ and } \frac{(a-b)(b-c)(a-c)}{abc}>0,$$

$$\text{whence } \frac{a}{c}-\frac{c}{a}+\frac{b}{a}-\frac{a}{b}+\frac{c}{b}-\frac{b}{c}>0, \text{ and } \frac{a}{c}+\frac{b}{a}+\frac{c}{b}>\frac{c}{a}+\frac{a}{b}+\frac{b}{c}.$$

12. Let $b+c-a=x$, $c+a-b=y$, $a+b-c=z$;

$$\text{by substitution } \frac{1}{x}+\frac{1}{y}+\frac{1}{z}>\frac{y}{x+y+z}.$$

But $x^2+y^2>2xy$, $y^2+z^2>2yz$, $x^2+z^2>2xz$; and $x^2z+y^2z>2xyz$, $y^2x+z^2x>2xyz$, $x^2y+z^2y>2xyz$, also $3xyz=3xyz$. Hence by addition

$$x^2y+xy^2+x^2z+xz^2+y^2z+yz^2+3xyz>9xyz,$$

$$\text{or } (x+y+z)(xy+xz+yz)>9xyz, \text{ whence } \frac{xy+xz+yz}{xyz}>\frac{9}{x+y+z},$$

$$\text{or } \frac{1}{x}+\frac{1}{y}+\frac{1}{z}>\frac{9}{x+y+z}.$$

13. Let $b+c-a=x$, $c+a-b=y$, $a+b-c=z$; then $a+b+c=x+y+z$,

and $2a=y+z$, $2b=x+z$, $2c=x+y$.

$$\text{Hence } \frac{a}{b+c-a}+\frac{b}{c+a-b}+\frac{c}{a+b-c}=\frac{1}{2}\left\{\frac{y+z}{x}+\frac{x+z}{y}+\frac{x+y}{z}\right\}.$$

Suppose x, y, z in order of magnitude, and shew the aggregate to be greater than 3.

XXVI.

1. Here $a=\frac{b-1}{b}$ and $a=\frac{1}{1-c}$, $\therefore\ \frac{b-1}{b}=\frac{1}{1-c}$ from which the required expression may be found.

2. Let $a-\frac{bc}{a}=b-\frac{ac}{b}=p$; then $a^2-bc=pa$ and $b^2-ac=pb$,

$$\therefore\ (a^2-b^2)+c(a-b)=(a-b)p \text{ and } p=a+b+c.$$

But $a-\frac{bc}{a}=b-\frac{ac}{b}$, $\therefore\ a-b=\frac{bc}{a}-\frac{ac}{b}$, from which $a+b=-\frac{ab}{c}$, $\therefore\ p=c-\frac{ab}{c}$.

3. Equate the values of b from each of the given expeessions.

4. Here $ax+a'=bx+b'$, $bx+b'=cx+c'$; from the first $x=\frac{b'-a'}{a-b}$,

from the second, $x=\frac{a'-b'}{b-c}$; equate these values of x.

5. Since $\frac{a-b}{1+ab}+\frac{c-d}{1+cd}=0$, then $\frac{a-b}{1+ab}=\frac{d-c}{1+cd}$, and $(a-b)(1+cd)=(1+ab)(d-c)$,
$\therefore$ $a+abc-d-abc=b+bad-c-cad$, or $a(1+bc)-d(1+bc)=b(1+ad)-c(1+ad)$,
or $(a-d)(1+bc)=(b-c)(1+ad)$, $\therefore$ $\frac{b-c}{1+bc}=\frac{a-d}{1+ad}$, and $\frac{b-c}{1+bc}+\frac{d-a}{1+ad}=0$.

6. $\frac{y}{b}=1-\frac{c}{z}$, $\frac{z}{c}=1-\frac{a}{x}$, and $\frac{x}{a}=\frac{c}{c-z}$;

$$\therefore \frac{xyz}{abc}=\frac{c}{c-z}\cdot\frac{x-a}{x}\cdot\frac{z-c}{z}=-\frac{c}{z}\left\{1-\frac{a}{x}\right\}=-\frac{c}{z}\cdot\frac{z}{c}=-1.$$

7. First $1-x=1-\frac{e+z}{1+ez}=\frac{1-e-z+ez}{1+ez}=\frac{(1-e)(1-z)}{1+ez}$;

next $1+x=1+\frac{e+z}{1+ez}=\frac{1+e+z+ez}{1+ez}=\frac{(1+e)(1+z)}{1+ez}$.

Thirdly, by dividing the former by the latter, $\frac{1-x}{1+x}=\frac{1-e}{1+e}\cdot\frac{1-z}{1+z}$.

8. If the value of y be substituted for y in $z=\frac{a+by}{c+dy}$, and the expression reduced, and divided by $b+c$, the following result will be obtained:
$\left(dz-\frac{ad+b^2}{b+c}\right)x=a-\frac{(c^2+ad)z}{b+c}$; in which if $\frac{ad+b^2}{b+c}=-b$ and $\frac{c^2+ad}{b+c}=-c$,
the provision will hold good.

9. Here $\frac{a^2(b-c)}{a-d}=\frac{b^2(a-c)}{b-d}$, clearing from fractions by multiplying each side by $(a-d)(b-d)$, then $a^2\left\{b^2-(c+d)b+cd\right\}=b^2\left\{a^2-(c+d)a+cd\right\}$,

whence $(c+d)(b-a)ab=cd(b^2-a^2)$,

$(c+d)ab=cd(b+a)$, dividing by $b-a$,

and $\frac{(c+d)ab}{abcd}=\frac{cd(b+a)}{abcd}$, dividing by $abcd$,

or $\frac{1}{d}+\frac{1}{c}=\frac{1}{a}+\frac{1}{b}$.

The converse requires only the reversion of the steps.

10. $b^2x^2=1-a^2x^2=(1+ax)(1-ax)$, $a^2x^2=1-b^2x^2=(1+bx)(1-bx)$,

whence $\frac{b^2}{1+ax}=\frac{1-ax}{x^2}$, and $\frac{b^3}{1+ax}=\frac{b+abx}{x^2}$,

and $\frac{a^2}{1-bx}=\frac{1+bx}{x^2}$, and $\frac{a^3}{1-bx}=\frac{a-abx}{x^2}$,

therefore $\frac{a^3}{1-bx}+\frac{b^3}{1+ax}=\frac{a+b}{x^2}=(a+b)(a^2+b^2)$.

11. $\frac{x}{a}=\frac{(a+b)b}{y}$, $\frac{y}{b}=\frac{(a+b)a}{x}$, and $\frac{x}{a}-\frac{y}{b}=\frac{(a+b)(bx-ay)}{xy}$:

$\frac{x}{b}=\frac{(a+b)a}{y}$, $\frac{y}{a}=\frac{(a+b)b}{x}$, and $\frac{x}{b}-\frac{y}{a}=\frac{(a+b)(ax-by)}{xy}$;

$$\therefore \left\{\frac{x}{a}-\frac{y}{b}\right\}\cdot\left\{\frac{x}{b}-\frac{y}{a}\right\}=\frac{(a+b)^2\left\{ab(x^2+y^2)-(a^2+b^2)xy\right\}}{x^2y^2}.$$

But $\frac{x^2-xy+y^2}{xy}=\frac{a^2-ab+b^2}{ab}$ $\therefore$ $ab(x^2+y^2)-(a^2+b^2)xy=abxy-abxy=0$.

12. $\frac{z(x^2-y^2)}{xy}=a-b$, $\frac{x(y^2-z^2)}{yz}=b-c$, then $\frac{z(x^2-y^2)}{xy}+\frac{x(y^2-z^2)}{yz}=a-c$

$$\text{or } \frac{y^2(x^2-z^2)}{xyz}=a-c, \text{ whence } \frac{x^2-z^2}{a-c}=\frac{xz}{y}.$$

13. Put each fraction equal to p, then $2p=a+b$, and $2p=c+d$.

14. Let each expression be denoted by p,

$$\text{then } x^2+z^2=y^2xzp,\ y^2+z^2=x^2yzp,\ x^2+y^2=z^2xyp,$$

$$\text{and } 2(x^2+y^2+z^2)=pxyz(x^2+y^2+z^2)\ \therefore p=\frac{2}{xyz}.$$

15. $x=-\left\{\frac{(a-b)y}{b-c}+\frac{(a-b)z}{c-a}\right\}$, $y=-\left\{\frac{(b-c)x}{a-b}+\frac{(b-c)z}{c-a}\right\}$,

$$z=-\left\{\frac{(c-a)x}{a-b}+\frac{(c-a)y}{b-c}\right\},$$

$$\therefore x+y+z=-\left\{\frac{(a-b)y}{b-c}+\frac{(a-c)z}{c-a}+\frac{(b-a)x}{a-b}\right\}=\&c$$

17. Let $\frac{x}{b-c}=p$, then $x=(b-c)p$ and $ax=a(b-c)p$,

$$\text{similarly } by=b(c-a)p,\ cz=c(a-b)p.$$

Hence by addition $ax+by+cz=p(ab-ac+bc-ab+ac-bc)=0$.

Otherwise, $\frac{ax}{y}=\frac{(b-c)a}{c-a}$, $\frac{by}{y}=\frac{(c-a)b}{c-a}$, $\frac{cz}{y}=\frac{(a-b)c}{c-a}$, $\therefore \frac{ax+by+cz}{y}=\frac{0}{c-a}=0$.

18. Since $ab+ac+bc=0$, $\therefore a^2-bc=a^2+ab+ac$.

Hence $\frac{1}{a^2-bc}+\frac{1}{b^2-ac}+\frac{1}{c^2-ab}=\frac{1}{a+b+c}.\left\{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right\}=\frac{1}{a+b+c}.\frac{ab+ac+bc}{abc}=0$.

19. Here $abc=\frac{xyz}{(x+y)(y+z)(z+x)}$, and $a-abc=\frac{xyz-x(x+y)(x+z)}{(x+y)(x+z)(y+z)}$,

$$\therefore \frac{1}{a(1-bc)}=-\frac{(x+y)(x+z)(y+z)}{x^2(x+y+z)} \text{ and } \frac{x^2}{a(1-bc)}=-\frac{(x+y)(x+z)(y+z)}{x+y+z}.$$

By a similar process $\frac{y^2}{b(1-ac)}$ and $\frac{z^2}{c(1-ab)}$ may be shewn equal to the same quantity.

20. $\frac{x+y-z}{ax+by-cz}=\frac{y+z-x}{by+cz-cx}$ $\therefore \frac{x+y-z}{y+z-x}=\frac{ax+by-cz}{by+cz-ax}$ add 1 to each and reduce to improper fractions, then $\frac{2y}{y+z-x}=\frac{2by}{by+cz-ax}$, and $\frac{y+z-x}{by+cz-ax}=\frac{1}{b}$,

$$\text{similarly } \frac{x+z-y}{ax+cz-by}=\frac{1}{a}, \text{ and } \frac{x+y-z}{ax+by-cz}=\frac{1}{c}.$$

Otherwise let the reciprocals of each $=p$,

then $ax+by-cz=p(x+y-z)$
$by+cz-ax=p(y+z-x)$
$ax+cz-by=p(x+z-y)$
then adding the first and second; next the first and third; and, lastly, the second and third, and the results are, $p=b$, $p=a$, $p=c$, and $a=b=c$.

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the extent, at least, of the Six Books, was published. From the manner of arranging the Demonstrations, this edition has the advantages of the symbolical form, and it is at the same time free from the manifold objections to which that form is open. The duodecimo edition of this Work, comprising only the first Six Books of Euclid, with Deductions from them, having been introduced at this Institution as a text book, now renders any other Treatise on Plane Geometry unnecessary in our course of Mathematics."—*Preface to Descriptive Geometry, &c. for the Use of the Royal Military Academy, by S. Hunter Christie, M.A., of Trinity College, Cambridge, late Secretary of the Royal Society, &c., Professor of Mathematics in the Royal Military Academy, Woolwich.* (1847.)

"Mr. Potts, by the publication of his Edition of *Euclid*, with its most valuable notes and problems, and the solutions and commentaries, has recalled the attention of Englishmen to the subject:—first in his own and the Sister Universities, then in the public schools, and, finally, in most Scholastic Establishments in the Country.—His Euclid is one of our own text-books in the Royal Military Academy, and we find its arrangements and additions exceedingly conducive to the acquisition of a thorough understanding of the subject by the Gentlemen Cadets."—*T. S. Davies, Professor of Mathematics in the Royal Military Academy, Woolwich.* (1848.)

"The Edition of the *Elements of Euclid* which Mr. Potts has published, is confessedly the best which has yet appeared."—*John Phillips Higman, M.A., F.R.S., late Fellow and Tutor of Trinity College, Cambridge.* (1848.)

"Mr. Potts has lately published an Edition of *Euclid's Elements of Geometry*, which he has illustrated with a collection of Examples. I consider that he has performed his task with great care and judgment, and that the work seems to bid fair to possess a larger share of popular favour than any edition of *Euclid* yet published."—*R. Buston, B.D., Fellow and Tutor of Emmanuel College.* (1848.)

"I consider Mr. Potts' Edition of *Euclid* to be a most valuable addition to our Cambridge Mathematical literature, and especially to the department of Geometry; and look to it as a great help towards keeping up, and indeed reviving, the true spirit and feeling for Geometry, which of late years had been too much neglected among us."—*W. Williamson, B.D., Fellow and Tutor of Clare College.* (1848.)

"I believe there is a general opinion in this University that the Principles of Euclid and Elementary Geometry cannot possibly be presented to the mind of a commencing student in a better form, nor be accompanied by a more judicious selection of problems, with hints for their solution, than occurs in the pages of Mr. Potts' publications. By combining symmetry of arrangement with simplicity of language, and by restoring the syllogism to its plain and simple form, so as to make an introduction to Geometry serve at the same time as an exercise in logic (an advantage which has been quite lost sight of in many of the abbreviated editions with which this University had previously been deluged), I consider that Mr. Potts has done good service to the cause of education."—*J. Power, M.A., Fellow of Clare College, and University Librarian.* (1848.)

"Mr. Potts has maintained the text of Simson, and secured the very spirit of Euclid's Geometry, by means which are simply mechanical. It consists in printing the syllogism in a separate paragraph, and the members of it in separate subdivisions, each, for the most part, occupying a single line. The divisions of a proposition are therefore seen at once without requiring an instant's thought. Were this the only advantage of Mr. Potts' Edition, the great convenience which it affords in tuition would give it a claim to become the Geometrical text-book of England. This, however, is not its only merit."—*Philosophical Magazine, January*, 1848.

"If we may judge from the solutions we have sketched of a few of them [the Geometrical Exercises], we should be led to consider them admirably adapted to improve the taste as well as the skill of the Student. As a series of judicious exercises, indeed, we do not think there exists one at all comparable to it in our language—viewed either in reference to the student or teacher.—*Mechanics' Magazine*, No. 1175.

"The 'Hints' are not to be understood as propositions worked out at length, in the manner of Bland's Problems, or like those worthless things called 'Keys,' as generally 'forged and filed,'—mere books for the dull and the lazy. In some cases references only are made to the Propositions on which a solution depends; in others, we have a step or two of the process indicated; in one case the analysis is briefly given to find the construction or demonstration; in another case the reverse of this. Occasionally, though seldom, the entire process is given as a model; but most commonly, just so much is suggested as will enable a student of average ability to complete the whole solution—in short, just so much (and no more) assistance is afforded as would, and *must be*, afforded by a tutor to his pupil. Mr. Potts appears to us to have hit the 'golden mean' of Geometrical tutorship."—*Mechanics' Magazine*, No. 1270.

"We can most conscientiously recommend it [The School Edition] to our own younger readers, as the *best* edition of the *best* book on Geometry with which we are acquainted."—*Mechaics' Magazine*, No. 1227.

LONDON: LONGMANS & CO., PATERNOSTER ROW.

ELEMENTARY ALGEBRA,

WITH BRIEF NOTICES OF ITS HISTORY.

SECTION VI.

INVOLUTION, EVOLUTION, AND SURD NUMBERS.

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INVOLUTION AND EVOLUTION, AND SURD NUMBERS.

ART. I. The words Involution and Evolution are used in various senses, according to the subjects respecting which they are employed.

In this part of mathematical science, the word *Involution* is employed to denote the process by which the powers of quantities are expressed or determined; and *Evolution* is the reverse process, by which the roots of quantities are expressed or determined.

The word *power* is used to denote the product of two or more equal numbers or quantities; and the word *root* denotes the reverse of that denoted by the word *power*.

The second power of any number or quantity is the product of two equal numbers or quantities; and conversely, the second root of any number or quantity is that number or quantity of which the second power will produce the given number or quantity.

The second power of a number is also named the square* of that number, because the product of the two equal numbers which express the number of lineal units in the adjacent sides of a square, denotes the number of square units in the area of the square.

In the same manner, the third power of any number or quantity is the product of three equal numbers or quantities; and the third root of any number or quantity is that number or quantity of which the third power is equal to the given number or quantity.

The third power of any given number is also called the cube of that number, because the product of the three equal numbers which express the number of lineal units in the three adjacent edges of a cube, will denote the number of cubic units contained in the volume of the cube; and the third root of any given number is called the cube root, because if a given number denote the number of cubic units in the volume of a cube, the cube root of the given number will denote the number of lineal units in one of the edges of the cube.

As the second and third powers of any quantity a have been assumed to be denoted by a^2 and a^3, so the second and third roots by a consistent analogy will be denoted by $a^{\frac{1}{2}}$ and $a^{\frac{1}{3}}$.

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* The words "square" and "cube" are the names of geometrical forms of a surface and a volume—modes of continued quantity, and cannot with strict propriety be applied to numbers. The proper analogous terms to the geometrical names *square* and *cube* are the *second power* and *third power* of numbers. But as the same words are applied both to the geometrical figures and the products of two and of three equal numbers, it is necessary that the distinction should be understood.

2. *To find a rule for the formation of the square of any algebraical polynomial.*

Here $(a+b)^2 = a^2+2ab+b^2$,

$(a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc$,

$(a+b+c+d)^2 = a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd$,

$$\begin{aligned}(a+b+c+d+e)^2 &= a^2+2a(b+c+d+e),\\ &\quad +b^2+2b(c+d+e),\\ &\quad +c^2+2c(d+e),\\ &\quad +d^2+2de+e^2,\\ &= a^2+b^2+c^2+d^2+e^2+2ab+2ac+2ad+2ae+2bc+2bd\\ &\quad +2be+2cd+2ce+2de.\end{aligned}$$

And so on, whatever may be the number of terms.

Hence, it appears that the square of any polynomial consists of the sum of the squares of each term and twice the product of all the terms taken two and two.

If there be n terms in the polynomial the rule may be thus expressed :—

Square the first term, and take the product of twice the first term and the sum of the remaining $n-1$ terms.

Next square the second term, and take the product of twice the second term and the sum of the remaining $n-2$ terms,

And so on to the square of the last two of the n terms.

Since $(a \pm b)^2 = a^2 \pm 2ab + b^2$, it is obvious that $4 \times a^2 \times b^2 = (\pm 2ab)^2$, or the square of every binomial consists of three terms, such that four times the product of the first and third is equal to the square of the second term. And, conversely, if four times the product of two terms of a trinomial be equal to the square of the remaining term, the trinomial is a complete square.*

3. *Every positive quantity has two equal square roots, one positive and the other negative.*

Since the square of $+a$, or $(+a)^2 = +a \times +a = +a^2$,

and the square of $-a$, or $(-a)^2 = -a \times -a = +a^2$.

Hence, conversely,

the square root of $+a^2$, or $\{(+a)^2\}^{\frac{1}{2}} = (+a)^{\frac{2}{2}} = +a$, and the square root of $+a^2$, or $\{(-a)^2\}^{\frac{1}{2}} = (-a)^{\frac{2}{2}} = -a$,

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* If in $(a+b)^2 = a^2+2ab+b^2$, $2a$ be written for a and 1 for b, then $(2a+1)^2 = 4a^2+4a+1 = 4a(a+1)+1$, which expresses the theorem.

The square of the sum of any two consecutive numbers is equal to four times their product increased by unity.

This theorem will be found useful in finding the squares of the odd natural numbers, by dividing the numbers into two parts whose difference is unity, as for instances :—

$(19)^2 = 4 \times 10 \times 9 + 1 = 360 + 1 = 361$, $(23)^2 = 4.12.11 + 1 = 528 + 1 = 529$, &c.

And if in $a^2 - b^2 = (a+b)(a-b)$, $a+1$ be put for a and a for b, then $(a+1)^2 - a^2 = a^2 + 2a + 1 - a^2 = 2a + 1$, which expressed in words is, The difference of the squares of two consecutive numbers is equal to twice the smaller number increased by unity.

$\therefore$ the square root of a^2 or $(a^2)^{\frac{1}{2}} = \pm a$.

In the same manner, if there be more terms than one,

The square of $a-b$, or $(a-b)^2 = a^2 - 2ab + b^2$,

and the square of $b-a$, or $(b-a)^2 = b^2 - 2ab + b^2$.

Hence, conversely, the square root of $a^2 - 2ab + b^2$ is $a-b$ and $b-a$, or $(a^2 - 2ab + b^2)^{\frac{1}{2}} = \pm(a-b)$.

It will be manifest from these conclusions that a negative quantity, as $-a^2$, cannot have a square root. The square root, however, of a negative quantity admits of *expression*, but not of *determination*, the negative sign indicating the extraction impossible to be performed. The square root of $-a^2$ may be expressed in the forms $(-a^2)^{\frac{1}{2}}$ and $a(-1)^{\frac{1}{2}}$, or $\sqrt{-a^2}$ and $a\sqrt{-1}$. The expression $(-a^2)^{\frac{1}{2}}$ has no intelligible meaning until it has been restored to $-a^2$ by the process of involution. When the data of a problem are incompatible with each other, the quæsita will always involve this expression; and, conversely, when this form of expression appears in the solution, the data of the problem are incompatible with each other.

The square root of a negative quantity has been named an imaginary or impossible quantity, and when such an expression occurs in the solution of a problem, it indicates that the problem is impossible under the proposed conditions. As for instance if it be proposed to divide 12 into two parts such that their product shall be 40. If x denote one part, then $12-x$ will denote the other part, and $x(12-x) = 40$. The solution of this equation gives $6 + \sqrt{-4}$ and $6 - \sqrt{-4}$ for the two parts of 12; and these parts satisfy the equation, for their sum is 12, and their product is 40. But if the problem itself be considered, it will be seen that 12 can be divided six ways into two parts; namely, 1 and 11, 2 and 10, 3 and 9, 4 and 8, 5 and 7, 6 and 6; the respective products of these parts are 11, 20, 27, 32, 35, 36; and 36 is the greatest possible of all the products. It is therefore clear that 12 cannot be divided into two integral numbers whose product shall be 40, and this is indicated in the results obtained from the equation. It has also been judiciously remarked with respect to impossible expressions that "these symbols, although the parts of them are incapable of being numerically computed, yet, without the violation of perspicuity or logical accuracy, may be employed in calculation, since their origin and derivation, and consequently what they represent, are known." *

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* In the second section of the first chapter of the Vija Ganita, the author states:—"The square of an affirmative or of a negative quantity is affirmative, and the root of an affirmative quantity is twofold, positive and negative. There is no square root of a negative quantity, for it is not a square." One of the commentators has given the following note on the text: "For, if it be maintained that a negative quantity may be a square, it must be shewn what it can be the square of. Now it

4. Prop. *Every square number ends in the figures* 1, 4, 5, 6, 9, *or* 00.

For every number ends in 1, 2, 3, 4, 5, 6, 7, 8, 9, or 0. And their squares end in 1, 4, 9, 6, 5, 6, 9, 4, 1, or 00 respectively. Therefore every square number ends in 1, 4, 5, 6, 9, or 00. It also follows that every number ending in 2, 3, 7, 8, 0, or an odd number of ciphers, cannot be a square number.

The converse of the proposition, however, is not always true, for there are other numbers besides square numbers which end in the figures 1, 4, 5, 6, 9, or 00.

5. *In the series of the natural numbers, the square numbers occur at intervals in the series, and the other numbers between the successive square numbers occur in groups, each consisting of two, four, six, &c., numbers between the first and second square numbers, the second and third, the third and fourth, &c., respectively.*

The squares of the first ten natural numbers are 1, 4, 9, 16, 25, 36, 49, 64, 81, 100; and the square roots of these are respectively 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. Hence in the first hundred natural numbers there are ten square numbers, and ninety which are not square numbers.

And further, it will appear on examination that between the square numbers 1 and 4 there are two numbers not square numbers; between 4 and 9, four numbers; between 9 and 16, six numbers; between 16 and 25, eight numbers; and so on, each successive group containing two more than the preceding.*

6. *If an integral number consist of* 1, 2, 3, *&c., digits, its square or second power will consist of* 1 *or* 2, 3 *or* 4, 5 *or* 6, *&c., digits respectively, and generally if a number consist of n digits its square will consist of* $2n-1$ *or* $2n$ *digits.*

Since the squares of 1, 10, 100, 1,000, &c., the least numbers composed of 1, 2, 3, 4, &c., digits, are 1, 100, 10,000, 1,000,000, &c., respectively, it follows that

The square of a number of 1 digit may consist of 1 or 2 digits.
„ „ 2 digits „ 3 or 4 digits.
„ „ 3 digits „ 5 or 6 digits.

And so on, and generally.

The square of a number of n digits will consist of $2n-1$ or $2n$ digits. For since the square of every number of n digits is less than

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cannot be the square of an affirmative quantity, for a square is the product of the multiplication of two like quantities, and if an affirmative one be multiplied by an affirmative one, the product is affirmative. Nor can it be the square of a negative quantity, for a negative quantity also, multiplied by a negative one, is positive. Therefore we do not perceive any quantity such as that its square can be negative."

* The process of deducing the law of a series, or the general form of any algebraical expression, from the consideration of particular cases, is usually called *induction*.

10^n, but not less than 10^{n-1}; the square of a number of n digits is less than 10^{2n}, but not less than 10^{2n-2}. But 10^{2n} is the least number which consists of $2n+1$ digits, and 10^{2n-2} the least number of $2n-1$ digits. Hence the square of every number of n digits will consist of less than $2n+1$ digits, but of not less than $2n-1$ digits, and consequently will consist of $2n-1$ or $2n$ digits.*

Conversely. If an integral number consist of $2n-1$ or $2n$ digits, its square root will consist of n digits. Hence, if a point be placed over every second digit beginning at the unit's place, the number of points will indicate the number of digits in the square root of the number.

Also since the square of a decimal of n places will consist of $2n$ places, it follows conversely that the square root of a decimal of $2n$ places will consist of n places of figures.

If, however, the given decimal whose square root is required consist of $2n-1$, or of an odd number of places, it will be necessary to annex a cipher to the right-hand figure of the given decimal, in order that the number of places may be $2n$, a multiple of 2.

7. *To explain the process of finding the square root of a given quantity.*

First. Let the given root, $a+b$, consist of two terms. Then, since the square of $a+b$ is $a^2+2ab+b^2$.†

Conversely, the square root of $a^2+2ab+b^2$ is $a+b$; and the object is to find how $a+b$ may be derived from $a^2+2ab+b^2$.

It is obvious that a, the first term, is the square root of a^2, the first term of the given square.

If a^2 be subtracted from $a^2+2ab+b^2$, there remains $+2ab+b^2$, from which to find b, the second term of the root.

Now b is contained $+2a+b$ times in $+2ab+b^2$; hence b, the second term of the root, is found by dividing the first term, $+2ab$, of the new dividend by $2a$, or twice the first term of the root.

Increase this divisor $2a$ by $+b$, the second term of the root, and multiply the divisor so increased by the second term of the root, and subtract the product from the new dividend, and the remainder is zero.

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* It may be noted that 10^0, 10^1, 10^2, 10^3, . . . 10^{n-1} denote the *least numbers* which consist of 1, 2, 3, 4, ... n digits, respectively; and 10^1-1, 10^2-1, 10^3-1, ... 10^n-1, or 99, 999, 9,999, &c., denote the *greatest numbers* which consist of 1, 2, 3, 4, ... n digits respectively.

† This expression is geometrically illustrated by the diagram in Euc. II. 4.

If the lines AC, CB contain a, b linear units respectively, then AB contains $a+b$ units, and $ABED$, the square on the line AB, is made up of the parts $ACGH$, $CBKG$, $HGFD$, $GKEF$.

Here $ABED$ is denoted by $(a+b)^2$, $HGFD$ by a^2, $CBKG$ by b^2, $ACGH$ by ab, $GKEF$ by ab.

If the line be divided into three parts a, b, c,

$$\text{then } (a+b+c)^2 = a^2+b^2+c^2+2ab+2ac+2bc$$

may be similarly illustrated by describing a square on the whole line and drawing the necessary lines through the points of division, as in the diagram to Euc. II. 4.

But if there be a remainder, the given number is not a square number, and its exact square root cannot be expressed by any number whatever.

Continued approximations, however, can be made to the square root of any such number, by annexing periods of ciphers as decimals to the given number, and continuing the extraction to any number of decimals required.

The square root of a fraction is found by dividing the square root of the numerator by the square root of the denominator.

When the numerator and denominator are square numbers, the quotient of these square roots is the exact square root of the fraction.

But when the numerator and denominator are not square numbers, the given fraction can be reduced to a decimal, and by extracting the square root of this decimal, an approximation to the square root of the fraction can be determined to any required degree of accuracy.

The 4th, 8th, 16th, and generally the 2^nth roots of any numbers can be found by means of the square root; as the fourth root of a number is the square root of the square root of that number; and the eighth root is the square root of the square root of the square root of the number, and so on for other roots.

8. *The cube of a positive quantity is positive, and the cube of a negative quantity is negative; and conversely, the cube root of a positive quantity is positive, and the cube root of a negative quantity is negative.*

For $(+a)^3 = +a \times +a \times +a = +a^3$;

and $(-a)^3 = -a \times -a \times -a = -a^3$.

That is, the cube of $+a$ is a^3, and the cube of $-a$ is $-a^3$.

Conversely $(+a^3)^{\frac{1}{3}} = +a$; and $(-a^3)^{\frac{1}{3}} = -a$.

Also $(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$,

and $(b-a)^3 = b^3 - 3b^2a + 3ba^2 - a^3 = -(a^3 - 3a^2b + 3ab^2 - b^3)$,

and it is obvious that $(b-a)^3$ is not equal to $(a-b)^3$ in the same manner as $(b-a)^2$ was shewn to be equal to $(a-b)^2$.

The cube of $a+b$ may be put into the form $a^3+b^3+3ab(a+b)$, and the cube of $a-b$ into the form $a^3-b^3-3ab(a-b)$.

Also since $(a+b)^3 = a^3+3a^2b+3ab^2+b^2$,

it appears that $3a^2b \times 3ab^2 = 9 \times a^3b^3$.

Or when four quantities are equal to the cube of a binomial, the product of the second and third terms is equal to nine times the product of the first and fourth.

The cubes of $a+b+c$, $a+b+c+d$, &c., may be found in the same manner as the squares of $a+b+c$, $a+b+c+d$, &c.

9. *In the series of the cube numbers each ends in one of the nine digits or in three ciphers.*

For every number ends in 1, 2, 3, 4, 5, 6, 7, 8, 9, or 0. The cubes of the first ten natural numbers are

1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, it will be seen that every cube number ends in the figures

1, 8, 7, 4, 5, 6, 3, 2, 9, or 000;

that is, every cube number ends in one of the nine digits or in three ciphers, and it is obvious that any number ending in one or two ciphers cannot be a cube number.*

10. *In the series of natural numbers, the cube numbers occur at intervals, and the other numbers between the successive cube numbers occur in groups, the numbers in each successive group being connected by a certain law.*

Since the cubes of the first ten natural numbers are

1, 8, 27, 64, 125, 216, 343, 512, 729, 1000;

it will be found on inspection, between the first and second, the second and third, and the successive cube numbers, that the groups of other numbers will contain 6, 18, 36, 60, 90, 126, 168, 216, 272, &c., numbers respectively. And the successive differences of the numbers in each group being 12, 18, 24, 30, 36, 42, 48, 56, &c., indicate the law of the successive groups.

11. *If an integral number consist of* 1, 2, 3, 4, *&c., digits, its cube or third power will consist of* 1, 2, 3; *or* 4, 5, 6; *or* 7, 8, 9, *&c., digits respectively: and generally, if a number consist of n digits, its cube will consist of* $3n-2$, $3n-1$, *or* $3n$ *digits.*

Since the cubes of 1, 10, 100, 1000, &c., the least numbers composed of 1, 2, 3, 4, &c., digits, are 1, 1,000, 1,000,000, 1,000,000,000, &c., respectively, it appears that—

The cube of a number of 1 digit may consist of 1, 2 or 3 digits,
" " 2 digits " 4, 5 or 6 "
" " 3 digits " 7, 8 or 9 "

And generally the cube of a number of n digits will consist of $3n-2$, $3n-1$, or $3n$ digits. For every number of n digits is less than

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* "Cubic numbers ending in 1, 2, 3, 4, 5, 6, 7, 8, 9, or 0 followed by 1
have their roots ending in 7, 4, 1, 8, 5, 2, 9, 6, 3, or 0 followed by 1.
Cube numbers ending in 1, 2, 3, 4, 5, 6, 7, 8, 9, or 0 followed by 3
have their roots ending in 1, 4, 7, 0, 3, 6, 9, 2, 5, or 8 followed by 7.
Cube numbers ending in 1, 2, 3, 4, 5, 6, 7, 8, 9, or 0 followed by 7
have their roots ending in 7, 0, 3, 6, 9, 2, 5, 8, 1, or 4 followed by 3.
Cube numbers ending in 1, 2, 3, 4, 5, 6, 7, 8, 9, or 0 followed by 9
have their roots ending in 3, 0, 7, 4, 1, 8, 5, 2, 9, or 6 followed by 9.

Hence every cubic number ending in 11 has 71 as the ending of its cube root; and every cubic number ending in 21 has 41 for the last two figures of its cube root, and so on.

Also it may be seen that the last two figures of any cubic number ending in 1, 3, 7, or 9 may be known if the figure before 1, 3, 7, or 9 be also known. For instance, the cube root of the cubic number 185,193 may be known to be 57, because all cubic numbers ending in 93 have their roots ending in 57. Thus we may know the cube root of the cubic number 3,869,893 by knowing only 3, ... 93. The first figure of the root must be 1 and the next two 57." *This note is taken from the Analyst, Vol. I., p. 75.*

10^n, but not less than 10^{n-1}; the cube of a number of n digits is less than 10^{3n}, but not less than 10^{3n-3}. But 10^{3n} is the least number which consists of $3n+1$ digits, and 10^{3n-3} is the least number which consists of $3n-2$ digits. Hence the cube of a number of n digits will consist of less than $3n+1$ digits, but of not less than $3n-2$ digits, and consequently must consist of $3n-2$, $3n-1$, or $3n$ digits.

Conversely, if an integral number consist of $3n-2$, $3n-1$, or $3n$ digits, its cube root will consist of n digits.

Hence if a point be placed over every third digit of a number beginning at the unit's place, the number of points will indicate the number of digits in the cube root of the number.

And since the cube of a decimal of n places will consist of $3n$ places; it follows conversely that the cube root of a decimal of $3n$ places will consist of n places of figures.

If a given decimal whose cube root is required consist of $3n-2$ or $3n-1$ places, it will be necessary to annex two ciphers or one to the right-hand digit of the decimal, in order that the number of places may be $3n$, a multiple of 3.

12. *To explain the process for finding the cube root of a given quantity.*

First suppose the root to consist of two terms, $a+b$.

Since the cube of $a+b$ is $a^3+3a^2b+3ab^2+b^3$.

Conversely, the cube root of $a^3+3a^2b+3ab^2+b^3$ is $a+b$, and it is required to be shewn how $a+b$ may be deduced from

$$a^3+3a^2b+3ab^2+b^3.$$

The first term of the root is a, the cube root of a^3 the first term of the given cube, and from this subtracting a^3, there remains

$$3a^2b+3ab^2+b^3.$$

The second term of the root is b, a factor of $3a^2b+3ab^2+b^3$; and if the first term, $3a^2b$, be divided by $3a^2$, three times the square of the first term of the root, the second term b is obtained.

And if the sum of three times the square of the first term multiplied by the second, three times the first multiplied by the square of the second, and the cube of the third term, be subtracted from $3a^2b+3ab^2+b^3$, the whole cube of $a+b$ has been subtracted, and $a+b$, the cube root, is determined.*

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* The following process may be in some instances found more commodious when several terms of the root are required :—

For example. Find the cube root of $x^6+9x^5+39x^4+483x^3+155x^2+144x+64$.

$$\begin{array}{rl}
 & x^6+9x^5+39x^4+483x^3+155x^2+144x+64\,(x^2+3x+4 \\
 & \underline{x^6} \\
 & \underline{3x^4)\,9x^5} \\
(x^2+3x)^3 = & \underline{x^6+9x^5+27x^4+27x^3} \\
 & \qquad 3x^4)\,12x^4 \\
(x^2+3x+4)^3 = & \overline{\underline{x^6+9x^5+39x^4+483x^3+155x^2+144x+64}} \\
 & \qquad\qquad 0
\end{array}$$

The process may be thus exhibited—

$$\begin{array}{l}
\qquad a^3+3a^2b+3ab^2+b^3(a+b \\
\qquad a^3 \\ \hline
3a^2)3a^2b+3ab^2+b^3 \\ \hline
\left.\begin{array}{rl}
3\times a^2\times b= & 3a^2b \\
3\times a\times b^2= & \qquad +3ab^2 \\
b^3= & \qquad\qquad +b^3
\end{array}\right\} \\ \hline
\qquad 3a^2b+3ab^2+b^3. \\ \hline
\qquad 0
\end{array}$$

If the root consist of three terms $a+b+c$, then since the cube of $a+b+c$ is $(a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3$, the first two terms of the root being already found, the third term is found by dividing the first term of the next dividend by $3(a+b)^2$, or three times the square of the first two terms of the root, and then proceeding in the same manner as before.

The process stands thus:—

$$\begin{array}{l}
\qquad (a+b)^3+3(a+b)^2c+3(a+b)c^2+c^3(a+b \\
\qquad (a+b)^3 \\ \hline
3\times(a+b)^2=3(a+b)^2)\ 3(a+b)^2c+3(a+b)c^2+c^3 \\ \hline
\left.\begin{array}{rl}
3\times(a+b)^2\times c= & 3(a+b)^2c \\
3\times(a+b)\times c^2= & \qquad +3(a+b)c^2 \\
c^3= & \qquad\qquad +c^3
\end{array}\right\} \\ \hline
\qquad 3(a+b)^2c+3(a+b)c^2+c^3 \\ \hline
\qquad 0
\end{array}$$

And the same process is repeated if there be more than three terms in the cube root of the given quantity.

This process can be applied to find the cube root of numbers.

If a, b be the digits of a number, then the number is denoted by $10a+b$, and the cube of $10a+b$ is $1000a^3+300a^2b+30ab^2+b^3$, and the process of finding the cube root is thus exhibited.

$$\begin{array}{l}
\qquad 1000a^3+300a^2b+30ab^2+b^3(10a+b \\
\qquad 1000a^3 \\ \hline
3\times(10a)^2=300a^2)300a^2b+30ab^2+b^3 \\ \hline
\left.\begin{array}{rl}
3\times(10a)^2\times b= & 300a^2b \\
3\times(10a)\ \times b^2= & \qquad +30ab^2 \\
b^3= & \qquad\qquad +b^3
\end{array}\right\} \\ \hline
\qquad 300a^2b+30ab^2+b^3 \\ \hline
\qquad 0
\end{array}$$

If $a=3$, $b=4$, then $(30+4)^3=27000+10800+1440+64$, and the process is thus exhibited:—

$$\begin{array}{rl}
 & 27000+10800+1440+64(30+4 \\
 & 27000 \\
\hline
3\times(30)^2=2700) & 10800+1440+64 \\
\hline
\left.\begin{array}{rl} 3\times(30)^2\times4= & 10800 \\ 3\times\ 30\ \times4^2= & \qquad +1440 \\ 4^3= & \qquad\qquad +64 \end{array}\right\} & \\
\hline
 & 10800+1440+64 \\
\hline
 & 0
\end{array}
\qquad
\begin{array}{rl}
 & 3\dot{9}30\dot{4}(34 \\
 & 27 \\
\hline
3\times3^2=27) & 12304 \\
\hline
\left.\begin{array}{rl} 3\times3^2\times4= & 108 \\ 3\times3\ \times4^2= & 144 \\ 4^3= & 64 \end{array}\right\} & \\
\hline
 & 12304 \\
\hline
 & 0
\end{array}$$

And in a similar manner if the root consist of three digits, the first two digits having been determined, the third is found by taking for the next divisor three times the square of the root already determined, as in the following process of finding the cube root of

$$1000(30+4)^3+300(30+4)^2 5+30(30+4)25+125$$

or 41063625, which is the cube of 300+40+5, a number of three digits.

$$\begin{array}{rl}
 & 4\dot{1}06\dot{3}62\dot{5}(345 \\
 & 27 \\
\hline
3\times3^2=27) & 14063 \\
\hline
\left.\begin{array}{rl} 3\times3^2\times4= & 108 \\ 3\times3\times4^2= & 144 \\ 4^3= & 64 \end{array}\right\} & \\
\hline
 & 12304 \\
\hline
3\times(34)^2=3468) & 1759625 \\
\hline
\left.\begin{array}{rl} 3\times(34)^2\times5= & 17340 \\ 3\times34\times5^2= & 2550 \\ 5^3= & 125 \end{array}\right\} & \\
\hline
 & 1759625 \\
\hline
 & 0
\end{array}$$

The process for finding the cube root of numbers may be expressed by the following rule:—

1. Divide the number into periods by placing a point over every third figure, beginning at the place of units.

2. Find the greatest number the cube of which is less than the number of the last period on the left; this number will be the first figure of the root.

3. Subtract the cube of this number from the last period, and to the remainder annex the next period for a dividend.

4. Divide this number (omitting the last two figures) by three times the square of the first figure of the root, the quotient will be the second figure.

5. Then subtract from this dividend the sum of the three following

products, namely, three times the square of the first multiplied by the second figure of the root, three times the first multiplied by the square of the second, and the cube of the second; placing the unit's figure of the second product one place to the right of the first, and that of the third product one place to the right of the second.

6. To the remainder annex the third period for a new dividend.

7. To find the third figure of the root, take three times the square of the root already found for a new trial divisor, and proceed as in finding the second figure of the root. The fourth and other figures of the root are found in the same manner, taking for each successive trial divisor three times the square of the root already found.

If there be a remainder after the integral part of the root has been obtained, the process may be continued, and a nearer approximation to the true value of the root may be obtained by annexing periods of decimals, and continuing the operation for any required number of decimal places in the root.

The cube root of a fraction is found by extracting the cube root of the numerator and denominator; but if the numerator and denominator be not cubic numbers, it will be found more convenient to reduce the fraction to a decimal, and extract the cube root of the decimal.

The 9th, 27th, 81st, and generally the 3^nth roots of numbers can be found by means of the cube root.

Also the 6th, 12th, 18th, and generally the $2^m.3^n$th roots can be determined by the square and cube roots.

13. The square roots of the numbers which are not complete squares have been named surd numbers.

The square roots of the numbers 2 and 3, which lie between 1 and 4, are each greater than 1 but less than 2; as also the square roots of 5, 6, 7, 8, which lie between 4 and 9, are each greater than 2 but less than 3; and a similar remark applies to all the numbers which lie between any two consecutive square numbers. The square roots of these numbers can be *denoted*, but cannot be exactly expressed by any definite number integral or fractional. Approximate values of the square roots of these numbers can be determined to any required degree of accuracy.*

The cube root of numbers which are not cubic numbers, are also called surd numbers, and the cube roots of all numbers which lie between the successive cubic numbers, may be *denoted*, but cannot be *exactly expressed* by any numbers whatever. The cube roots of the numbers 2, 3, 4, 5, 6, 7, which lie between 1 and 8, are each greater than 1 but less than two; and approximate values of the cube roots of these numbers can be obtained to any required degree of accuracy. And so of all the numbers which lie between the consecutive cubic numbers.

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* The square root of any incomplete square number may be continued to any assigned number of places of decimals, and each successive decimal figure obtained in the root gives a nearer approximation to the true value of the root.

For example. Let the square root of 2 be extracted to ten places of decimals.

The square root of 2 or $2^{\frac{1}{2}} = 1{\cdot}4142135623\ldots\ldots$

If 1·4, 1·41, 1·414, 1·4142, 1·41421, 1·414213, &c., be respectively taken for the square root of 2, the successive differences of these numbers are ·01, ·004, ·0002, ·0001, ·000003, &c., which successively decrease, and shew that each succeeding approximation to the square root of 2 is nearer to the true value than each of those which precede it.

The sum or difference of quadratic surd numbers may be expressed by connecting them by means of the signs of addition and subtraction; but if they can be reduced so as to involve the same surd number, the sum or difference can be expressed as a single quantity.*

In the same manner the sum or difference of cubic surd numbers can be expressed.

The product or quotient of two quadratic surd numbers is obtained by multiplying or dividing the numbers, and retaining the common index. And the product or quotient of two cubic surd numbers is expressed in the same manner.

If the indices of the surd numbers be different, the indices must be reduced to the same denominator before the operations can be performed.†

The powers or roots of surd numbers are expressed, the former by multiplying, the latter by dividing the index of the surd number by the index of the required power or root.‡

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* To find the sum and difference of $\sqrt{7}$ and $2\sqrt{5}$.
Here the sum is $3\sqrt{7}+2\sqrt{5}$ and the difference is $3\sqrt{7}-2\sqrt{5}$.
To find the sum and difference of $\sqrt{75}$ and $\sqrt{48}$.
Here $\sqrt{75}=\sqrt{(25\times 3)}=\sqrt{25}\times\sqrt{3}=4\sqrt{3}$,
$$\text{and } \sqrt{48}=\sqrt{(16\times 3)}=\sqrt{16}\times\sqrt{3}=4\sqrt{3},$$
$$\therefore \sqrt{75}+\sqrt{48}=5\sqrt{3}+4\sqrt{3}=9\sqrt{3}, \text{ and } \sqrt{75}-\sqrt{48}=5\sqrt{3}-4\sqrt{3}=\sqrt{3}.$$

† Find the product and quotient of $3\sqrt{15}$ and $2\sqrt{5}$.
Here $3\sqrt{15}\times 2\sqrt{5}=6\sqrt{75}=6\sqrt{(25\times 3)}=6\sqrt{25}\times\sqrt{3}=6\times 5\times\sqrt{3}=30\sqrt{3}$,
$$\text{and } 3\sqrt{15}\div 2\sqrt{5}=\frac{3\sqrt{15}}{2\sqrt{5}}=\tfrac{3}{2}\sqrt{3}.$$
Find the product and quotient of $\sqrt{18}$ and $\sqrt[3]{12}$.
Here $\sqrt{18}=(18)^{\frac{1}{2}}=(18)^{\frac{3}{6}}=(18^3)^{\frac{1}{6}}=(5832)^{\frac{1}{6}}$,
$$\text{and } \sqrt[3]{12}=(12)^{\frac{1}{3}}=(12)^{\frac{2}{6}}=(12^2)^{\frac{1}{6}}=(144)^{\frac{1}{6}}.$$
$$\text{Then } \sqrt{18}\times\sqrt[3]{12}=(5832)^{\frac{1}{6}}\times(144)^{\frac{1}{6}}=(5832\times 144)^{\frac{1}{6}}=(839808)^{\frac{1}{6}},$$
$$\text{and } \sqrt{18}\div\sqrt[3]{12}=\frac{(18^3)^{\frac{1}{6}}}{(12^2)^{\frac{1}{6}}}=\left\{\frac{81}{2}\right\}^{\frac{1}{6}}.$$

‡ Find the second power and the second root of $2\sqrt{5}$.
Here $(2\sqrt{5})^2=(2\times 5^{\frac{1}{2}})^2=2^2\times 5=20$,
$$\text{and } (2\times\sqrt{5})^{\frac{1}{2}}=(2\times 5^{\frac{1}{2}})^{\frac{1}{2}}=2^{\frac{1}{2}}\times 5^{\frac{1}{4}}=4^{\frac{1}{4}}\times 5^{\frac{1}{4}}=(20)^{\frac{1}{4}}.$$
Find the third power and the third root of $4\sqrt[3]{7}$.
Here $(4\sqrt[3]{7})^3=(4\times 7^{\frac{1}{3}})^3=4^3\times 7=448$,
$$\text{and } (4\sqrt[3]{7})^{\frac{1}{3}}=(4\times 7^{\frac{1}{3}})^{\frac{1}{3}}=4^{\frac{1}{3}}\times 7^{\frac{1}{9}}=4^{\frac{3}{9}}\times 7^{\frac{1}{9}}=(4^3\times 7)^{\frac{1}{9}}=(448)^{\frac{1}{9}}.$$

14. *If an equation consist partly of rational and partly of quadratic surd quantities, as $x+\sqrt{y}=a+\sqrt{b}$, then the rational parts on each side are equal, as also the irrational parts.*

If x be not equal to a, suppose $x=a\pm m$, then $a\pm m+\sqrt{y}=a+\sqrt{b}$, and $\pm m+\sqrt{y}=\sqrt{b}$, or the sum or difference of a rational and irrational quantity is equal to a rational quantity, which is impossible; and therefore $x=a$ and $\sqrt{y}=\sqrt{b}$.

This property can be applied in some cases to find the square root of a binomial when one or both of its terms contains quadratic surds.

Let $a+\sqrt{b}$ denote the given binomial, and let $x+\sqrt{y}$ its square root.

$$\text{Then } x+\sqrt{y}=\sqrt{(a+\sqrt{b})}$$

$$\text{and } x^2+2x\sqrt{y}+y=a+\sqrt{b}, \text{ by squaring each side,}$$

$$\therefore x^2+y=a, \text{ and } 2x\sqrt{y}=\sqrt{b}, \text{ and } x^2-2x\sqrt{y}+y=a-\sqrt{b},$$

$$\therefore x-\sqrt{y}=\sqrt{(a-\sqrt{b})} \text{ by extracting the square root,}$$

$$\text{but } x+\sqrt{y}=\sqrt{(a+\sqrt{b})},$$

$$\therefore x^2-y=\sqrt{(a^2-b)}.$$

If a^2-b be a complete square, the square root of $a+\sqrt{b}$ can be expressed in the form of $x+\sqrt{y}$.

$$\text{And since } x^2+y=a, \text{ and } x^2-y=\sqrt{(a^2-b)};$$

$$\therefore 2x^2=a+\sqrt{(a^2-b)}, \text{ and } x=\left\{\frac{a+\sqrt{(a^2-b)}}{2}\right\}^{\frac{1}{2}},$$

$$2y=a-\sqrt{(a^2-b)}, \text{ and } \sqrt{y}=\left\{\frac{a-\sqrt{(a^2-b)}}{2}\right\}^{\frac{1}{2}},$$

$$\therefore x+\sqrt{y}=\left\{\frac{a+\sqrt{(a^2-b)}}{2}\right\}^{\frac{1}{2}}+\left\{\frac{a-\sqrt{(a^2-b)}}{2}\right\}^{\frac{1}{2}}.\text{*}$$

Since a is the sum of the squares of the two terms of the binomial, and $\sqrt{b}$ twice the product of these terms; in most cases the square root of a binomial surd, when it is possible, may generally be found by inspection.

In the same manner may be found the square root of an impossible expression, which will always involve the impossible symbol $\sqrt{-1}$.

Let the square root of $a+b\sqrt{-1}$ be required.

---

* Find the square root of $79+20\sqrt{3}$.

Let $x+\sqrt{y}=\sqrt{(79+20\sqrt{3})}$, $x^2+2x\sqrt{y}+y=79+20\sqrt{3}$;

$$\therefore x^2+y=79, \text{ and } 2x\sqrt{y}=20\sqrt{3}, \; x^2-2x\sqrt{y}+y=79-20\sqrt{3},$$

$$x-\sqrt{y}=\sqrt{(79-20\sqrt{3})}, \text{ but } x+\sqrt{y}=\sqrt{(79+20\sqrt{3})};$$

$$\therefore x^2-y=\sqrt{(6241-1200)}=\sqrt{5041}=71.$$

Hence the square root $79+20\sqrt{3}$ can be expressed in the form $x+\sqrt{y}$.

Since $x^2+y=79$ and $x^2-y=71$, $\therefore 2x^2=150$, $\therefore x^2=75$, and $x=\sqrt{75}=5\sqrt{3}$,

$$\text{and } 2y=8, \; y=4, \text{ and } \sqrt{y}=\sqrt{4}=2, \; \therefore \sqrt{(79+20\sqrt{3})}=2+5\sqrt{3}.$$

By a like process the square root of $-5+12\sqrt{-1}$ will be found to be $2+3\sqrt{-1}$.

Assume $x+y=\sqrt{(a+b\sqrt{-1})}$, then $x^2+2xy+y^2=a+b\sqrt{-1}$,
and $x^2+y^2=a$, $2xy=b\sqrt{-1}$, $\therefore\ x^2-2xy+y^2=a-b\sqrt{-1}$,
and $x-y=\sqrt{(a-b\sqrt{-1})}$, but $x+y=\sqrt{(a+b\sqrt{-1})}$,

$$\therefore\ x^2-y^2=\sqrt{(a^2+b^2)},$$

and if a^2+b^2 be a complete square, the square root of $a+b\sqrt{-1}$ can be expressed in the form required.

But since $x^2+y^2=a$, and $x^2-y^2=\sqrt{(a^2+b^2)}$,

$$2x^2=a+\sqrt{(a^2+b^2)},\text{ and } x=\left\{\frac{a+\sqrt{(a^2+b^2)}}{2}\right\}^{\frac{1}{2}},$$

$$-2y^2=-a+\sqrt{(a^2+b^2)},\text{ and } y=\left\{\frac{a-\sqrt{(a^2+b^2)}}{2}\right\}^{\frac{1}{2}}.\sqrt{-1}.$$

$$\therefore\ x+y=\left\{\frac{a+\sqrt{(a^2+b^2)}}{2}\right\}^{\frac{1}{2}}+\left\{\frac{a-\sqrt{(a^2+b^2)}}{2}\right\}^{\frac{1}{2}}.\sqrt{-1}.$$

15. It has been already pointed out that a^n has been assumed to denote the product of n quantities each equal to a, and the expression a^n is named the nth power of a.

So conversely, the expression $a^{\frac{1}{n}}$ by a consistent analogy has been assumed to denote the reverse process of that denoted by a^n, or the nth root of a.

By assuming the principle of indices to be generally true,

$$\text{The } n\text{th power of } a^{\frac{1}{n}}=a^{\frac{1}{n}}\times a^{\frac{1}{n}}\times a^{\frac{1}{n}}\ldots\text{ to } n \text{ factors.}$$

$$=a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+\ldots\text{ to } n \text{ terms.}}$$

$$=a^{\frac{n}{n}}$$

$$=a^1 \text{ or } a$$

Since $a^{\frac{1}{n}}$ is such a quantity, which when raised to the nth power, produces a, it follows that $a^{\frac{1}{n}}$ is a proper and consistent expression for the nth root of a.

Also the expression $a^{\frac{m}{n}}$ will properly and consistently denote the mth power of the nth root of a, or the nth root of the mth power of a, and the expression may be written in the form $(a^{\frac{1}{n}})^m$ or $(a^m)^{\frac{1}{n}}$.*

It has also been shewn that $a^m.a^n=a^{m+n}$, and $a^m\div a^n=a^{m-n}$, when m and n are integers.

The same forms of exponential expressions are also true when the

---

* The truth of this may be otherwise exhibited:—

If $(a^m)^{\frac{1}{n}}=a^x$, in which x is unknown,
By raising each side of this equality to the nth power,

$$\text{then } a^m=a^{nx},\ \therefore\ nx=m,\text{ and } x=\frac{m}{n}.$$

$$\therefore\ (a^m)^{\frac{1}{n}}=a^{\frac{m}{n}}.$$

indices are fractions, namely, $a^{\frac{m}{n}} \times a^{\frac{p}{q}} = a^{\frac{m}{n}+\frac{p}{q}}$, and $a^{\frac{m}{n}} \div a^{\frac{p}{q}} = a^{\frac{m}{n}-\frac{p}{q}}$.

For $a^{\frac{m}{n}} = a^{\frac{mq}{nq}} = \{a^{\frac{1}{nq}}\}^{mq}$, and $a^{\frac{p}{q}} = a^{\frac{np}{nq}} = \{a^{\frac{1}{nq}}\}^{np}$,

$$\therefore\ a^{\frac{m}{n}} \times a^{\frac{p}{q}} = \{a^{\frac{1}{nq}}\}^{mq} \times \{a^{\frac{1}{nq}}\}^{np} = \{a^{\frac{1}{nq}}\}^{mq+np} = a^{\frac{mq+np}{nq}} = a^{\frac{m}{n}+\frac{p}{q}}.$$

And so may be shewn $a^{\frac{m}{n}} \div a^{\frac{p}{q}} = a^{\frac{m}{n}-\frac{p}{q}}$.

16. *The even and odd powers of a positive quantity are positive; and the even powers of a negative quantity are positive, and the odd powers negative.*

For $(+a)^2 = +a^2$, $(+a)^3 = +a^3$, $(+a)^4 = +a^4$, &c.

Generally $(+a)^{2n} = (+a^2)^n = +a^{2n}$, and $(+a)^{2n+1} = +a^{2n}.a = +a^{2n+1}$, or any power, odd or even, of a positive quantity is positive.

Again $(-a)^2 = +a^2$, $(-a)^3 = -a^3$, $(-a)^4 = +a^4$, &c.

Generally $(-a)^{2n} = \{(-a)^2\}^n = (+a^2)^n = +a^{2n}$,

$$\text{and } (-a)^{2n+1} = (-a)^{2n}.(-a) = +a^{2n}.(-a) = -a^{2n+1},$$

or, the even powers of a negative quantity are positive, and the odd powers are negative.

17. *The even roots of a positive quantity are positive and negative, and the odd roots of a positive quantity are positive, and of a negative quantity are negative.*

For the $2n$th power of $+a^{\frac{1}{2n}}$ is $+a$,

and the $2n$th power of $-a^{\frac{1}{2n}}$ is $+a$.

Hence, conversely, the $2n$th root of $+a$ is $+a^{\frac{1}{2n}}$ and $-a^{\frac{1}{2n}}$;

$$\text{that is } (+a)^{\frac{1}{2n}} = \pm a^{\frac{1}{2n}}.$$

Again, the $(2n+1)$th power of $+a^{\frac{1}{2n+1}}$ is $+a$,

and the $(2n+1)$th power of $-a^{\frac{1}{2n+1}}$ is $-a$.

Hence, conversely, the $(2n+1)$th root of $+a$ is $+a^{\frac{1}{2n+1}}$,

and the $(2n+1)$th root of $-a$ is $-a^{\frac{1}{2n+1}}$;

$$\text{that is } \{+a\}^{\frac{1}{2n+1}} = +a^{\frac{1}{2n+1}}, \text{ and } \{-a\}^{\frac{1}{2n+1}} = -a^{\frac{1}{2n+1}}.$$

18. *The even roots of negative quantities are impossible expressions.*

For the $2n$th root of $-a^{2n}$ cannot be $+a$ or $-a$.

But since $-a^{2n} = +a^{2n} \times -1$,

the $2n$th root of $-a^{2n}$ may be thus expressed:

$$\{-a^{2n}\}^{\frac{1}{2n}} = \{+a^{2n} \times -1\}^{\frac{1}{2n}} = \{+a^{2n}\}^{\frac{1}{2n}} \times (-1)^{\frac{1}{2n}} = \pm a(-1)^{\frac{1}{2n}}.*$$

---

* To find $(+\sqrt{-1})^n$ and $(-\sqrt{-1})^n$.
First, the nth power of $+\sqrt{-1}$,

An exponential quantity is raised to any power by multiplying the exponent of the quantity by that of the power; and conversely, any root of an exponential quantity is extracted by dividing the exponent of the quantity by that of the root.

For $(a^m)^2 = a^m.a^m = a^{2m}$, $(a^m)^3 = a^m.a^m.a^m = a^{3m}$, $(a^m)^4 = a^{4m}$, &c.

And generally $(a^m)^n = a^m.a^m.a^m \ldots\ldots$ to n factors,

$$= a^{m+m+m+\ldots\ldots n \text{ terms.}}$$

$$= a^{mn}.$$

Or the nth power of a^m is a^{mn}, and is found by multiplying m, the exponent of the quantity, by n, the exponent of the power.

In the same manner, $(a^n)^m = a^{mn}$. Hence $(a^m)^n = (a^n)^m$.

Conversely, $(a^{mn})^{\frac{1}{n}} = (a^{\frac{1}{n}})^{mn} = a^{\frac{mn}{n}} = a^m$.

Or the nth root of a^{mn} is a^m, and is found by dividing mn, the exponent of the quantity, by n, the exponent of the root.

Also it may be shewn that $(a^m)^{-n} = a^{-mn}$, and $(a^{-m})^{-n} = (a^{-n})^{-m}$.

Any power of the product, and of the quotient of two quantities is equal to the product and quotient respectively of the powers of the quantities; that is, $(ab)^m = a^m b^m$ and $\left\{\frac{a}{b}\right\}^m = \frac{a^m}{b^m}$.

Also any root of the product, and of the quotient of two quantities is equal to the product and the quotient respectively of the roots of the quantities; that is, $(ab)^{\frac{1}{n}} = a^{\frac{1}{n}} b^{\frac{1}{n}}$, and $\left\{\frac{a}{b}\right\}^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}$.

$(+\sqrt{-1})^2 = (-1)^{\frac{1}{2}}.(-1)^{\frac{1}{2}} = (-1)^{\frac{1}{2}+\frac{1}{2}} = (-1)^{\frac{2}{2}} = -1.$
$(+\sqrt{-1})^3 = (-1).(-1)^{\frac{1}{2}} = -\sqrt{-1}.$
$(+\sqrt{-1})^4 = -(-1)^{\frac{1}{2}}.(-1)^{\frac{1}{2}} = -(-1)^{\frac{1}{2}+\frac{1}{2}} = -(-1)^{\frac{2}{2}} = -(-1) = +1.$
$(+\sqrt{-1})^5 = +1.(-1)_{\frac{1}{2}} = +\sqrt{-1}.$

&c., &c. Hence they recur after the 4th power.

Generally, since every integral number is of the form $4m$, $4m+1$, $4m+2$, $4m+3$,

$\therefore (+\sqrt{-1})^{4m} = [\{+(-1)^{\frac{1}{2}}\}]^{4m} = \{(-1)^2\}^m = (+1)^m = +1.$
$(+\sqrt{-1})^{4m+1} = (\sqrt{-1})^{4m}.\sqrt{-1} = +1 \times \sqrt{-1} = +\sqrt{-1}.$
$(+\sqrt{-1})^{4m+2} = (\sqrt{-1})^{4m}.(\sqrt{-1})^2 = +1 \times -1 = -1.$
$(+\sqrt{-1})^{4m+3} = (\sqrt{-1})^{4m+2}\sqrt{-1} = -1 \times \sqrt{-1} = -\sqrt{-1}.$

Hence when n is of the forms $4m$ and $4m+2$, $(+\sqrt{-1})^n = \pm 1$,
and when n is of the forms $4m+1$ and $4m+3$, $(+\sqrt{-1})^n = \pm\sqrt{-1}$.

In a similar way may be found the forms of the nth powers of $-\sqrt{-1}$.

$(-\sqrt{-1})^{4m} = [\{-(-1)^{\frac{1}{2}}\}^4]^m = \{+(-1)^2\}^m = (+1)^m = +1.$
$(-\sqrt{-1})^{4m+1} = (-\sqrt{-1})^{4m} \times (-\sqrt{-1}) = +1 \times -\sqrt{-1} = -\sqrt{-1}.$
$(-\sqrt{-1})^{4m+2} = (-\sqrt{-1})^{4m}.\{-(-1)^{\frac{1}{2}}\}^2 = +1.(-1)^{\frac{2}{2}} = +1 \times -1 = -1.$
$(-\sqrt{-1})^{4m+3} = (-\sqrt{-1})^{4m+2} \times (-\sqrt{-1}) = -1 \times -\sqrt{-1} = +\sqrt{-1}.$

Hence when n is of the form $4m$ and $4m+2$, $(-\sqrt{-1})^n = \pm 1$; and when n is of the form $4m+1$, and $4m+3$, $(-\sqrt{-1})^n = \mp\sqrt{-1}$.

EXERCISES ON INVOLUTION, EVOLUTION, ETC.

I.

Extract the square roots of—

1. $x^2+12x+36$. 2. $x^2-8x+16$. 3. $4a^2x^2+4abx+b^2$.

4. $x^2+x^{-2}\pm 2$. 5. $x^4+2x^3+3x^2+2x+1$.

6. $4x^4-4x^3-3x^2+2x+1$. 7. $x^4-10ax^3+33a^2x^2-40a^3x+16a^4$.

8. $49x^4+56x^3y+30x^2y^2+8xy^3+y^4$.

9. $a^6+2a^5b+3a^4b^2+4a^3b^3+3a^2b^4+2ab^5+b^6$.

10. $a^8+6a^7+13a^6+14a^5+12a^4+10a^3+5a^2+2a+1$.

11. $\{(x-1)(x-2)+3\}\cdot\{(x-2)(x-3)+4\}+8x^3+55x+175$.

12. $16a^3(a+b+c)+4abc(b+c)+4a^2(b^2+c^2)+16a^2bc+b^2c^2$.

13. $3(3a^2-2ab+b^2)(a^2+3b^2)+b^2(a+4b)^2$.

14. $a^2(a-5b)(a-b)+b^2(3a-b)^2-3a^2b^2$.

15. $4a^2b^2+(a^2+b^2)^2+4ab(a^2+b^2)$.

16. $a^4+b^4+c^4+d^4+2a^2(b^2+d^2)+2b^2(c^2+d^2)+2c^2(a^2+d^2)$.

17. $(a-b)^4+2(a^4+b^4)-2(a^2+b^2)(a^2-b^2)$.

18. $4a(2a^3-a^2-1)+(2a^3+a^2+1)^2$.

19. $ax(ax+1)(ax+2)(ax+3)+1$.

20. $a^2(a^2+b^2-c^2)+2(a+b)(b+c)ac+2a^2(ab+ac+bc)+b^2c^2$.

21. $(a^2+b^2+c^2)^3+2(ab+ac+bc)^3-6(a^2+b^2+c^2)(ab+bc+ac)^2$.

II.

Extract the square roots of the following numbers—

1. 625, 2704, 65536, 425104, 6589489, 58553104, 152399025.

2. ·25, ·0025, ·000025, 2500, 250000, 10, ·1, ·01, ·001, ·0001, ·000001.

3. ·025, 520, ·121, 1·2101, 4·9, ·049, 490, ·9, 1·01, ·144, $\cdot\dot{9}$, $\dot{5}\dot{4}$.

4. $1\frac{7}{9}$, $20\frac{1}{4}$, $12\frac{1}{4}$, $7\frac{9}{16}$, $498\frac{7}{9}$, $\frac{582169}{956484}$.

5. $\frac{2}{5}$, $2\frac{2}{3}$, $1\frac{1}{2}$, $17\frac{5}{8}$, $1\frac{1}{125}$, $3\frac{1}{3}$.

6. 1·64, 1·064, 1·0064, 1·00064, 1·000064, 1·0000064.

7. 6·4, 0·64, ·064, ·0064, ·00064, ·000064, 64·00064, 64,000064.

8. 76212900, ·00762129, 7·62129, 76212·9, 7621·29, 76·2129.

III.

1. Shew that every quantity has two equal square roots, one positive and the other negative. Find the square roots of $a^2-2ab+b^2$, and shew when a trinomial of this form is a complete square, that four times the product of the first and third terms is equal to the square of the second.

2. Shew that $(a+b+c)^2=a^2+b^2+c^2+2ab+2ac+2bc$: and from this

result deduce the squares of $a+b-c$, $a-b+c$, $-a+b+c$, $a-b-c$ respectively, without performing the operation of multiplication.

3. Write down the algebraical expression for the product of the square root of the sum of two quantities a and b, and the cube of their sum; and calculate its value to seven places of decimals when $a=1$ and $b=2$.

4. State in words the several operations to be performed in order to obtain the result expressed by the following algebraical expression, and find its value when $a=b=4$.

$$\left\{\frac{ma^2+nb^2}{m+n}\right\}^{\frac{1}{2}}$$

5. If a^4 arise from the products of $+a^2\times+a^2$, and $-a^2\times-a^2$; can a^2 be considered as the only square root of a^4?

6. Express $\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}=\left\{\frac{a}{b}\right\}^{\frac{1}{2}}$ in words, and find the value of each expression separately, when $a=225$ and $b=25$.

7. Extract the square root of $\frac{a^2}{9}-\frac{2ax}{21}+\frac{x^2}{49}-\frac{ay}{6}+\frac{xy}{14}+\frac{y^2}{16}$, and explain why, if the terms be arranged in the reverse order, and the root be then extracted, a result will be obtained differing only in the sign of the whole quantity from that obtained in the first instance.

8. Find the square root of $x^2-2xy+4y^2$ in terms of a and b, where $x=9a^2+12ab$, and $y=2b^2+6ab$.

9. Shew that the square root of $(x-x^{-1})^2+(y-y^{-1})^2$ is a rational function of a and b, if $x=\frac{1}{4}\left\{\frac{a}{b}+\frac{b}{a}\right\}$ and $y=\frac{ax-b}{a-bx}$.

IV.

1. What relations must subsist between a, b, c in order that ax^2+bx+c may be a complete square?

2. Shew that if $x=\frac{m^2-cn^2}{bn^2-2amn}$, then a^2x^2+bx+c is a perfect square.

3. After performing the operation for the extraction of the square root, find a value of x which will make $x^4+6x^3+11x^2+3x+31$ a perfect square number.

4. Shew that $x^4+2ax^3+b^2x^2+c^3x+d^4$ is a perfect square if

$$x=\frac{(a^2-b^2)^2-4d^4}{4a(a^2-b^2)+c^3}.$$

5. Find the values of a, b, and c which will make the expression

$x^6-8x^5+ax^4+bx^3+cx^2-44x+4$ a perfect square.

6. If $x^4+px^3+qx^2+rx+s$ be a perfect square, then shall $r^2=p^2s$,

$p^2+8s^{\frac{1}{2}}=4q$, and $p^3+8r=4pq$.

7. $ax^2+2bxy+cy^2+2dx+2ey+f$ is a perfect square if $b^2=ac$, $d^2=af$, $e^2=cf$.

8. If $ax^4+bx^3+cx^2$ be subtracted from $(x^2+2x+4)^2$, and the remainder be a perfect square; determine the values of a, b, and c.

V.

1. Find the number of which $1{\cdot}4142\ldots$ is the approximate square root.

2. Find the least number which when multiplied into 850 will make the result a perfect square.

3. Find the sum, difference, product, and quotient of $\sqrt{\cdot 9}+\sqrt{\cdot 4}$ and $\sqrt{\cdot 9}-\sqrt{\cdot 4}$.

4. Shew that the integral part of the square root of 111111 is 333 with a remainder 222.

5. Shew that the square root of 3 differs from $1\frac{3}{4}$ by less than $\frac{1}{50}$.

6. The difference between two square integral numbers is equal to the sum of their square roots, together with twice the sum of all the intermediate numbers between those square roots.

7. Shew whether the square root of any integer which is not a perfect square number can be expressed by means of an integer and a rational fraction.

8. Explain why no integer or terminating decimal can have a recurring decimal for its square root. In what case can a recurring decimal have such a root? Give an example.

9. In extracting the square root of any number, if more than half of the required number of digits be obtained by extraction, the remaining digits may be found by division: and determine the limit which the remainder cannot exceed after any step of the process.

VI.

1. Shew that every square number ends in 1, 4, 5, 6, 9 or 00, and that the fourth powers of all numbers prime to 2 and 5 end in unity.

2. The square of 10129 is 102596641; find the square of 101293 without going through the operation of the multiplication.

3. Write an algebraical expression by means of which it may be shewn how to find a series of square numbers, each of which shall be the sum of two square numbers.

4. If the difference of any two numbers be unity, the difference of their squares is equal to the sum of the two numbers.

5. The difference of the squares of a number composed of two digits and the number formed by reversing these digits, is equal to 99 times the difference of the squares of the digits.

6. Shew that the square of every odd number diminished by unity is divisible by 8, as also the difference of the squares of any two odd numbers.

7. In any square number 4 is the only digit which can occupy both the place of units and of tens.

8. Shew that any odd square number when divided by 4 will leave unity for the remainder. Is this property true of any other than square numbers?

9. Prove that if any number which is a perfect square be divided by 3 it can never leave 2 for a remainder.

10. If the last digit but one in a perfect square be an odd number, the last digit must be 6.

11. Every square number is divisible by 5, or becomes so when increased or diminished by 1.

12. If a number end in 5, shew that the last two digits of the square of the number are 25, and the number composed of the remaining digits is equal to the product of two consecutive numbers.

13. A number consisting of four digits of which the two middle ones are zeros; prove that the difference of the squares of the number and of the number formed by reversing the digits is equal to 999999 times the difference of the squares of the extreme digits.

14. If two numbers are equidistant from 25, the square of the greater exceeds the square of the less by as many hundreds as the number itself exceeds 25; also if two numbers be equidistant from 50, the square of the greater exceeds that of the less by twice as many hundreds as the number itself exceeds 50. Shew how to apply these properties in readily finding the squares of all numbers from 13 to 100.

VII.

1. Shew that the product of the sum of any two square numbers by the sum of two other square numbers can always be expressed by the sum of two square numbers (*Diophantus*).

2. If the product of any four consecutive natural numbers be increased by unity, and the product of any four consecutive odd numbers be increased by 16, the sums are square numbers.

3. If any three consecutive whole numbers be taken, prove that the sum of the squares of the greatest and least is greater by 2 than twice the square of the middle number.

4. If there be three numbers, one of which is the sum of the other two; twice the sum of their fourth powers is a square number.

5. Of any six consecutive numbers, the first being odd, the difference between the sum of the squares of the even and odd numbers is equal to three times the sum of the third and fourth numbers.

6. Shew that the prodnct of six consecutive numbers cannot be a complete square.

7. If a, b, c be integers and $a^2+b^2=c^2$, then abc is always divisible by 60.

8. In how many ways can $(v^2-a^2)(x^2-b^2)(y^2-c^2)(z^2-d^2)$ be expressed in the form of the difference of two squares?

VIII.

1. If the diagonal of a square be 30 feet, what is the length of a side, and if 30 feet be the length of a side, what is the length of the diagonal?

2. If the adjacent sides of a rectangular parallelogram be 970·25 and 250·4 yards, what is the side of the equivalent square?

3. If the perimeters of a square and of a rectangle be 16 feet; how much longer is the diagonal of the rectangle than that of the square, supposing the longer side of the rectangle to be one foot more than a side of the square?

4. Determine the area and the side of that square which shall be equal to that of the sum of the areas of three squares whose sides are 3, 4, 5 feet respectively.

5. One side of a right-angled triangle is 4 inches, and the other side is as much less than 4 inches as the hypotenuse exceeds 4 inches; find the hypotenuse and the other side.

6. A side of a square is found to be 10 feet, and if the length made by error of measurement be one-tenth of an inch too much or too little; within what limits can the area of the square be considered known?

7. If unity represent a line of any given length, by help of the forty-seventh proposition of the first book of Euclid represent by lines the square roots of the first ten natural numbers.

8. Shew that if a^2+1 denote the hypothenuse and a^2-1 the base, then $2a$ will denote the perpendicular of a right-angled triangle; and hence may be found a series of right-angled triangles whose sides are integers or rational fractions.

9. The adjacent sides of a rectangular parallelogram are respectively equal to the hypotenuses of two right-angled triangles, whose sides are commensurable with the unit of linear measurement; prove that its area will be commensurable or incommensurable with the corresponding unit of square measurement according as its sides are or are not commensurable with each other.

IX.

1. The area of a rectangular field whose length is three times its breadth, is 6 acres 960 yards; find its perimeter and the diagonal.

2. The length of a room is twice its breadth, and the area is 1,152 square feet; what is its length?

3. If a square garden with its pathway contain an acre of land, what would be the width of the pathway if it be made (1) half-way

round, (2) completely round the garden, supposing the pathway occupies one eighth of the space?

4. Two persons start from the place where two roads meet at right angles to each other, one walks 3 miles an hour and the other 4; how far distant are they from each other 2 hours and 45 minutes after they started?

5. Find the length of a footpath which crosses diagonally a rectangular field 143 yards long and 116 broad.

X.

Determine the cube roots of the following expressions—

1. a^3-3a^2+3a-1. 2. $x^3+x^{-3}+3(x+x^{-1})$.
3. $a^6-3a^5+5a^3-3a-1$. 4. $(3a-a^3)^2+(1-3a^2)^2$.
5. $a^3x^3-b^3y^3-3abxy(ax-by)$. 6. $8a^6-36a^4b^2+54a^2b^4-27b^6$.
7. $x^6-6a^5+15a^4-20a^3+15a^2-6a+1$.
8. $27a^6-108a^5x+171a^4x^2-136a^3x^3+57a^2x^4-12ax^5+x^6$.
9. $8x^6-36x^5+66x^4-63x^3+33x^2-9x+1$. 10. $x^6-\frac{8}{x^3}+12-6x^3$.
11. $\frac{a^3}{8}-\frac{8}{27a^6}+\frac{2}{3a^3}-\frac{1}{2}$. 12. $\frac{a^3}{b^3}-\frac{b^3}{a^3}-3\left\{\frac{a^2}{b^2}+\frac{b^2}{a^4}\right\}+5$.

XI.

Find the cube roots of the following numbers—

1. 262144, 531441, 1953125, 3048625, 4492125, 5177717.
2. 15625, 140608, 1677216, 277167808, 16915218263, 448048351808.
3. ·064, ·000064, ·009261, ·000405224, ·000830584, ·000027270901.
4. 1·01, 9·6, ·4, ·04, ·004, 21·1, 3·43, 8·88, 9·04, $1\cdot\dot{9}1\dot{2}$, $10\cdot\dot{0}0\dot{1}$.
5. 1·030301, 884·736, 9393·931, 40·353607, 700·227072, 738763·264
6. 2, 7, 10, 100, 3·14159, each to three places of decimals.
7. $\frac{1}{2}$, $\frac{7}{8}$, $\frac{4}{7}$, $3\frac{3}{8}$, $3\frac{1}{3}$, $1000\frac{1}{2}$, $2\frac{10}{27}$, $\frac{2}{3}$.

XII.

1. The second differences of the cubes of the natural numbers form a series of numbers whose differences are a constant number. What is that number?

2. The cube root of every number greater than 3 is greater than the fourth root of the number increased by unity.

3. Prove that no number, the sum of whose digits is 6, can be either a perfect square, a perfect cube, or the difference between two perfect cubes.

4. The cube of every odd number greater than unity can be expressed in two different ways by the difference between two numbers which are perfect squares.

5. Shew that the cube root of an exact cube containing four, five or six figures can be determined by inspection, by taking the cube root of the first figure, or of the first two or three figures, and observing with what two figures the number ends.

6. Shew that the cube root of a number not a complete cube cannot be represented by any rational fraction, and if it be reduced to a decimal the latter cannot have a recurring period.

7. When will the cube root of a recurring decimal be a recurring decimal? Give examples.

8. Find the fourth root of ·00028561, the sixth root of ·000004826809, and the ninth root of ·000000010604499373.

XIII.

1. Every integral number consisting of n digits has $3n$, $3n-1$, or $3n-2$ digits in its cube.

2. The difference of the cube of the sum of any three numbers, and the sum of the cubes of the numbers, is always exactly divisible by the sum of any two of the numbers.

3. If two numbers differ by a unit, their product, together with the sum of their squares, is equal to the difference of the cubes of the two numbers.

4. Shew that any cube number when divided by 4 or by 7 cannot leave 2 for a remainder.

5. Shew that the product of any three consecutive numbers increased by the second number is always a perfect cube number.

6. The sum of the cubes of two integral numbers increased by unity is greater than three times their product.

7. Decompose 15750 into its prime factors, and find the number into which it must be multiplied to make it a perfect cube.

8. Determine an algebraical expression by which a series of cube numbers can be found, each of which shall be the sum of three cube numbers.

XIV.

1. If the edge of a cube be four lineal inches, what is the number of square inches in its surface, and the number of cubic inches in its volume?

2. If the volume of a cube contain 1,000 cubic inches, find its surface, the length of one of its edges, the diagonal of one of its faces, and the diagonal of the cube itself.

3. If the edge of a cube be 12·75 inches, what is the length of the edge of another cube which is twice the magnitude of the first?

4. The diagonal of a cube is one foot longer than each of the edges; what is the content of the cube in inches?

5. What is the volume of a cube whose diagonal is 12 lineal inches? and what is the diagonal of a cube whose volume is 12 cubic inches?

6. If the surface of a cube be 50 square inches, what is its volume? and if the volume be 50 cubic inches, what is the surface?

7. The three conterminous edges of a rectangular parallelopiped are 36, 75, and 80 inches respectively; find the side of a cube which shall be of the same capacity.

8. What is the content of that cube, the surface of which is 900 square feet 54 inches?

9. How many cubes having the area of one side $2\frac{1}{4}$ square inches, are equivalent to that cube, an edge of which is 36 inches?

10. Having given a cube whose side is 100 inches; find (within the tenth of an inch) the side of a cube whose magnitude is double of the former cube. Why cannot the problem be solved exactly?

11. Find the length of the edge of a cube whose volume shall be equal to three cubes whose surfaces are respectively 2,400, 5,400, and 15,000 square inches.

12. If the lengths of the edges of three cubes be a, b, $a+b$; shew that $\frac{(a+b)^3}{(a+b)^3-(a^3+b^3)}=\frac{1}{3}\left\{\sqrt{\frac{a}{b}}+\sqrt{\frac{b}{a}}\right\}^2$.

XV.

1. What must be the dimensions of a chest whose length is double its breath, and the breadth double its depth, that its content may be a million cubic inches?

2. The content of a cube is 1,012,199,273,930,125 cubic feet; how many square yards of canvas would be required to cover it?

3. The internal surface of the bottom of a box one foot deep, contains eight square feet; find the edge of a cubical box of the same content.

4. The breadth of a room is twice its height, and half its length; and the content is 4,096 cubic feet; find the dimensions of the room.

5. The area of floor of a room is 768 square feet, that of each of the two opposite walls 576 square feet, and that of each of the other two walls 432 feet; find the height and content of the room.

6. What must be the length of the side of a square cistern four feet deep, in order that it may contain 5,000 gallons of water, supposing a gallon to contain 272·274 cubic inches?

7. If a cubic foot of water weigh 1,000 ounces; find the dimensions of a cubical tank what shall contain a ton of water.

8. If the external edge of a cubical chest be a inches, and the chest be composed of boards m inches thick; find the difference between the

external and internal surfaces of the chest, and the content of the boards of which it is composed.

9. Find the length of the edge of a cubical block of coal which weighs 17 tons 12 cwt., if a cubic foot of coal weigh 77 pounds.

10. The cost of a cubic mass of metal is £8,584 18s. 7d. at £4 3s. 4d. a cubic inch; what is the area of one face of the mass?

11. A hollow cubical box made of material $1\frac{1}{3}$ inches thick has an interior capacity of 50·653 cubic feet; determine the length of the outside edge of the box.

12. A cubical box a inches deep is filled with layers of spherical balls, each one inch in diameter; find the number contained in the box, and what portion of space in the box is vacant. Find also what portions of the space would be vacant, (1) when each of the balls is half an inch, and (2) when one-third of an inch in diameter.

XVI.

Reduce the following expressions to their simplest forms:—

1. $a^{m+n}\times a^{m-n}$. 2. $a^{m+n}\div a^{m-n}$. 3. $a^{m-n}\div a^{m+n}$.
4. $a^{m-n}\times a^{n-p}\times a^{p-m}$. 5. $a^{m-n}\times a^{n-p}\div a^{p-q}$. 6. $a^{2m}\times a^{-3m}\times a^{m}$.
7. $a^{3m}\div a^{2m-n}$. 8. $a^{m-3}\div a^{n-m}$. 9. $(\pm a)^{2m}\div(\pm a)^{-3m}$.
10. $(a^{-m})^{n}$. 11. $(a^{-m})^{-n}$. 12. $(a^{0})^{-n}$. 13. $(a^{-m})^{\frac{1}{n}}$.
14. $(a^{m})^{\frac{-1}{n}}$. 15. $(a^{-m})^{\frac{-1}{n}}$. 16. $(a^{m}b^{n})^{\frac{1}{m}}\times(a^{n}b^{m})^{\frac{1}{n}}$.
17. $(a^{-m}b^{n})^{\frac{-1}{m}}\times(a^{m}b^{-n})^{\frac{-1}{n}}$. 18. $(a^{m}b^{-n})^{\frac{-1}{m}}\times(a^{-m}b^{n})^{\frac{1}{n}}$.
19. $\dfrac{\{(ab)^{m}\}^{n}}{(a^{n}b^{m})^{m}}\div\dfrac{a^{m-1}b^{2n}}{a^{2m}b^{n-1}}$. 20. $\dfrac{\{(ab)^{-m}\}^{n}}{(a^{n}b^{m})^{-m}}$. 21. $\dfrac{\{(ab)^{m}\}^{\frac{-1}{n}}}{(a^{-n}b^{-m})^{\frac{1}{m}}}$.
22. $\{(a^{m})^{-n}\}^{p}\times\{(a^{-2m})^{3n}\}^{-2p}$. 23. $\left\{a.a^{\frac{-(m+n)}{n}}\right\}^{\frac{n}{m+n}}\times(a^{m+n}a^{m})^{\frac{1}{m}}$.
24. $(a^{-m}a^{m+n})^{\frac{-1}{n}}\div(a^{-n}a^{m-n})^{\frac{-1}{m}}$.

XVII.

Verify the correctness of the following equivalents:—

1. $\{-a^{5}.(-b)^{4}.(-c)^{-3}\}^{5}=\left\{\dfrac{a^{5}b^{4}}{c^{3}}\right\}^{5}$.
2. $\{x^{3}(-y^{5})^{\frac{1}{3}}\div-y^{3}(-x^{3})^{\frac{1}{5}}\}^{6}=-x^{8}y^{5}$.
3. $\{a^{4}b^{2}c(a^{3}b^{2}c)^{\frac{1}{2}}\}^{\frac{1}{3}}=a^{\frac{11}{6}}b^{1}c^{\frac{1}{2}}$. 4. $\{(ab^{2})^{\frac{1}{2}}.(a^{3}b^{3})^{\frac{1}{3}}.(a^{2}b^{4})^{\frac{1}{4}}\}^{\frac{1}{12}}=a^{\frac{1}{6}}b^{\frac{1}{4}}$.
5. $\{(ab^{2}).(ab^{3})^{\frac{1}{2}}.(ab^{4})^{\frac{1}{3}}.(ab^{5})^{\frac{1}{4}}\}^{\frac{1}{2}}=(a^{49}.b^{73})^{\frac{1}{24}}$.
6. $\{(ab^{2})^{\frac{1}{3}}.(a^{2}b)^{\frac{1}{3}}.(a^{5}b)^{\frac{1}{4}}.(a^{7}b^{10})^{\frac{1}{6}}\}=(a^{27}.b^{17})^{\frac{1}{6}}$.
7. $\left\{\dfrac{ay}{x}\right\}^{\frac{1}{2}}.\left\{\dfrac{bx}{y^{2}}\right\}^{\frac{1}{3}}.\left\{\dfrac{y^{3}}{b^{2}a^{3}}\right\}^{\frac{1}{6}}=\dfrac{y^{\frac{1}{3}}}{x^{\frac{1}{6}}}$.
8. $\left\{\dfrac{3^{9}.a^{-6}.(a+b)^{3}.(a+x)^{-6}}{27.a^{6}x^{-3}}\right\}^{\frac{1}{3}}=\dfrac{9(a+b)x}{a^{4}(a+x)^{2}}$.

XVIII.

Find the following products:—

1. $(a+b^{\frac12})^2$. 2. $(a^{\frac12}-b^{\frac12})^2$. 3. $(a^{\frac12}+b^{\frac12})(a^{\frac12}-b^{\frac12})$.
4. $(a^{\frac12}+a^{\frac14}b^{\frac14}+b^{\frac12})(a^{\frac12}-a^{\frac14}b^{\frac14}+b^{\frac12})$. 5. $(a^{\frac13}+b^{\frac13})^3$. 6. $(a^{\frac13}-b^{\frac13})^3$.
7. $(a^{\frac13}+b^{\frac13})(a^{\frac23}-a^{\frac13}b^{\frac13}+b^{\frac23})$. 8. $(a^{\frac13}-b^{\frac13})(a^{\frac23}+a^{\frac13}b^{\frac13}+b^{\frac23})$.
9. $(a^{\frac23}+a^{\frac13}b^{\frac13}+b^{\frac23})(a^{\frac23}-a^{\frac13}b^{\frac13}+b^{\frac23})$. 10. $(x^{\frac23}-2x^{\frac13}+3)(x^{\frac23}+2x^{\frac13}+3)$.
11. $(a^{\frac52}+a^2b^{\frac12}+a^{\frac32}b+ab+a^{\frac12}b^2+b^{\frac52})(a^{\frac12}-b^{\frac12})$.
12. $(x+2y^{\frac12}+3z^{\frac13})(x-2y^{\frac12}+3z^{\frac13})$. 13. $(a^{\frac23}+3a^{\frac13}b^{\frac13}+9b^{\frac23})(a^{\frac13}-3b^{\frac13})$.
14. $(x^{\frac32}-3x^{\frac12}+6-3x^{-\frac12}+x^{-\frac32})^2$.

XIX.

Determine the following quotients, and verify the truth of them:—

1. $a-b$ by $a^{\frac12}+b^{\frac12}$. 2. a^3-b^3 by $a^{\frac32}-b^{\frac32}$. 3. a^2+b^2 by $a+2^{\frac12}a^{\frac12}b^{\frac12}+b$.
4. $a^{\frac34}+b^{\frac34}$ by $a^{\frac14}+b^{\frac14}$. 5. $a^{\frac43}-b^{\frac43}$ by $a^{\frac13}+b^{\frac13}$. 6. $3a^2-4a^{\frac32}b^{\frac12}+b^2$ by $(a^{\frac12}-b^{\frac12})^2$.
7. $x^{\frac32}-3^{\frac12}x+9x^{\frac12}-3^{\frac52}$ by $x-3$. 8. $x^{\frac23}+y^{\frac23}-z^{\frac23}-2x^{\frac13}y^{\frac13}$ by $x^{\frac13}-y^{\frac13}+z^{\frac13}$.
9. $x^{\frac52}-x^2y^{\frac12}+x^{\frac32}y-xy+x^{\frac12}y^2-y^{\frac52}$ by x^2+xy+y^2.
10. a^2-ab+b^2 by $a-3^{\frac12}a^{\frac12}b^{\frac12}+b$. 11. $x^{\frac43}-4x^{\frac23}+12x^{\frac13}-9$ by $x^{\frac23}-2x^{\frac13}+3$.
12. x^4+1 by $x^2-2^{\frac12}x+1$. 13. $2x^3-6x+5$ by $2^{\frac13}x+2^{\frac23}+1$.
14. $x^{\frac72}-y^{\frac72}$ by $x^{\frac12}-y^{\frac12}$.

Divide x^5-y^5 by $x-y$, and $x-y$ by $x^{\frac15}-y^{\frac15}$, and from the results infer the quotient of x^5-y^5 divided by $x^{\frac15}-y^{\frac15}$.

XX.

Verify the following equivalent expressions:—

1. $(x-2^{\frac12}x^{\frac12}+1)(x+2^{\frac12}x^{\frac12}+1)=x^2+1$.
2. $(a^2+2^{\frac12}ax+x^2)(a^2-2^{\frac12}ax+x^2)=a^4+x^4$.
3. $(x^2+3^{\frac12}x+1)(x^2-3^{\frac12}x+1)(x-2^{\frac12}x^{\frac12}+1)(x+2^{\frac12}x^{\frac12}+1)=x^6+1$.
4. $(x^2+2^{\frac12}x+1)(x^2-2^{\frac12}x+1)(x^2-2^{\frac12}x-1)(x^2+2^{\frac12}x-1)$
$=x^8-4x^6+2x^4-4x^2+1$.
5. $(x-1)\{x+(1-\surd 3)\}.\{x+(1-\surd 3)\}=x^3+x^2-4x+2$.
6. $\{x^2+\frac12(1+\surd 5)x+1\}.\{x^2+\frac12(1-\surd 5)+1\}=x^4+3x^2+1$.
7. $(x+\surd 2-\surd 3)(x-\surd 2-\surd 3)(x+\surd 2+\surd 3)(x-\surd 2+\surd 3)$
$=x^4-16x^2+1$.

XXI.

Extract the square roots of

1. $x^2-2x^{\frac32}+3x-2x^{\frac12}+1$. 2. $x^4+x^3-2x^{\frac52}-2x^{\frac32}+2x^2+1$.
3. $(a-b)^2-4(a-b)a^{\frac12}b^{\frac12}$. 4. $\frac{a}{b}+\frac{b}{a}-\frac{2a^{\frac12}}{b^{\frac12}}+\frac{2b^{\frac12}}{a^{\frac12}}-1$.
5. $\frac{x^3}{16}-\frac{x^2}{6}-\frac{x^{\frac32}}{4}+\frac{x}{9}+\frac{x^{\frac12}}{3}+\frac{1}{4}$. 6. $1-x^{\frac12}+\frac{x}{4}+\frac{2ax^{\frac32}}{3}-\frac{ax^2}{3}+\frac{a^2x^3}{9}$.
7. $(x-1)(x+2^{\frac12}x^{\frac12}+1)+2$. 8. $\frac{x}{y}+\frac{5}{2}+\frac{y}{x}-\frac{2^{\frac12}(x+y)}{x^{\frac12}y^{\frac12}}$.
9. $(2a^{\frac12}c^{\frac12}+b^2)x^2+2a^{\frac12}bx^{\frac52}+ax^3+2bc^{\frac12}x^{\frac32}+c^2x$.
10. $1+x$ to four terms. 11. $1+x+x^2$ to five terms.
12. a^2+ab+b^2 to four terms.

XXII.

Extract the cube roots of

1. $a^{3}-3a^{2}b^{\frac{1}{2}}+3ab-b^{\frac{3}{2}}$.
2. $x^{3}+3x^{\frac{5}{2}}-5x^{\frac{3}{2}}+3x^{\frac{1}{2}}-1$.
3. $x^{3}+15x^{2}+6x^{\frac{5}{2}}+20x^{\frac{3}{2}}+6x^{\frac{1}{2}}+15x+1$.
4. $(1-x^{3})-24x(1+x)-9x^{\frac{1}{2}}(1+x+x^{2})$.
5. $a^{3}+x^{3}$, to three terms.
6. $(1+2x^{\frac{1}{2}})(1+16x^{2}-8x)(1-2x^{\frac{1}{2}})$.

XXIII.

Verify the truth of the following identical expressions:—

1. $(x^{\frac{3}{2}}-1+x^{-\frac{3}{2}})(x^{\frac{3}{2}}+1+x^{-\frac{3}{2}})=x^{3}+x^{-3}+1$.
2. $(2x+y^{-1})(2y+x^{-1})=(2x^{\frac{1}{2}}y^{\frac{1}{2}}+x^{-\frac{1}{2}}y^{-\frac{1}{2}})^{2}$.
3. $(x^{\frac{1}{2}}-3y^{\frac{1}{2}})(x^{\frac{1}{2}}+y^{\frac{1}{2}})(x^{\frac{1}{2}}+2y^{\frac{1}{2}})(x+3y^{\frac{1}{2}})(x^{\frac{1}{2}}-y^{\frac{1}{2}})(x^{\frac{1}{2}}-2y^{\frac{1}{2}})$
$=x^{3}-14x^{2}y+49xy^{2}-36y^{3}$.
4. $\{(1+x)(1-x)^{\frac{1}{2}}-(1-x)(1+x)^{\frac{1}{2}}\}^{2}-\{(1-x)^{\frac{1}{2}}-1\}^{2}$
$=x^{2}\{2(1-x^{2})^{\frac{1}{2}}-1\}$.
5. $\{xy+2x(xy-x^{2})^{\frac{1}{2}}\}^{\frac{1}{2}}+\{xy-2x(xy-x^{2})^{\frac{1}{2}}\}=(2x^{2}-xy)^{\frac{1}{2}}$.
6. $\{2x^{2}+2(x^{4}-y^{4})^{\frac{1}{2}}\}^{\frac{1}{2}}-\{2x^{2}-2(x^{4}-y^{4})^{\frac{1}{2}}\}^{\frac{1}{2}}=2(x^{2}-y^{2})^{\frac{1}{2}}$.
7. $(a^{\frac{1}{2}}+b^{\frac{1}{2}}-c^{\frac{1}{2}})(a^{\frac{1}{2}}+b^{\frac{1}{2}}+c^{\frac{1}{2}})+(a^{\frac{1}{2}}-b^{\frac{1}{2}}-c^{\frac{1}{2}})(a^{\frac{1}{2}}-b^{\frac{1}{2}}+c^{\frac{1}{2}})=2(a+b+c)$.
8. $\{(ab)^{\frac{1}{2}}+(ac)^{\frac{1}{2}}+(bd)^{\frac{1}{2}}+(cd)^{\frac{1}{2}}\}.\{(ab)^{\frac{1}{2}}-(ac)^{\frac{1}{2}}-(bd)^{\frac{1}{2}}+(cd)^{\frac{1}{2}}\}$
$=(a-d)(b-c)$.
9. $(a^{6}+a^{4}b^{2})^{\frac{1}{2}}+(b^{6}+b^{4}a^{2})^{\frac{1}{2}}=(a^{2}+b^{2})^{\frac{3}{2}}$.
10. $(a^{4}b+a^{2}bc)^{\frac{1}{2}}+(ab^{4}+b^{4}c)^{\frac{1}{2}}=(a+b)(a+c)^{\frac{1}{2}}b^{\frac{1}{2}}$.
11. $(a^{2}+a^{\frac{4}{3}}b^{\frac{2}{3}})^{\frac{1}{2}}+(b^{2}+b^{\frac{4}{3}}a^{\frac{2}{3}})^{\frac{1}{2}}=(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{3}{2}}$.
12. $\{(a+b)^{\frac{2}{3}}+(a-b)^{\frac{2}{3}}\}(a+b)^{\frac{4}{3}}-\{(a+b)^{\frac{4}{3}}-(a-b)^{\frac{4}{3}}\}(a-b)^{\frac{2}{3}}=2(a^{2}+b^{2})$.

XXIV.

Find the values of the following expressions:—

1. $\{(x^{2}+y^{2})^{\frac{1}{2}}+z\}.\{(x^{2}+y^{2})^{\frac{1}{2}}-z\}$, when $x=4$, $y=5$, $z=6$.
2. $(1-x)^{\frac{1}{2}}+\{1-x+(1+x)^{\frac{1}{2}}\}^{\frac{1}{2}}$, when $x=\dfrac{24}{25}$.
3. $x^{3}-2x^{2}+x+18$, when $x=2+\sqrt{-5}$.
4. $\{x^{4}+2x^{3}+3x^{2}+2x+5\}^{\frac{1}{2}}$, when $x^{2}+x+1=0$.
5. $x+y+(x^{2}-xy+y^{2})^{\frac{1}{2}}$, when $x=\sqrt{2}+1$ and $y=\sqrt{2}-1$.
6. $\dfrac{a+(a^{2}+b^{2})^{\frac{1}{2}}}{a^{3}-2b(a^{2}-b^{2})}$, when $a=-4$ and $b=-3$.
7. $\dfrac{1+x}{1+(1+x)^{\frac{1}{2}}}+\dfrac{1-x}{1+(1-x)^{\frac{1}{2}}}$, when $x=\dfrac{\sqrt{3}}{2}$.
8. $\dfrac{x-y}{x+y}+\dfrac{x+y}{x-y}$, when $x=2+\sqrt{3}$ and $y=2-\sqrt{3}$.
9. $\dfrac{x^{2}+xy+y^{2}}{x^{2}-xy+y^{2}}$, when $x=\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$ and $y=\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$.
10. $\dfrac{(x^{2}+a^{2})^{\frac{1}{2}}+(x^{2}-a^{2})^{\frac{1}{2}}}{(x^{2}+a^{2})^{\frac{1}{2}}-(x^{2}-a^{2})^{\frac{1}{2}}}$, when $x=\frac{1}{2}(a^{2}+1)^{\frac{1}{2}}$.

11. $\dfrac{\{(a^2-x^2)(a+x)\}^{\frac12}-\{(a^3+x^3)(a-x)\}^{\frac13}}{\{(a+x)^2(a-x)\}^{\frac13}-\{(a-x)^2(a+x)\}^{\frac13}}$, when $x=a$.

XXV.

Reduce the following expressions to their simplest forms :—

1. $\dfrac{1+x^{\frac12}-x^{\frac32}-x^2}{x^{\frac32}+6x^2+5x^{\frac52}}$. 2. $\dfrac{x^{\frac32}+2x^{\frac12}-x^{-\frac12}}{x^{\frac52}+2x^{\frac32}-x^{-\frac12}}$.

3. $\dfrac{42x^{\frac32}-43y^{\frac12}x^2+36yx^{\frac32}-5xy^{\frac32}}{36x^{\frac32}-48xy^{\frac12}+37x^{\frac12}y^2-5y^{\frac32}}$.

4. $\dfrac{x^2-2x-1}{x^3-2x^2-6x-3-2x(x+1)(x+1)^{\frac12}}$.

5. $\dfrac{x^3-3x+(x^2-1)(x^2-4)^{\frac12}-2}{x^3-3x+(x^2-1)(x^2-4)^{\frac12}+2}$.

6. $\dfrac{2(a^6+2b)-7a^{\frac32}b^{\frac12}(a^3+2b^{\frac12})+15a^3b^{\frac12}}{2(a^6-b)+5a^{\frac32}b^{\frac12}(a^3+2b^{\frac12})-15a^3b^{\frac12}}$. 7. $\dfrac{x-2x^{\frac34}y^{\frac14}+2x^{\frac14}y^{\frac34}-y}{x-2x^{\frac12}y^{\frac12}+y}$.

8. $\dfrac{(a+x)^{\frac12}+(a-x)^{\frac12}}{(a+x)^{\frac12}-(a-x)^{\frac12}}$. 9. $\dfrac{(a^{2n+1}+a^{2n}+1}{(a^{2n}-a^{n-1}+1}$.

XXVI.

Verify the correctness of the following equivalents :—

1. $xy\left\{\dfrac{x^2+y^2}{x^2}+\dfrac{y^2+x^2}{y^2}\right\}^{\frac12}=x^2+y^2$.

2. $\dfrac{xy}{x+y}\cdot\left\{\dfrac{x+y}{x}+\dfrac{x-y}{y}\right\}^{\frac12}=\dfrac{x^{\frac12}y^{\frac12}(x^2+y^2)^{\frac12}}{x+y}$.

3. $\dfrac{ab}{b-c}\pm\left\{\dfrac{a^2b^2}{(b-c)^2}-\dfrac{a^2b}{b-c}\right\}^{\frac12}=\dfrac{ab^{\frac12}}{b^{\frac12}\mp c^{\frac12}}$.

4. $\left\{\dfrac{27a^4-18a^2b^2-b^4}{8ab^2}\right\}^2+\left\{\dfrac{(9a^2-b^2)^{\frac12}.(b^2-a^2)^{\frac32}}{8ab^2}\right\}^2=b^2$.

5. $\dfrac{(a^2+x^2)^{\frac12}+(a^2-x^2)^{\frac12}}{(a^2+x^2)^{\frac12}-(a^2-x^2)^{\frac12}}-\left\{\dfrac{a^4}{x^4}-1\right\}^{\frac12}=\dfrac{a^2}{x^2}$.

6. $\dfrac{(x+y)x^{\frac12}}{y^{\frac12}}-\dfrac{(x-y)y^{\frac12}}{x^{\frac12}}=\dfrac{(x+y)y^{\frac12}}{x^{\frac12}}+\dfrac{(x-y)x^{\frac12}}{y^{\frac12}}=\dfrac{x^{\frac32}}{y^{\frac12}}+\dfrac{y^{\frac32}}{x^{\frac12}}$.

7. $\dfrac{x^3}{(1+x)^{\frac52}}+\dfrac{x^2}{(1+x)^{\frac32}}+\dfrac{x}{(1+x)^{\frac12}}=\dfrac{x+3x^2+3x^3}{(1+x)^2}\cdot\dfrac{1}{(1+x)^{\frac12}}$.

8. $\dfrac{\sqrt2-1}{2\sqrt2}\cdot\dfrac{x+1}{x^2+2^{\frac12}x+1}+\dfrac{\sqrt2+1}{2\sqrt2}\cdot\dfrac{x+1}{x^2-2^{\frac12}x+1}=\dfrac{x^3+2x^2+2x+1}{x^4+1}$.

9. $\dfrac{\frac12(1+\sqrt5)x+2}{x^2-\frac12(1+\sqrt5)x+1}+\dfrac{\frac12(1-\sqrt5)x-2}{x^2-\frac12(1-\sqrt5)x+1}=\dfrac{x^3+2x^2+(1+2\sqrt5)x}{x^4-x^3+x^2-x+1}$.

10. $\dfrac{x+1}{x-1}\times\left\{\dfrac{x-1}{x+1}\right\}^{\frac12}\times\left\{\dfrac{x-1}{x+1}\right\}^{\frac13}=\left\{\dfrac{x+1}{x-1}\right\}^{\frac16}$.

11. $\left\{\dfrac{(b+x)^7}{(a^2-x^2)^7}\right\}^{\frac13}\times\left\{\dfrac{a+x}{b+x}\right\}^{\frac34}\times\dfrac{(a-x)^{\frac34}}{(b+x)^{\frac14}}=\dfrac{(b+x)^{\frac43}}{(a^2-x^2)^{\frac{19}{12}}}$.

12. $\left\{\left(\frac{1+x}{1-x}\right)^{\frac{3}{2}}-\left(\frac{1-x}{1+x}\right)^{\frac{3}{2}}\right\}\div\left(\frac{1+x}{1-x}\right)^{\frac{3}{2}}=\frac{3(1+x^2)x}{1+3x^2}.$

13. $\left\{ab-\left(\frac{1-a^2}{1+a^2}\cdot\frac{1-b^2}{1+b^2}\right)^{\frac{1}{2}}\right\}\div\left\{1+ab\left(\frac{1-a^2}{1+a^2}\cdot\frac{1-b^2}{1+b^2}\right)^{\frac{1}{2}}\right\}.$

14. $\{(a+b)^2-4ab\}^{\frac{1}{4}}.\left\{\frac{a^4-b^4}{a-b}+2ab(a+b)\right\}^{\frac{1}{6}}=(a^2-b^2)^{\frac{1}{2}}.$

15. $\{(a-b)^2+4ab\}^{\frac{1}{2}}.\{(a+b)^2-4ab\}^2.\left\{\frac{a^3-b^3}{a-b}-3ab\right\}^{-\frac{3}{2}}=a^2-b^2.$

16. $\frac{2-2x^2}{(1-x^2)^{\frac{5}{2}}}-\frac{1}{(1-x^2)^{\frac{1}{2}}}=\frac{1+x^2}{(1-x^2)^{\frac{3}{2}}}.$

17. $\frac{(1-x^2)^{\frac{3}{2}}(1+x)^2}{2(1+x)^{\frac{1}{2}}}\left\{\frac{(1-x)^{\frac{1}{2}}}{(1+x)^{\frac{1}{2}}}+\frac{(1+x)^{\frac{1}{2}}}{(1-x)^{\frac{1}{2}}}\right\}=(1-x^2)(1+x)^{\frac{3}{2}}.$

XXVII.

1. For what reason is the negative sign necessary for distinguishing *algebraical* from *arithmetical* calculation? and how does the algebraical representation of impossible quantities result necessarily from this use of the negative sign?

2. Is it always possible to express the sum of the squares of two algebraical expressions in the form of the product of a sum and difference?

3. Shew that in the addition, subtraction, multiplication, and division of quantities of the form $a+b\sqrt{-1}$, and also in the involution and evolution of such quantities, the results will always be of the form $A\pm B\sqrt{-1}$.

Express $\frac{2+3\sqrt{-1}}{4+5\sqrt{-1}}$ and $\frac{a+b\sqrt{-1}}{c+d\sqrt{-1}}$ in this form.

4. Explain why the introduction of imaginary expressions does not vitiate a process of algebraical reasoning.

5. Shew that $\{\sqrt{-1}\}^m$ will have four different values according to the forms of the number m, and exhibit the values of $(1+\sqrt{-3})^{\frac{1}{2}}$.

6. Shew that $(-a)^{\frac{1}{4m}}\times(-b)^{\frac{1}{4m}}$ is always an impossible expression, and $(-a)^{\frac{1}{4m+2}}\times(-b)^{\frac{1}{4m+2}}$ is always a possible expression, when m is any integral number.

7. Are the quantities $\sqrt[4]{-a}\times\sqrt[4]{-b}$, and $\sqrt[6]{-a}\times\sqrt[6]{-b}$, rational or irrational?

8. Shew the absurdity of the following reasoning:—

$$(-1)^{\frac{1}{3}}=(-1)^{\frac{2}{6}}=\{(-1)^2\}^{\frac{1}{6}}=(+1)^{\frac{1}{6}}=+1.$$

9. Point out the fallacy in the statement

$$\sqrt{(-1)}\times\sqrt{(-1)}=\sqrt{\{(-1)\times(-1)\}}=\sqrt{+1}=1.$$

10. Find the third power of $\frac{-1\pm\sqrt{-3}}{2}$, and of $\frac{+1\pm\sqrt{-3}}{2}$; what inferences may be deduced from the results of the operation? Shew

also that $\frac{-1+\sqrt{-3}}{2}$ is the receprocal of $\frac{-1-\sqrt{3}}{2}$, and $\frac{+1+\sqrt{-3}}{2}$ of $\frac{+1-\sqrt{-3}}{2}$.

11. Shew that any power of a cube root of unity, is itself a cube root of unity. Exemplify this property in the fourth and fifth powers.

12. Is the following argument legitimate? If $a^0=1$ and $b^0=1$; therefore $a^0=b^0$, and consequently $a=b$.

13. Establish the equalities, in which ω is an imaginary cube root of unity,

$$\begin{aligned}(y-z)^2(y+z-2x)+(z-x)^2(z+x-2y)+(x-y)^2(x+y-2z)\\ =-(y+z-2x)(z+x-2y)(x+y-2z)\\ =(x+\omega y+\omega^2 z)^3+(x+\omega^2 y+\omega z)^3.\end{aligned}$$

XXVIII.

Verify the correctness of the following expressions:—

1. $\sqrt{(-a^2)}.\sqrt{(-b^2)}=-ab$,
and $\sqrt{(-a^2)}.\sqrt{(-b^2)}.\sqrt{(-c^2)}=-abc\sqrt{-1}$.
2. $\sqrt[3]{(-a^3)}.\sqrt[3]{(-b^3)}=+ab$, and $\sqrt[3]{(-a^3)}.\sqrt[3]{(-b^3)}.\sqrt[3]{(-c^3)}=-abc$.
3. $\{-\sqrt{(-2\sqrt{-3})}\}^4=-12$.
4. $(a+b\sqrt{-1})(a-b\sqrt{-1})=(a+2^{\frac{1}{2}}a^{\frac{1}{2}}b^{\frac{1}{2}}+b)(a-2^{\frac{1}{2}}a^{\frac{1}{2}}b^{\frac{1}{2}}+b)$.
5. $(a^2+b^2\sqrt{-1})(a^2-b^2\sqrt{-1})=(a^2+2^{\frac{1}{2}}ab+b^2)(a^2-2^{\frac{1}{2}}ab+b^2)$.
6. $(a+b\sqrt{-1})(c+d\sqrt{-1})(c-d\sqrt{-1})(a-b\sqrt{-1})$
$=(ac+bd)^2+(ad-bc)^2$.
7. $(x-\sqrt{2}-\sqrt{-3})(x-\sqrt{2}+\sqrt{-3})(x+\sqrt{2}-\sqrt{-3})$
$(x+\sqrt{2}+\sqrt{-3})=x^4+2x^2+25$.
8. $(1+\sqrt{-1})^2\pm(1-\sqrt{-1})^2=0$ and $4\sqrt{-1}$.
9. $\sqrt{(16+30\sqrt{-1})}+\sqrt{(16-30\sqrt{-1})}=10$.
10. $\sqrt{\{a+b\sqrt{-1}\}}+\sqrt{\{a-b\sqrt{-1}\}}=\sqrt{\{2a+2\sqrt{(a^2+b^2)}\}}$.
11. $(a+b\sqrt{-1})^3\pm(a-b\sqrt{-1})^3=2a(a^2-3b^2)$, and $2b(3a^2-b^2)\sqrt{-1}$.

XXIX.

1. $\frac{a^2+b^2}{a-b\sqrt{-1}}=a+b\sqrt{-1}$.
2. $\frac{a^3-(1-\sqrt{-1})a^2b+(1-\sqrt{-1})ab^2+b^3\sqrt{-1}}{a+b\sqrt{-1}}=a^2-ab+b^2$.
3. $\frac{a-b\sqrt{-1}}{a+b\sqrt{-1}}+\frac{a+b\sqrt{-1}}{a-b\sqrt{-1}}=\frac{2(a^2-b^2)}{a^2+b^2}$.
4. $\frac{1+2\sqrt{-1}}{1+2\sqrt{-1}}-\frac{1+2\sqrt{-1}}{1-2\sqrt{-1}}=\pm\frac{8\sqrt{-1}}{5}$.
5. $\frac{1+(1-\sqrt{2})\sqrt{-1}}{1-(1-\sqrt{2})\sqrt{-1}}+\frac{1-(1-\sqrt{2})\sqrt{-1}}{1+(1+\sqrt{2})\sqrt{-1}}=\sqrt{2}$.

6. $$\frac{xy-x^2\sqrt{-1}}{xy-2x^2\sqrt{-1}}\times\frac{y^2-2xy\sqrt{-1}}{y^2+2xy\sqrt{-1}}\times\frac{2x-y\sqrt{-1}}{x-y\sqrt{-1}}\times\frac{2xy+(x^2-y^2)\sqrt{-1}}{\sqrt{-x^2-y^2}}=-(x^2y+{}^2)^{\frac{1}{2}}.$$

7. $$\left\{\frac{-1+\sqrt{-3}}{2}\right\}^3+\left\{\frac{-1-\sqrt{-3}}{2}\right\}^3=2.$$

8. $$\left\{\frac{-2+36\sqrt{-1}}{2+3\sqrt{-1}}\right\}^{\frac{1}{2}}=3+\sqrt{-1}.$$

XXX.

1. If $x=-1+2\sqrt{-1}$, find the numerical value of x^4-12x.
2. If $a+b\sqrt{-1}=(a-b\sqrt{-1})(x+y\sqrt{-1})$, what are x and y?
3. If $\alpha=\frac{1}{2}(-1+\sqrt{-3})$,
shew that $(a+\alpha b+\alpha^2c)(a+\alpha^2b+\alpha c)=a^2+b^2+c^2-ab-ac-bc$.
4. If $\alpha=\frac{1}{2}(-1+\sqrt{-3})$ and $\beta=\frac{1}{2}(-1-\sqrt{-3})$; shew that
$(x+y+z)(x+\alpha y+\beta z)(x+\beta y+\alpha z)=x^3+y^3+z^3-3xyz$.
5. If $\alpha=\dfrac{1+\sqrt{-1}}{\sqrt{2}}$, then $\alpha^6+\alpha^4+\alpha^2+1=0$.
6. If $\alpha=\frac{1}{2}(1+\sqrt{-3})$, and $\beta=\frac{1}{2}(1-\sqrt{-3})$, express
$(1+\alpha)^{\frac{3}{2}}+(1+\beta)^{\frac{3}{2}}$ in the simplest form.

XXXI.

Find the sums and differences of the following surd numbers:—

1. $\sqrt{27}$ and $\sqrt{48}$.
2. $14\sqrt{147}$ and $13\sqrt{75}$.
3. $\sqrt{180}$ and $\sqrt{405}$.
4. $\frac{7}{5}\sqrt{50}$ and $\frac{1}{7}\sqrt{72}$.
5. $3\sqrt{\frac{2}{5}}$ and $\frac{1}{2}\sqrt{\frac{1}{10}}$.
6. $9\sqrt{\frac{5}{27}}$ and $4\sqrt{\frac{3}{4}}$.
7. $\sqrt[3]{500}$ and $\sqrt[3]{108}$.
8. $\sqrt[3]{40}$ and $\sqrt[3]{135}$.
9. $2\sqrt[3]{81}$ and $7\sqrt[3]{3}$.
10. $3\sqrt[3]{\frac{1}{3}}$ and $\sqrt[3]{72}$.
11. $\frac{1}{7}\sqrt[3]{500}$ and $\frac{1}{5}\sqrt[3]{108}$.
12. $\frac{3}{4}\sqrt[3]{\frac{2}{3}}$ and $\frac{2}{5}\sqrt{\frac{9}{32}}$.
13. $\sqrt{12}$ and $\sqrt{27}$.
14. $\sqrt{12}$ and $\sqrt{20}$.

XXXII.

Determine the products and quotients of the following surds:—

1. $\sqrt{448}$ and $\sqrt{112}$.
2. $4\sqrt{8}$ and $3\sqrt{6}$.
3. $12\sqrt{200}$ and $13\sqrt{150}$.
4. $\frac{1}{17}\sqrt{108}$ and $\frac{1}{3}\sqrt{16}$.
5. $\sqrt{\frac{3}{5}}$ and $\sqrt{\frac{5}{27}}$.
6. $\frac{3}{4}\sqrt{\frac{2}{3}}$ and $\frac{2}{5}\sqrt{\frac{1}{6}}$.
7. $9\sqrt{8}$ and $3\sqrt{5}$.
8. $4\sqrt[3]{72}$ and $13\sqrt[3]{2}$.
9. $12\sqrt[3]{18}$ and $\sqrt[3]{520}$.
10. $\frac{1}{2}\sqrt[3]{20}$ and $\frac{1}{3}\sqrt[3]{5}$.
11. $\frac{1}{2}\sqrt[3]{\frac{2}{3}}$ and $\frac{3}{4}\sqrt[3]{\frac{5}{6}}$.
12. $\frac{3}{5}\sqrt[3]{\frac{1}{135}}$ and $\frac{2}{7}\sqrt[3]{\frac{1}{5}}$.
13. $\frac{1}{2}\sqrt[3]{6}$ and $\frac{2}{3}\sqrt[3]{18}$.
14. $\frac{5}{7}\sqrt[3]{\frac{2}{3}}$ and $\frac{2}{5}\sqrt[3]{\frac{3}{4}}$.
15. $3\sqrt[3]{\frac{1}{3}}$ and $4\sqrt[3]{72}$.
16. $3\sqrt{2}$ and $2\sqrt[3]{3}$.
17. $5\sqrt{3}$ and $\frac{3}{5}\sqrt[3]{18}$.
18. $\frac{1}{2}\sqrt[3]{6}$ and $\frac{2}{3}\sqrt{18}$.

XXXIII.

Find the second, third, fourth, and fifth powers of the following quantities:—

1. $\sqrt{3}$. 2. $5\sqrt{3}$. 3. $\sqrt{12}$.
4. $\frac{1}{2}\sqrt{5}$. 5. $\frac{2}{3}\sqrt{7}$. 6. $3\sqrt{8}$.
7. $3\sqrt[3]{3}$. 8. $\frac{1}{3}\sqrt[3]{6}$. 9. $5\sqrt[4]{5}$.
10. $2\pm\sqrt{3}$. 11. $\sqrt{2}\pm\sqrt{3}$. 12. $5\sqrt{2}+3\sqrt{3}$.

XXXIV.

Extract the square roots and the cube roots of the following expressions $\sqrt{2}$, $\sqrt{3}$, $5\sqrt{5}$, $5\sqrt{27}$, $\sqrt[3]{2}$, $\sqrt[3]{3}$, $25\sqrt{125}$; and the square roots of $12+6\sqrt{3}$, $2+\sqrt{3}$, $35-12\sqrt{6}$, $-5+12\sqrt{-1}$.

XXXV.

Reduce to their simplest form the following surd quantities:—

1. $3\sqrt{147}-3\sqrt{75}-3\sqrt{\frac{1}{3}}$.
2. $8\sqrt{\frac{3}{4}}-\frac{1}{2}\sqrt{12}+4\sqrt{27}-2\sqrt{\frac{3}{16}}$.
3. $(15+19\sqrt{2}-2\sqrt{3}-12\sqrt{6})\div(3+\sqrt{2}+2\sqrt{3})$.
4. $(\sqrt{2}+\sqrt{3})(\sqrt{3}+\sqrt{5})(\sqrt{5}+\sqrt{2})$.
5. $(9+2\sqrt{3}+2\sqrt{5}+2\sqrt{15})^{\frac{1}{2}}$.
6. $\sqrt{(2+\sqrt{3})}-\sqrt{(2-\sqrt{3})}$.
7. $\{3\sqrt{3}+2\sqrt{6}\}^{\frac{1}{2}}-\{3\sqrt{3}-2\sqrt{6}\}^{\frac{1}{2}}$.
8. $7\sqrt[3]{54}+9\sqrt[3]{250}+\sqrt[3]{2}+2\sqrt[3]{128}$.
9. $\sqrt[3]{192}-\sqrt[3]{81}-\sqrt[3]{16}+\sqrt[3]{128}$.
10. $\sqrt[3]{24}+\sqrt[3]{81}+\sqrt[3]{\frac{64}{9}}$.
11. $(\sqrt{5}+2)^{\frac{1}{3}}+(\sqrt{5}-2)^{\frac{1}{3}}$.
12. $\{(10\cdot4)^{2}+(14\cdot31)^{2}\}^{\frac{1}{2}}$.
13. $(96)^{\frac{1}{3}}\times(243)^{\frac{1}{4}}\times(75)^{\frac{1}{6}}\times(3)^{\frac{1}{10}}$.

XXXVI.

Verify the correctness of the following equivalents:—

1. $\dfrac{8-5\sqrt{2}}{3-2\sqrt{2}}=4+\sqrt{2}$. 2. $\dfrac{\sqrt{20}+\sqrt{12}}{\sqrt{5}-\sqrt{3}}=8+2\sqrt{15}$.
3. $\dfrac{6\sqrt{5}-5\sqrt{3}}{4\sqrt{5}-3\sqrt{3}}=5(9+2\sqrt{15})$.
4. $\dfrac{1+\sqrt{3}}{3\sqrt{3}-2\sqrt{2}}=1+\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{3}}+\dfrac{\sqrt{3}}{\sqrt{2}}$.
5. $\left\{\dfrac{\sqrt{2}+1}{\sqrt{2}-1}\right\}^{\frac{1}{2}}=\sqrt{2}+1$. 6. $\left\{\dfrac{2-\sqrt{3}}{2+\sqrt{3}}\right\}^{\frac{1}{4}}=\dfrac{\sqrt{2}}{1+\sqrt{3}}$.
7. $\dfrac{3\sqrt{8}+2\sqrt{7}}{\sqrt{8}+\sqrt{7}}=2\cdot51\ldots$ 8. $\dfrac{\sqrt{7}+2\sqrt{5}}{\sqrt{7}-\sqrt{5}}=\dfrac{35}{2}$ nearly.
9. $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}=10$ nearly. 10. $\dfrac{\frac{1}{2}\sqrt{\frac{1}{2}}}{\sqrt{2}+3\sqrt{\frac{1}{2}}}=\dfrac{1}{10}$.

11. $\dfrac{(\sqrt{5}+1)}{\sqrt{2}\sqrt{(5-\sqrt{5})}}=\sqrt{\{1+\frac{2}{5}\sqrt{5}\}}$.

12. $\dfrac{1}{\sqrt{2}+\sqrt{3}+\sqrt{5}}=\dfrac{3\sqrt{2}+2\sqrt{3}-\sqrt{30}}{12}$.

13. $\dfrac{1}{\sqrt{2}+\sqrt{3}-\sqrt{5}}=\dfrac{3\sqrt{2}+2\sqrt{3}+\sqrt{30}}{12}$.

14. $\dfrac{1+\sqrt{2}}{1-\sqrt{2}+\sqrt{3}}=\dfrac{\sqrt{2}+\sqrt{6}+\sqrt{12}}{4}$.

15. $\dfrac{2\sqrt{2}-2\sqrt{3}}{4+\sqrt{6}-\sqrt{2}}=(\sqrt{3}-\sqrt{2})(\sqrt{2}+1)$.

16. $\dfrac{(5{\cdot}12)^{\frac{1}{3}}+({\cdot}03375)^{\frac{1}{3}}}{(80)^{\frac{1}{3}}-({\cdot}01)^{\frac{1}{3}}}=\dfrac{1}{2}$.

XXXVII.

Verify the correctness of the following equivalents:—

1. $\dfrac{\sqrt{245}+\sqrt{75}}{\sqrt{5}-\sqrt{3}}+\dfrac{\sqrt{245}-\sqrt{75}}{\sqrt{5}+\sqrt{3}}=50$.

2. $\dfrac{6-2\sqrt{2}-\sqrt{(11-6\sqrt{2})}}{2\sqrt{(11+6\sqrt{2})}-3-\sqrt{2}}=\dfrac{11-6\sqrt{2}}{7}$.

3. $\dfrac{\sqrt{2}}{\sqrt{3}-\sqrt{2}}-\dfrac{\sqrt{3}}{\sqrt{3}+\sqrt{2}}=2\sqrt{6}-1$.

4. $\dfrac{2+\sqrt{3}}{\sqrt{2}+\sqrt{(2+\sqrt{3})}}+\dfrac{2-\sqrt{3}}{\sqrt{2}-\sqrt{(2-\sqrt{3})}}=\sqrt{2}$.

5. $\left\{\dfrac{2-\sqrt{3}}{2}\right\}^{-\frac{3}{2}}-\left\{\dfrac{2+\sqrt{3}}{2}\right\}^{\frac{3}{2}}=7.\left\{\dfrac{\sqrt{3}+1}{2}\right\}^{3}$.

6. $\dfrac{\sqrt{5}-1}{3}.\left\{\dfrac{3-\sqrt{5}}{2}\right\}^{\frac{3}{2}}+\dfrac{4}{15}.\left\{\dfrac{3-\sqrt{5}}{2}\right\}^{\frac{5}{2}}+\dfrac{1}{3}.\left\{\dfrac{\sqrt{5}-1}{2}\right\}^{3}=\dfrac{1}{5}$.

XXXVIII.

1. Arrange in order of magnitude $2^{\frac{1}{2}}$, $3^{\frac{1}{3}}$, $4^{\frac{1}{4}}$, $5^{\frac{1}{5}}$, $6^{\frac{1}{6}}$, without extracting the roots.

2. Find the values of $\dfrac{2}{\sqrt{3}}$, $\dfrac{2\sqrt{3}}{\sqrt{2}}$, $\left(\dfrac{3}{5}\right)^{\frac{1}{2}}$, $\left(\dfrac{3}{2}\right)^{\frac{2}{3}}$, $\left(\dfrac{2}{3}\right)^{\frac{3}{2}}$,

$8^{-\frac{2}{3}}$, $7^{\frac{1}{3}}$, $({\cdot}\dot{0}3\dot{7})^{\frac{2}{3}}$, each to four places of decimals.

3. Which is the greater in each pair of the following expressions?

$\sqrt[3]{4}$ or $\sqrt[4]{5}$: $2^{\frac{1}{2}}$ or $3^{\frac{1}{3}}$: $3\sqrt[3]{2}$ or $2\sqrt{3}$:

$(5^2)^{\frac{1}{3}}$ or $(5^3)^{\frac{1}{4}}$ or $5^{2^{\frac{1}{3}}}$: $\sqrt{7}$ or $2\sqrt[4]{3}$:

$\dfrac{1}{2}$ or $\left\{\dfrac{2}{3}\right\}^{\frac{3}{2}}$: $\dfrac{\sqrt[3]{3}}{\sqrt{2}}$ or $\dfrac{\sqrt{3}}{\sqrt[3]{5}}$:

4. Which is the greater (1) $\sqrt{10}+\sqrt{7}$ or $\sqrt{19}+\sqrt{3}$;
(2) $\sqrt{2}+\sqrt{7}$ or $\sqrt{3}+\sqrt{5}$; (3) $\sqrt{5}+\sqrt{14}$ or $\sqrt{3}+3\sqrt{2}$?
(4) $\sqrt{6}-\sqrt{5}$ or $\sqrt{8}-\sqrt{7}$? (5) $2^{\frac{1}{2}}+5^{\frac{1}{3}}$ or 3?

5. Can the three lines whose lengths are $3\sqrt{3}$, $5\sqrt{5}$, $7\sqrt{7}$ be the sides of the triangle?

6. If a denote the length of the edge of each of the five regular solids, shew that (1) the surface of the tetrahedron is $a^2\surd 3$: (2) of the hexahedron or cube is $6a^2$: (3) of the octahedron $2a^2\surd 3$: (4) of the dodecahedron $15a^2\surd\{1+\frac{2}{5}\surd 5\}$: and of the icosahedron $5a^2\surd 3$. And (1) the volume of the tetrahedron is $\frac{\surd 2}{12}.a^3$: (2) of the hexahedron or cube, a^3: (3) of the octahedron $\frac{\surd 2}{3}.a^3$: (4) of the dodecahedron $\frac{\surd 5}{2\surd 2}.\{47+2\surd 5\}^{\frac{1}{2}}.a^3$: (5) of the icosahedron $\frac{5}{6\surd 2}\{7+3\surd 5\}^{\frac{1}{2}}.a^3$.

XXXIX.

1. If $(a^2+b^2)^{\frac{1}{2}}+a=bx$, find $x-x^{-1}$ and $x+x^{-1}$ in terms of a and b.

2. $m=x+x^{-1}$, $n=y+y^{-1}$;
$$\text{then } mn+\surd(m^2-4)(n^2-4)=2(xy+x^{-1}y^{-1}).$$

3. If $c=a\sqrt{1-b^2}+b\sqrt{1-a^2}$,
$$\text{then shall } (a+b+c)(a+b-c)(a+c-b)(b+c-a)=4a^2b^2c^2.$$

4. If $(ay^2-a^2)^{\frac{1}{2}}=yz$, and $(ax^2-a^2)^{\frac{1}{2}}=xy$, then shall $(az^2-a^2)^{\frac{1}{2}}=xz$.

5. If $x(a^2-y^2)^{\frac{1}{2}}+y(a^2-z^2)^{\frac{1}{2}}=a^2$, then $x^2+y^2=a^2$.

6. If $x^{\frac{1}{2}}+y^{\frac{1}{2}}=a^{\frac{1}{2}}$, then $(x+y+a)^2=2(x^2+y^2+a^2)$.

7. If $x^{\frac{1}{2}}+y^{\frac{1}{2}}+z^{\frac{1}{2}}=a^{\frac{1}{2}}$,
$$\text{then } 64axy^2=\{2(a^2+x^2+y^2+z^2)-(a+x+y+z)^2\}^2.$$

8. Shew that $\{a+b^{\frac{1}{2}}+c^{\frac{1}{2}}+d^{\frac{1}{2}}\}^{\frac{1}{2}}$ can be found in the form $x^{\frac{1}{2}}+y^{\frac{1}{2}}+z^{\frac{1}{2}}$, when the condition $2a(bcd)^{\frac{1}{2}}=bc+bd+cd$ is satisfied.

9. If $x=\{r+(r^2+q^3)^{\frac{1}{2}}\}^{\frac{1}{3}}+\{r-(r^2+q^3)^{\frac{1}{2}}\}^{\frac{1}{3}}$;
$$\text{then shall } x^3+3qx-2r=0.$$

10. Shew that $ax^2+by^2+cz^2=(a^{\frac{1}{3}}+b^{\frac{1}{3}}+c^{\frac{1}{3}})^3$, when $ax^3=by^3=cz^3$,
$$\text{and } x^{-1}+y^{-1}+z^{-1}=1.$$

11. If $x^{\frac{1}{3}}+y^{\frac{1}{3}}+z^{\frac{1}{3}}=0$, then shall $(x+y+z)^3=27xyz$.

12. If $x^{\frac{1}{4}}+y^{\frac{1}{4}}+z^{\frac{1}{4}}=0$, then $(x^2+y^2+z^2-2xy-2xz-2yz)^2$
$$=128xyz(x+y+z).$$

XL.

1. If $x^2-x^{-2}=y$, express $\frac{1+x^4}{1-x^4}$ in terms of y.

2. Express with negative indices
$$\sqrt[4]{a^2b^8}+a(\surd b)^6+\sqrt[5]{(a^{-2}b^{10})}+\surd a^{-3}b^4.$$

3. If $\beta=\left\{\frac{1-e}{1+e}\right\}^{\frac{1}{2}}$, then shall $\frac{1-\beta}{1+\beta}=\frac{e}{1+\surd(1-e^2)}$.

4. If $a(b-c)^2-c(b+c)^2=0$, then $\frac{c}{b}\cdot\frac{a^{\frac{1}{2}}+c^{\frac{1}{2}}}{a^{\frac{1}{2}}-c^{\frac{1}{2}}}=1$.

5. Shew that $\left\{\frac{(a+x)(b+x)}{(a-x)(b-x)}\right\}^2=\left\{\frac{a+b^{\frac{1}{2}}}{a-b^{\frac{1}{2}}}\right\}^4$ when $x^2=ab$.

6. If $\frac{(a^2+1)(b^2+1)}{(ab+1)^2}=\frac{(c^2+1)(d^2+1)}{(cd+1)^2}$; then shall $\frac{cd+1}{ab+1}=\frac{\pm(c-d)}{a-b}$.

7. Shew that $\frac{1+ax+by}{\{(1+a^2+b^2)(1+x^2+y^2)\}^{\frac{1}{2}}}$ cannot be equal to ±1, unless $a=x$ and $b=y$; a, b, x, y being positive quantities.

8. Shew that the two expressions $x-y$ and $\sqrt{1-x^2}-\sqrt{1-y^2}$ fulfil the condition, that the difference of their squares is divisible by the sum of their squares.

9. If $yz+xz+xy=1$, prove that

$$x\left\{\frac{(1+y^2)(1+z)^2}{1+x^2}\right\}^{\frac{1}{2}}+y\left\{\frac{(1+z^2)(1+x^2)}{1+y^2}\right\}^{\frac{1}{2}}+z\left\{\frac{(1+x^2)(1+y^2)}{1+z^2}\right\}^{\frac{1}{2}}=2.$$

10. If $x^2+y^2+z^2+2xyz=1$; prove that

(1) $\{(1-y^2)(1-z^2)\}^{\frac{1}{2}}+\{(1-z^2)(1-x^2)\}^{\frac{1}{2}}+\{(1-x^2)(1-y^2)\}^{\frac{1}{2}}$
$=z(1+x)+y(1+z)+x(1+y)$.

(2) $\left\{\frac{1-x}{1+x}\cdot\frac{1-y}{1+y}\right\}^{\frac{1}{2}}+\left\{\frac{1-y}{1+y}\cdot\frac{1-z}{1+z}\right\}^{\frac{1}{2}}+\left\{\frac{1-z}{1+z}\cdot\frac{1-x}{1+x}\right\}^{\frac{1}{2}}=1.$

XLI.

1. Shew that it is not an arbitrary assumption to express the nth root of a by the symbol $a^{\frac{1}{n}}$, if a^n be assumed to denote the nth power of a.

2. If a denote any quantity numerically greater than b, does it follow that $(\pm a)^n$ is always algebraically greater than $(\pm b)^n$? If not, specify the exception or exceptions.

3. Shew that if any number a be greater than x, $a+\frac{x}{2a}$ is greater than, and $a+\frac{x}{2a+1}$ is less than the square root of a^2+x.

4. Is $(x+1)^n$ always greater than x^{n+1}?

5. Which is the greater $(x+a)(x^2+b^2)^{\frac{1}{2}}$, or $(x+b)(x^2+a^2)^{\frac{1}{2}}$, supposing a, b, x to be integral positive quantities?

6. Shew that $a^{\frac{3}{2}}+a^{\frac{1}{2}}b$ is greater or less than $ab^{\frac{1}{2}}+b^{\frac{3}{2}}$, according as a is greater or less than b.

7. If $a(1-b^2)^{\frac{1}{2}}+b(1-a^2)^{\frac{1}{2}}$ be less than unity, then a^2+b^2 shall be greater than unity.

8. If a, b, c, be unequal numbers, prove that $a+b+c$ is greater than $(ab)^{\frac{1}{2}}+(ac)^{\frac{1}{2}}+(bc)^{\frac{1}{2}}$.

9. If $x=a+b$, and $y=c+d$, which is the greater $(xy)^{\frac{1}{2}}$ or $(ac)^{\frac{1}{2}}+(bd)^{\frac{1}{2}}$?

10. If x, y, z be positive quantities, any two of which are together greater than the third, then any two of the three quantities $\{2(y^2+z^2)-x^2\}^{\frac{1}{2}}$, $\{2(z^2+x^2)-y^2\}^{\frac{1}{2}}$, $\{2(x^2+y^2)-z^2\}^{\frac{1}{2}}$, are together greater than the third.

11. If $(a+\sqrt{b})^{\frac{1}{2}}=x_1^{\frac{1}{2}}+y_1^{\frac{1}{2}}$, and $(c+\sqrt{d})^{\frac{1}{2}}=x_2^{\frac{1}{2}}+y_2^{\frac{1}{2}}$; shew that

$$\{ac+\sqrt{(a^2d+c^2b-bd)}\}^{\frac{1}{2}}=(x_1x_2+y_1y_2)^{\frac{1}{2}}+(x_2y_1-x_1y_2)^{\frac{1}{2}}.$$

12. Shew that $\frac{ab(x^2-y^2)+xy(a^2-b^2)}{(a^2+b^2)(x^2+y^2)}$ is less than $\frac{1}{2}$, unless $\frac{x}{y}=\frac{a+b}{a-b}$.

13. If a be greater than b, and b be greater than c,

Shew that $\left\{\frac{a^3}{bc}\right\}^{\frac{1}{2}}+\left\{\frac{b^3}{ac}\right\}^{\frac{1}{2}}+\left\{\frac{c^3}{ab}\right\}^{\frac{1}{2}}$ is never less than $a^{\frac{1}{2}}+b^{\frac{1}{2}}+c^{\frac{1}{2}}$.

XLII.

1. Shew that $\left\{\frac{x^m}{x^{m-n}}\right\}^{m+n}\times\left\{\frac{x^{m+n}}{x^n}\right\}^{m+n}=x^{(m+n)^2}$,

and $\left\{\frac{x^{m+n}}{x^n}\right\}^{m}\div\left\{\frac{x^n}{x^{n-m}}\right\}^{m-n}=x^{mn}$.

2. Simplify $\left\{a^{\frac{m}{n}}\right\}^{(n^2)^{\frac{1}{mn}}}$, $\left\{a^{\left(\frac{m}{n}\right)^{n^2}}\right\}^{\frac{1}{mn}}$, and $\left\{\left(a^{\frac{m}{n}}\right)^{n^2}\right\}^{\frac{1}{mn}}$.

3. Shew that $(x^{2m}+x^{2n})^{\frac{1}{mn}}=x^{\frac{1}{m}+\frac{1}{n}}(x^{m-n}+x^{n-m})^{\frac{1}{mn}}$,

and $x^{2n}-1>2n(x^{n+1}-x^{n-1})$.

4. Shew that $\frac{x^{\frac{m+n}{n}}-x^{\frac{n+m}{m}}}{x^{\frac{2m-n}{n}}-x^{\frac{2n-m}{m}}}=\frac{x^2}{x^{\frac{m}{n}}+x^{\frac{n}{m}}}$.

5. Determine when the divisions $\frac{a\pm b}{a^{\frac{1}{n}}\pm b^{\frac{1}{n}}}$, $\frac{a^{mn}\pm b^{mn}}{a^m\pm b^n}$, $\frac{a^m\pm b^n}{a^p\pm b^r}$, are respectively possible, without remainders.

6. Shew that $a^n-b^{\frac{n}{m}}$ and a^m-b have a common divisor, when m and n denote integral numbers.

7. Write the middle term of the quotients in the division of a^m-b^n by $a^{\frac{1}{n}}-b^{\frac{1}{m}}$, m and n being one or both even numbers.

8. If $(10{,}000)^{\frac{1}{x}}=10$, then $x=4$, and if $3x=9^{\frac{1}{4}}$, then $x=3^{-\frac{1}{2}}$.

9. If $x^z=z^x$, then shall $\left\{\frac{x}{z}\right\}^{\frac{x}{z}}=x^{\frac{x}{z}-1}$, and if $x=2z$, then $z=2$.

10. If $x=\left\{\frac{a+b}{a-b}\right\}^{\frac{2mn}{n-m}}$;

then shall $\frac{a^2-b^2}{2(a^2+b^2)}\left\{x^{\frac{1}{m}}+x^{\frac{1}{n}}\right\}=\left\{\frac{a+b}{a-b}\right\}^{\frac{n+m}{n-m}}$.

11. If $x=\left\{\frac{a}{b}\right\}^{\frac{2ab}{a^2-b^2}}$, then shall $\frac{ab}{a^2+b^2}\left\{x^{\frac{a}{b}}+x^{\frac{b}{a}}\right\}=\left\{\frac{a}{b}\right\}^{\frac{a^2+b^2}{a^2-b^2}}$.

RESULTS, HINTS, ETC., FOR THE EXERCISES ON INVOLUTION, EVOLUTION, AND SURD NUMBERS.

I.

The square roots are as follows :—

1. $x+6$. 2. $x-4$. 3. $2x+b$. 4. $x \pm x^{-1}$. 5. x^2+x+1.

6. $2x^2-x-1$. 7. $x^2-5ax+4a^2$. 8. $7x^2+4xy+y^2$.

9. $a^3+a^2b+ab^2+b^3$. 10. $a^4+3a^3+2a^2+a+1$. $3a^3+2a^2+2+1$.

11. x^2+15. 12. $4a^2+2(ab+ac+bc)$. 13. $3a^2-ab+5b^2$.

14. $a^2-3ab+b^2$. 15. $a^2+2ab+b^2$. 16. $a^2+b^2+c^2+d^2$.

17. This expression when reduced will be found to be not a complete square, and the square root of it may be continued to any number of terms.

18. $2a^3+a^2+2a-1$. 19. $a^2x^2+3ax+1$. 20. $a^2+ab+ac+bc$.

II.

1. 25, 52, 256, 652, 2567, 7652, 12345.

2. ·5, ·05, ·0025, 50, 500, 3·16227 . . ., ·31622 . . ., ·1, ·03162 . . ., ·01, ·001.

3. ·15811 . . ., 22·8035 . . ., ·34785 . . ., 1·1, 2·2135 . . ., 22·1359 . . ., ·94868 . . ., 1·0049 . . ., ·37960 . . ., ·91049 . . ., ·7385 . . .

4. $1\frac{1}{3}$, $4\frac{1}{2}$, $3\frac{1}{2}$, $2\frac{3}{4}$, $22\frac{1}{3}$, $\frac{763}{576}$.

5. 6324 . . ., 1·632 . . ., 1·2247 . . ., 4·1982 . . ., 1·0039 . . ., 1.8257 . . .

6. 1·2806 . . ., 1·0315 . . ., 1·0031 . . ., 1·0003 . . ., 1·00003 . . ., 1·000003 . . .

7. 2·5298 . . ., ·8, ·2529 . . ., ·08, ·02529 . . ., ·008, 8·000003 . . ., 8·0000003 . . .

8. 8730, ·0873, 2·7606 . . ., 276·06 . . ., 87·3, 8·73.

III.

1. See Art. 3, 2.

6. The quotient of the square roots of two numbers is equal to the square root of the quotient of the same two numbers.

7. The square roots are $\frac{a}{3}-\frac{x}{7}-\frac{y}{4}$ and $\frac{y}{4}+\frac{x}{7}-\frac{a}{3}$ found by performing the two operations. The results may be expressed by $\pm\left\{\frac{a}{3}-\frac{x}{7}-\frac{y}{4}\right\}$.

8. On making the substitutions for x and y, the expression becomes $81a^4+108a^3b+108a^2b^2+48ab^3+16b^4$, of which the square root is $9a^2+6ab+4b^2$.

IV.

2. On making the substitution and reducing

$$ax^2+bx+c=\left\{\frac{am^2-bmn+acn^2}{bn^2-2amn}\right\}^2.$$

3. After the extraction of the root, the remainder, being made equal to zero, gives two values of x, 3 and 1. These values when substituted for x in the given expression give 5776 and 64, both square numbers.

4. Extract the square root and put the remainder equal to zero.

8. The difference is $x^6+6x^5+(12-a)x^4+(56-b)x^3+(96-c)x^2+96x+64$. It is obvious that x^6, $6x^5$, and 64 are the first, second, and last terms of the square.

Suppose p to be the coefficient of the third term,
then $x^6+6x^5+(12-a)^4+(56-b)x^3+(96-c)x^2+96x+64=(x^3+3x^2+px+8)^2$, and when the latter is expanded, they become two identical expressions, and the coefficients of corresponding powers of x are equal, from which may be found

$$p=6,\ a=-9,\ b=-4, \text{ and } c=-12.$$

V.

1. The number is 2.
2. Here $850=2.5^2.17$. The multiplier required is 2.17.
3. The sum is $2\sqrt{\cdot 9}$: the difference $2\sqrt{\cdot 4}$: the product $\cdot 5$, and the quotient 5.
7. If possible, let $\sqrt{N}=a+\frac{p}{q}$, p and q being prime to each other,

$$\text{then } N=a^2+\frac{2ap}{q}+\frac{p^2}{q^2}\,;\ \therefore\ \frac{p^2}{q^2}=N-a^2-\frac{2ap}{q},$$

$$\text{and } \frac{p^2}{q}=qN-a^2q-2ap, \text{ an integer.}$$

But since p is prime to q ; $\therefore$ also p^2 is prime to q, $\frac{p^2}{q}$ is a fraction in its lowest terms ; but $\frac{p^2}{q}$ is an integer, which is absurd.

It therefore follows that it is not possible to express the square root of an incomplete square in the form $a+\frac{p}{q}$.

8. When the numerator and denominator of the fraction which produces a recurring decimal are square numbers, the square root of the fraction will produce a recurring decimal,

$$\text{as } \sqrt{\cdot\dot{4}}=\sqrt{\frac{4}{9}}=\frac{2}{3}=\cdot\dot{6}.$$

9. Let N be the given number whose square root consists of $2n+1$ digits ; p the number composed of the first $n+1$ digits found by extraction, q the number remaining of n digits.

$$\text{Then } N=(p\,.\,10^n+q)^2=p^2\,.\,10^{2n}+2\,.\,10^n\,.\,p\,.\,q+q^2\,;$$
$$\therefore\ N-p^2\,.\,10^{2n}=2\,.\,10^n\,.\,pq+q^2\,;$$
$$\therefore\ \frac{N-p^2.10^n}{2.10^n.p}=q+\frac{1}{2}.\frac{q}{10^n}.\frac{q}{p}.$$

Now $N-p^2.10^{2n}$ is the remainder after the first $n+1$ digits of the root have been found by extraction, and $2p.10^n$ is twice the portion of the root found.

And since q consists of n digits and 10^n of $n+1$ digits, $\therefore\ \frac{q}{10^n}$ is less than 1 ; also $\frac{q}{p}$ is less than 1, for p consists of $n+1$ digits.

Hence $\therefore\ \frac{1}{2}.\frac{q}{10^n}.\frac{p}{q}$ is less than 1, and may be omitted, as it will not cause an error of unity in the result ; $\therefore\ \frac{N-p^2.10^{2n}}{2.p.10^n}=q$, the number composed of the remaining n digits.

If p denote the nearest square root of N, r the remainder, then $N=p^2+r$, but N is less than $(p+1)^2$; $\therefore\ p^2+r$ is less than p^2+2p+1, and r is less than $2p+1$; that is, the remainder after any step in the extraction of the square root is always less than twice the root found increased by unity.

VI.

1. See Art. 4.

2. $(101293)^2=(101290+3)^2=(101290)^2+2\times 101290\times 3+3^2$.

3. By Euc. I. 47. $3^2+4^2=5^2$. Multiply these equals by x^2.

$\therefore (3x)^2+(4x)^2=(5x)^2$, in which x may any number whatever.

6. If x be any number, $2x+1$ is an odd number. And $(2x+1)^2=4x^2+4x+1$;

$\therefore (2x+1)^2-1=4x^2+4x=4x(x+1)$;

but since x is an odd, $x+1$ is an even number;

$\therefore (2x+1)^2-1$ is divisible by 8.

8. $(2x+1)^2=4(x^2+x)+1$. If $x=1, 2, 3$, &c., it will be found that other numbers besides odd square numbers when divided by 4 have unity for their remainders.

12. If 5 be the last digit of a number, and a denote the number composed of the other digits, then $10a+5$ is the number, and the square of $10a+5=100(a^2+a)+25$, of which $100(a^2+a)$ ends in two ciphers. Also $100(a^2+a)=100a(a+1)$ is the product of two consecutive numbers.

13. Let x the digit in the place of units, y in that of thousands, shew $(10^3y+x)^2-(10^3x+y)^2=(10^6-1)(y^2-x^2)$, and exemplify its truth with each of the nine digits.

VII.

1. If a, b be any two numbers, and c, d other two; shew that $(a^2+b^2)(c^2+d^2)$ may be expressed in two ways as the sum of two squares.

See Section IV., p. 30; X., Ex. 3.

2. If x, $x+1$, $x+2$, $x+3$ denote four consecutive numbers,

then $x(x+1)(x+2)(x+3)+1=(x^2+3x+1)^2$.

And if $2x+1$, $2x+3$, $2x+5$, $2x+7$ denote any four consecutive odd numbers, then $(2x+1)(2x+3)(2x+5)(2x+7)+16=(4x^2+16x+11)^2$. These may be verified by taking for x any number whatever.

7. The least numbers which satisfy the condition $a^2+b^2=c^2$ are 3, 4, 5 and $3^2+4^2=5^2$. And as 3, 4, 5 are prime to each other, their least common multiple is 60, which will always be a divisor of abc, when $a=3x$, $b=4x$, $c=5x$, x being any integral number whatever.

8. See Section IV., p. 30; X., Ex. 4.

VIII.

1. If the diagonal be 30 feet, then $\sqrt{450}$ is the length of a side; and if 30 be the side, then $\sqrt{1800}$ is the diagonal.

2. Side of the equivalent square is $\sqrt{(970{\cdot}25\times 250{\cdot}4)}$.

4. If a, b denote the lineal units in the sides of a right-angled triangle, and c those in the hypotenuse, then $a^2+b^2=c^2$. The area of the required square is 50 square feet, and the side is $5\sqrt{2}$ lineal feet. By a similar process a square may be found equal to any number of squares.

IX.

1. If x denote the breadth, then $3x$ is the length, and $3x^2=30{,}000$, the area of the field, $x^2=10{,}000$, $x=100$, the breadth.

3. If x be the side of the garden when the path goes half way round, then $4840-x^2=605$, and $x=\sqrt{4235}$, and $\sqrt{4840}-\sqrt{4235}=$ width of path. If the pathway go completely round the garden, then $(\sqrt{4840}-2x)^2=$ area of garden, and $4840=$ area of the path and garden;

$\therefore\ 4810-(\sqrt{4840}-2x)^2=$ area of path,
$=\frac{1}{8}$ of 4840;
$\therefore\ (\sqrt{4840}-2x)^2=4235$,
and $\sqrt{4840}-2x=\sqrt{4235}$,
$2x=\sqrt{4840}-\sqrt{4235}$,
$x=\frac{1}{2}(\sqrt{4840}-\sqrt{4235})$ the width of the path.

X.

The cube roots are the following :—

1. $a-1$. 2. $x+x^{-1}$. 3. a^2-a-1. 4. a^2+1. 5. $ax-by$.
6. $2a^2-3b^2$. 7. a^2-2a+1. 8. $3a^2-4ax+x^2$. 9. $2x^2-3x+1$.
10. $x^2-\frac{2}{x}$. 11. $\frac{a}{2}-\frac{2}{3a^2}$. 12. $\frac{a}{b}-1-\frac{b}{a}$.

XI.

The following are the cube roots of the numbers :—

1. 64, 81, 125, 145, 165, 173.
2. 25, 52, 256, 652, 2567, 7652.
3. ·4, ·04, ·21, ·074, ·094, ·0301.
4. 1·0033 . . . , 2·124 . . . , ·7368 . . . , ·3419 . . . , ·15874 . . . , 2·38 . . . , 1·508 . . . , 2·070 . . . , 2·083 . . . , 1·241 . . . , 2·155 . . .
5. 1·01, 9·6, 21·1, 3·43, 8·88, 90·4.
6. 1·259, 1·912, 2·154, 4·641, 1·464.
7. ·793 . . . , ·956 . . . , ·650 . . . , 1·5. 1·493 . . . , 10·102 . . . , $1\frac{1}{3}$, ·8735 . . .

XII.

2. Shew that $x^{\frac{1}{3}}$ is greater than $(x+1)^{\frac{1}{4}}$.
4. $(x+1)^3=\left\{\frac{(x+1)(x+2)}{1.2}\right\}^2-\left\{\frac{x(x+1)}{1.2}\right\}^2=x^3+3x^2+3x+1$.

See note, p. 14, Section X.

5. See p. 9, Art 9, note.
6. See V., Ex. 7.
7. See V., Ex. 8.
8. The fourth, sixth, and ninth roots of the given numbers are each ·13.

XIII.

1. See Art. 11.

2. If a, b, c be any three numbers, then shew that

$$(a+b+c)^3-(a^3+b^3+c^3)=3(a+b)(b+c)(c+d).$$

3. $(x+1)^2+x(x+1)+x^2=(x+1)^3-x^3$.

5. Let x, $x+1$, $x+2$ denote any three consecutive numbers, then $x(x+1)(x+2)+(x+1)=(x+1)^3$; also $(x-1)(x-2)(x-3)+(x-2)=(x-2)^3$.

6. If a, b be the numbers, shew that a^3+b^3+1 is greater then $3ab$. See Ex. 18, p. 45, Section IV.

7. The number expressed by its prime factors is $2.3^2.5^3.7$, and the required multiplier is $2^2.3.7^2$.

8. Here $3^3+4^3+5^3=6^3$; multiply these equals by x^3; then $(3x)^3+(4x)^3+(5x)^3=(6x)^3$, in which x may be taken equal to 1, 2, 3, 4, &c. Also, since $6^3+(20)^3+(36)^3=(38)^3$; then $(6x)^3+(20x)^3+(36x)^3=(38x)^3$, and x may be fractional as well as integral.

XIV.

1. 96 square inches in its surface, 216 cubic inches in its volume.

2. The edge of the cube is 10 inches, its surface is 600 square inches, the diagonal of one of the faces $10\sqrt{2}$, the diagonal of the cube $10\sqrt{3}$ inches.

4. The content is $\frac{1}{2}(\sqrt{3}+1)$ cubic inches.

7. The cube root of the number which expresses the content of the parallelopiped will be the edge of the equivalent cube.

11. Find first the content of the three cubes, and the cube root of their sum will be the edge of the cube required.

XV.

1. The dimensions are 50, 100, 200 inches respectively.

2. The cube root of the content gives the length of an edge of the cube, and 6 times the square of this number will give the number of square yards.

4. The length, breadth, and height are respectively 32, 16, and 8 feet.

5. The height is 9 feet, and the length and breadth are respectively 32 and 24 feet.

6. The length of a side is $\left\{\frac{1361370}{16\times1728}\right\}^{\frac{1}{3}}$ feet.

7. The length, breadth, and depth are respectively $\frac{(17920)^{\frac{1}{3}}}{10}$ feet.

8. Difference between external and internal surfaces is $24m(a+m)$ square inches, and content of the boards of which the chest is composed is $2m(3a^2-6am+4m^2)$ cubic inches.

9. The length of the edge is 8 feet.

10. The quotient arising from the division of £8,584 18s. 7d. by £4 3s. 4d. gives the number of cubic inches of the mass. The cube root of this number gives the lineal inches in the edge of the cube.

12. The balls are supposed to be in layers so that the diameters of the balls where they touch one another are in vertical and horizontal straight lines. And the number of balls which can be packed in the box is equal to the number of cubes which also could be packed in the box, if the edge of each cube be equal to the diameter of each ball. The portion of vacant space in the box is the difference between the content of the cubes and the content of the balls.

The content of a sphere is two-thirds of its circumscribing cylinder, or if r denote the radius of sphere, then the content of the sphere

$$=\frac{2}{3}\times(2\pi r^3) \text{ or } \frac{4\pi r^3}{3}, \text{ where } \pi=3{\cdot}14159\ldots$$

If the balls be each m inches in diameter the vacant space will be expressed by $a^3\left\{m^3-\frac{4\pi}{3}.\frac{m^3}{8}\right\}=a^3m^3\left(1-\frac{\pi}{6}\right)$.

In the question, suppose the edge of the box 25 inches, and find the vacant spaces in the three cases, when the balls are one inch, half an inch, and one-third of an inch in diameter.

XVI.

1. a^{2m}. 2. a^{2n}. 3. a^{-2n}. 4a^{0}. 5. a^{m-2p+q}. 6. a^{0}. 7. a^{m+n}.

8. a^{2m-n-3}. 9. $\pm a^{7m}$. 10. a^{-mn}. 11. a^{mn}. 12. a^{-mn}. 13. $a^{\frac{-m}{n}}$.

14. $a^{\frac{-m}{n}}$. 15. $a^{\frac{m}{n}}$. 16. $a^2 . b^{\frac{n^2+m^2}{mn}}$. 17. $a^{\frac{n-m}{n}} . b^{\frac{m-n}{m}}$.

18. $\left\{\frac{b^{\frac{1}{m}}}{a^{\frac{1}{n}}}\right\}^{m+n}$. 19. $\frac{a^{m+1}}{b^{m(m+n)+n+1}}$. 20. $b^{m(m-n)}$. 21. $a^{\frac{n}{m}-\frac{m}{n}}\cdot b^{1-\frac{m}{n}}$.

22. a^{11mnp}. 23. $a^{\frac{m^2-(m+n)^2}{m(m+n)}}$. 24. $a^{\frac{m^2+(m+n)^2}{m(m+n)}}$.

XVII.

These exercises require no remark.

XVIII.

These products differ from the exercises in Section IV. in this circumstance—that the indices of the quantities here are fractional and not integral.

XIX.

The same remark applies to these exercises.

XX.

These exercises require no remarks.

XXI.

1. $x - x^{\frac{1}{2}} + 1$. 2. $x^2 - x^{\frac{3}{2}} + 1$. 3. $a - 2a^{\frac{1}{2}}b^{\frac{1}{2}} - b$. 4. $\frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}} - 1 - \frac{b^{\frac{1}{2}}}{a^{\frac{1}{2}}}$.

5. $\frac{x^{\frac{2}{3}}}{4} - \frac{x^{\frac{1}{3}}}{3} - \frac{1}{2}$. 6. $1 - \frac{x^{\frac{1}{2}}}{2} + \frac{ax^{\frac{3}{4}}}{3}$. 7. $x + 2^{\frac{1}{2}}x^{\frac{1}{2}} - 1$. 8. $\frac{x^{\frac{1}{2}}}{y^{\frac{1}{2}}} - \frac{1}{2^{\frac{1}{2}}} + \frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}$.

9. $a^{\frac{1}{2}}x^{\frac{3}{2}} + bx + c^{\frac{1}{2}}x^{\frac{1}{2}}$. 10. $1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} +$ &c. 11. $1 + \frac{x}{2} + \frac{3x^2}{8} - \frac{3x^3}{16} + \frac{3x^4}{128} +$, &c.

12. $a + \frac{b}{2} + \frac{3b^2}{8a} - \frac{3b^3}{16a^2} +$, &c.

XXII.

1. $a - b^{\frac{1}{2}}$. 2. $x + x^{\frac{1}{2}} - 1$. 3. $x + 2x^{\frac{1}{2}} + 1$. 4. $1 - 3x^{\frac{1}{2}} - x$.

5. $a + \frac{x^3}{3a^2} - \frac{x^6}{9a^5} +$, &c. 6. $1 - 4x$.

XXIII.

The verification of these expressions involves the same operations as in p. 33, Section IV., the only difference being that in these examples the indices are fractional and not integral.

XXIV.

1. 5. 2. $1\frac{2}{3}$. 3. $x(x-1)^2 + 18 = 0$. 4. ±2. 5. $5 + 2\sqrt{2}$. 6. $\frac{-1}{22}$.

7. When $x = \frac{\sqrt{3}}{2}$, $(2+\sqrt{3})^{\frac{1}{2}} = \frac{1+\sqrt{3}}{\sqrt{2}}$, and $\frac{1+x}{1+(1+x)^{\frac{1}{2}}} + \frac{1-x}{1+(1-x)^{\frac{1}{2}}} = \frac{2(3-\sqrt{3})}{3}$.

8. $\frac{7\sqrt{3}}{6}$. 9. 3. 10. $\frac{a^2+1+(1+2a^2-15a^4)^{\frac{1}{2}}}{4a^2}$. 11. $\frac{(\sqrt[3]{3}-1)a^{\frac{1}{3}}}{\sqrt[3]{2}}$.

XXV.

These exercises are reduced in the same manner as those in pp. 13, 14 in Section V.

1. $\frac{1-x^{\frac{2}{3}}}{x^{\frac{1}{3}}+5x^2}$. 2. $\frac{x+1}{x^2+x-1}$. 4. $\frac{x^3-2x-1}{x^3-2x^2-6x-3-2x(x+1)(x+1)^{\frac{1}{2}}}$

$$=\frac{(x+1)(x^2-x-1)}{(x+1)(x^2-3x-3)-2x(x+1)(x+1)^{\frac{1}{2}}}=\frac{x^2-x-1}{x^2-3x-3-2x(x+1)^{\frac{1}{2}}}.$$

5. $\frac{(x^3-3x-2)+(x^2-1)(x^2-4)^{\frac{1}{2}}}{(x^3-3x+2)+(x^2-1)(x^2-4)^{\frac{1}{2}}}=\frac{(x^2-1)(x-2)+(x^2-1)(x^2-4)^{\frac{1}{2}}}{(x^2-1)(x+2)+(x^2-1)(x^2-4)^{\frac{1}{2}}}$

$$=\frac{(x^2-1)(x-2)^{\frac{1}{2}}.\left\{(x-2)^{\frac{1}{2}}+(x+2)^{\frac{1}{2}}\right\}}{(x^2-1)(x+2)^{\frac{1}{2}}.\left\{(x+2)^{\frac{1}{2}}+(x-2)^{\frac{1}{2}}\right\}}=\left\{\frac{x-2}{x+2}\right\}^{\frac{1}{2}}.$$

7. $\frac{x-2x^{\frac{3}{4}}y^{\frac{1}{4}}+2x^{\frac{1}{4}}y^{\frac{3}{4}}-y}{x-2x^{\frac{1}{2}}y^{\frac{1}{2}}+y}=\frac{(x^{\frac{1}{4}}-y^{\frac{1}{4}})(x^{\frac{3}{4}}+x^{\frac{1}{2}}y^{\frac{1}{4}}-x^{\frac{1}{4}}y^{\frac{1}{2}}+y^{\frac{3}{4}})}{(x^{\frac{1}{4}}-y^{\frac{1}{4}})(x^{\frac{3}{4}}-x^{\frac{1}{2}}y^{\frac{1}{4}}-x^{\frac{1}{4}}y^{\frac{1}{2}}-y^{\frac{3}{4}})}$.

8. $\frac{(a+x)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}}{(a+x)^{\frac{1}{2}}-(a-x)^{\frac{1}{2}}}=\frac{a+(a^2-x^2)^{\frac{1}{2}}}{x}=\frac{x}{a-(a^2-x^2)^{\frac{1}{2}}}$. 9. $a^{2^n}+a^{2^{n-1}}+1$.

XXVI.

The verification of these equivalents is effected by exactly the same processes as those in p. 24., Section V., the only difference being in the fractional indices.

In Example 13, one side of the equivalent is omitted. It should be

$$\frac{2ab-\left\{(1-a^4)(1-b^4)\right\}^{\frac{1}{2}}}{1+b^2+a^2(1-b^2)}$$

XXVII.

1. See Sect. IV., Art. 3, and notes pp. 2, 3; and in this Section VI., Art. 3, and notes pp. 3, 4.

2. $a^2+b^2=(a+b\sqrt{-1})(a-b\sqrt{-1})$.

3. $\frac{2+3\sqrt{-1}}{4+5\sqrt{-1}}=\frac{2+3\sqrt{-1}}{4+5\sqrt{-1}}\times\frac{4-5\sqrt{-1}}{4-5\sqrt{-1}}=\frac{23}{41}+\frac{2\sqrt{-1}}{41}$.

$$\frac{a+b\sqrt{-1}}{c+d\sqrt{-1}}=\frac{a+b\sqrt{-1}}{c+d\sqrt{-1}}\times\frac{c-d\sqrt{-1}}{c-d\sqrt{-1}}=\frac{ac+bd}{c^2+d^2}+\frac{(bc-ad)\sqrt{-1}}{c^2+d^2}.$$

4. See Art. 3, pp. 2, 3.

5. See Art. 18, and note pp. 17, 18.

6. First $(-a)^{\frac{1}{4n}}\times(-b)^{\frac{1}{4n}}=(a\times-1)^{\frac{1}{4n}}\times(b\times-1)^{\frac{1}{4n}}$

$$=a^{\frac{1}{4n}}.(-1)^{\frac{1}{4n}}.b^{\frac{1}{4n}}(-1)^{\frac{1}{4n}}$$

$$=(ab)^{\frac{1}{4n}}.(-1)^{\frac{1}{4n}+\frac{1}{4n}}$$

$$=(ab)^{\frac{1}{4n}}.(-1)^{\frac{1}{2n}}.$$

Hence the product involves $(-1)^{\frac{1}{2n}}$, an even root of -1, which is an impossible expression.

Next $(-a)^{\frac{1}{4n+2}}\times(-b)^{\frac{1}{4n+2}}=(a\times-1)^{\frac{1}{4n+2}}\times(b\times-1)^{\frac{1}{4n+2}}$

$$=(a^{\frac{1}{4n+2}}.(-1)^{\frac{1}{4n+2}}\times b^{\frac{1}{4n+2}}.(-1)^{\frac{1}{4n+2}}$$

$$=(ab)^{\frac{1}{4n+2}}.(-1)^{\frac{2}{4n+2}}$$

$$=(ab)^{\frac{1}{4n+2}}.(-1)^{\frac{1}{4n+1}}=-(ab)^{\frac{1}{4n+2}}.$$

Hence the product involving an odd root of -1 which is always possible and equal to -1.

10. The third power of $\frac{-1\pm\sqrt{-3}}{2}$ is $+1$; hence this expression exhibits the two symbolic cube roots of $+1$. And the third power of $\frac{+1\pm\sqrt{-3}}{2}$ is -1, and the expression gives the two symbolic cube roots of -1.

$$\text{And } \frac{-1+\sqrt{-3}}{2}=\frac{-1+\sqrt{-3}}{2}\times\frac{-1-\sqrt{-3}}{-1-\sqrt{-3}}=\frac{2}{-1-\sqrt{-3}}.$$

XXVIII.

The first eight examples offer no difficulty beyond the exact performance of the operations indicated.

9. This may be shown in two ways: by extracting the square roots of each quantity and taking the sum of the results, and by squaring each side of the expression, and showing that the result when reduced is $32+68=100$, an obvious identity.

XXIX.

The first five examples require no remark.

6. The expression may be put into the form

$$\frac{y-x\sqrt{-1}}{y-2x\sqrt{-1}}\times\frac{y-2x\sqrt{-1}}{y+2x\sqrt{-1}}\times\frac{2x-y\sqrt{-1}}{x-y\sqrt{-1}}\times\frac{(x-y\sqrt{-1})(y+x\sqrt{-1}}{\sqrt{(x^2+y^2)}\sqrt{-1}}$$

$$=\frac{y-x\sqrt{-1}}{y+2x\sqrt{-1}}\times\frac{(2x-y\sqrt{-1})(y+x\sqrt{-1})}{\sqrt{(x^2+y^2)}\sqrt{-1}},$$

$$=\frac{(y^2+x^2)}{-(2x-y\sqrt{-1})}\times\frac{(2x-y\sqrt{-1})}{\sqrt{(x^2+y^2)}}=\frac{-(y^2+x^2)}{\sqrt{(x^2+y^2)}}=-(x^2+y^2)^{\frac{1}{2}}.$$

XXX.

1. $x^4-12x=(-1+2\sqrt{-1})^4-12(-1+2\sqrt{-1})=5$.
2. $x=\frac{a^2-b^2}{a^2+b^2}$, $y=\frac{2ab}{a^2+b^2}$.

XXXI.

1. The sum is $7\sqrt{3}$; the difference $\sqrt{3}$.
2. $163\sqrt{3}$; $33\sqrt{3}$.
3. $15\sqrt{5}$; $3\sqrt{5}$.
4. $\frac{292}{63}\sqrt{2}$; $\frac{191}{63}\sqrt{2}$.
5. $\frac{13}{20}\sqrt{10}$; $\frac{11}{20}\sqrt{10}$.
6. $(2\pm\sqrt{5})\sqrt{3}$.
7. $8\sqrt[3]{4}$; $2\sqrt[3]{4}$.
8. $5\sqrt[3]{5}$; $\sqrt[3]{5}$.
9. $13\sqrt[3]{3}$; $\sqrt[3]{3}$.
10. $3\sqrt[3]{9}$; $\sqrt[3]{9}$.
11. $\frac{46}{35}\sqrt[3]{4}$; $\frac{4}{35}\sqrt[3]{4}$.
12. $\frac{7}{20}\sqrt[3]{18}$; $\frac{3}{20}\sqrt[3]{18}$.
13. $5\sqrt{3}$; $\sqrt{3}$.
14. $2(\sqrt{5}\pm\sqrt{3})$.

XXXII.

1. Product is 224; quotient 2.
2. $48\sqrt{3}$; $\frac{2}{3}\sqrt{3}$.
3. $15600\sqrt{3}$; $\frac{8}{13}\sqrt{3}$.
4. $\frac{2}{17}\sqrt{3}$; $\frac{2}{37}\sqrt{3}$.
5. $\frac{1}{3}$; $\frac{2}{3}$.
6. $\frac{1}{10}$; $\frac{47}{10}$.
7. $54\sqrt{10}$; $\frac{4}{5}\sqrt{10}$.
8. $104\sqrt[3]{18}$; $\frac{4}{13}\sqrt[3]{36}$.
9. $24\sqrt[3]{1170}$; $\frac{6}{35}\sqrt[3]{1170}$.
10. $\frac{1}{5}\sqrt[3]{100}$; $\frac{2}{5}\sqrt[3]{4}$.
11. $\frac{1}{5}\sqrt[3]{15}$; $\frac{2}{15}\sqrt[3]{100}$.
12. $\frac{2}{175}\sqrt[3]{5}$; $\frac{7}{10}$.
13. $\sqrt[3]{4}$; $\frac{1}{4}\sqrt[3]{9}$.
14. $\frac{1}{7}\sqrt[3]{4}$; $\frac{44}{21}\sqrt[3]{3}$.
15. $24\sqrt[3]{3}$; $\frac{3}{4}\sqrt[3]{6}$.
16. $6\sqrt[3]{72}$; $\frac{1}{2}\sqrt[3]{648}$.
17. $9\sqrt[3]{12}$; $\frac{21}{8}\sqrt[3]{96}$.
18. $\sqrt[3]{288}$; $\frac{1}{5}\sqrt[3]{288}$.

XXXIII.

1. $3, 3\sqrt{3}, 9, 9\sqrt{3}$.
2. $75, 375\sqrt{3}, 5625, 28125\sqrt{3}$.
3. $12, 24\sqrt{3}, 144, 288\sqrt{3}$.
4. $\frac{4}{3}, \frac{8}{15}\sqrt{5}, \frac{16}{15}, \frac{32}{75}\sqrt{5}$.
5. $\frac{28}{9}, \frac{56}{27}\sqrt{7}, \frac{784}{81}, \frac{1568}{243}\sqrt{7}$.
6. $72, 144\sqrt{2}, 576, 1152\sqrt{2}$.
7. $9\sqrt[3]{9}, 81, 243\sqrt[3]{3}, 729\sqrt[3]{9}$.
8. $\frac{1}{9}\sqrt[3]{36}, \frac{2}{9}, \frac{2}{27}\sqrt[3]{6}, \frac{2}{81}\sqrt[3]{36}$.
9. $25\sqrt[3]{25}, 625, 3125\sqrt[3]{5}, 15625\sqrt[3]{25}$.
10. $7\pm 4\sqrt{3}, 26\pm 15\sqrt{3}, 97\pm 56\sqrt{3}, 362\pm 209\sqrt{3}$.
11. $5\pm 2\sqrt{6}, 11\sqrt{2}\pm 9\sqrt{3}, 49\pm 20\sqrt{6}, 109\sqrt{2}\pm 89\sqrt{3}$.
12. $77+30\sqrt{6}, 655\sqrt{2}+531\sqrt{3}, 11329+4620\sqrt{6}, 98225\sqrt{2}+80187\sqrt{3}$.

XXXIV.

$$\{\sqrt{2}\}^{\frac{1}{2}}=(2^{\frac{1}{2}})^{\frac{1}{2}}=2^{\frac{1}{4}} \text{ or } \sqrt[4]{2}: \qquad \{\sqrt{2}\}^{\frac{1}{3}}=\{2^{\frac{1}{2}}\}^{\frac{1}{3}}=2^{\frac{1}{6}} \text{ or } \sqrt[6]{2}.$$

$$(5\sqrt{5})^{\frac{1}{2}}=(5.5^{\frac{1}{2}})^{\frac{1}{2}}=5^{\frac{1}{2}}.5^{\frac{1}{4}}=(5^2 5)^{\frac{1}{4}}=(125)^{\frac{1}{4}} \text{ or } \sqrt[4]{125},$$

$$(5\sqrt{5})^{\frac{1}{3}}=(5.5^{\frac{1}{2}})^{\frac{1}{3}}=5^{\frac{1}{3}}.5^{\frac{1}{6}}=(5^2.5)^{\frac{1}{6}}=(125)^{\frac{1}{6}}, \text{ or } \sqrt[6]{125}.$$

The square roots of the last four examples can be found as the example in the note, p. 15. The roots are respectively

$$3+\sqrt{3},\ \frac{\sqrt{3}+1}{\sqrt{2}},\ 3\sqrt{3}-2\sqrt{2},\ 2+3\sqrt{-1}.$$

XXXV.

1. Here $3\sqrt{147}=3\sqrt{(49\times 3)}=3\times 7\sqrt{3}=21\sqrt{3}$

$$-3\sqrt{75}=-3\sqrt{(25\times 3)}=-3\times 5\sqrt{3}=-15\sqrt{3}$$

$$-3\sqrt{\tfrac{1}{3}}=-3\sqrt{\tfrac{3}{9}}=-3\times\tfrac{1}{3}\sqrt{3}=-\sqrt{3}$$

$$\therefore\ 3\sqrt{147}-3\sqrt{75}-3\sqrt{\tfrac{1}{3}}=(21-15-1)\sqrt{3}=5\sqrt{3}.$$

2. $\frac{42}{5}\sqrt{3}$. 3. $18+20\sqrt{2}-12\sqrt{6}$. 4. $8\sqrt{2}+7\sqrt{3}+5\sqrt{5}+2\sqrt{30}$.

5. $1+\sqrt{3}+\sqrt{5}$. See XXXIX. 8, p. 36. 6. $\sqrt{2}$.

7. $2\sqrt[3]{3}$. 8. $75\sqrt[3]{2}$. 9. $\sqrt[3]{3}+2\sqrt[3]{2}$. 10. $\frac{12}{5}\sqrt[3]{3}$.

11. Let $(\sqrt{5}+2)^{\frac{1}{3}}+(\sqrt{5}-2)^{\frac{1}{3}}=x$, and put $(\sqrt{5}+2)^{\frac{1}{3}}=a$, $(\sqrt{5}-2)^{\frac{1}{3}}=b$,

$$\text{then } x=a+b,$$

$$\text{and } x^3=(a+b)^3=a^3+b^3+3ab(a+b),$$

$$\text{but } a^3+b^3=2\sqrt{5}, \text{ and } 3ab=3,$$

$$\therefore\ x^3=2\sqrt{5}+3x, \text{ and } x^3-3x=2\sqrt{5}.$$

To find the value of x requires the solution of a cubic equation.

12. 17·69. 13. 90.

XXXVI.

Since $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=a-b$, and $(a\sqrt{x}+b\sqrt{y})(a\sqrt{x}-b\sqrt{y})=a^2x-b^2y$; it is obvious that $\sqrt{a}+\sqrt{b}$ may be made rational by multiplying it by $\sqrt{a}-\sqrt{b}$; and $\sqrt{a}-\sqrt{b}$ by multiplying it by $\sqrt{a}+\sqrt{b}$.

Similarly for the factors $a\sqrt{x}+b\sqrt{y}$ and $a\sqrt{x}-b\sqrt{y}$.

When there are three terms, $\sqrt{a}+\sqrt{b}+\sqrt{c}$,

$$\text{since } \{(\sqrt{a}+\sqrt{b})+\sqrt{c}\}\cdot\{(\sqrt{a}+\sqrt{b})-\sqrt{c}\}=(\sqrt{a}+\sqrt{b})^2-c=(a+b-c)+2\sqrt{ab},$$

$$\text{and } \{(a+b-c)+2a\sqrt{b}\}.\{(a+b-c)-2\sqrt{ab}\}=(a+b-c)^2-4ab.$$

And two factors will be required in general to render such a denominator a rational quantity.

1. $\frac{8-5\sqrt{2}}{3-2\sqrt{2}}=\frac{8-5\sqrt{2}}{3-2\sqrt{2}}\times\frac{3+2\sqrt{2}}{3+2\sqrt{2}}=\frac{24+\sqrt{2}-20}{9-8}=4+\sqrt{2}$.

16. Here $(5.12)^{\frac{1}{3}}=\left\{\frac{8^3}{10^2}\right\}^{\frac{1}{3}}=\frac{8}{10^{\frac{2}{3}}}$

$$(\cdot 03375)^{\frac{1}{3}}=\left\{\frac{3^3.5^3}{10^5}\right\}^{\frac{1}{3}}=\frac{15}{10.10^{\frac{2}{3}}}=\frac{3}{2.10^{\frac{2}{3}}}$$

$$(80)^{\frac{1}{3}}=(8.10)^{\frac{1}{3}}=2.10^{\frac{1}{3}}=\frac{2\times 10}{10^{\frac{2}{3}}}=\frac{20}{10^{\frac{2}{3}}}$$

$$(\cdot 01)^{\frac{1}{3}}=\left\{\frac{1}{10^2}\right\}^{\frac{1}{3}}=\frac{1}{10^{\frac{2}{3}}}.$$

The given expression by these substitutions

$$\text{becomes }\frac{\frac{1}{10^{\frac{2}{3}}}\left\{8+\frac{3}{2}\right\}}{\frac{1}{10^{\frac{2}{3}}}\left\{20-1\right\}}=\frac{19}{2}\cdot\frac{1}{19}=\frac{1}{2}.$$

XXXVII.

1. These reductions offer no difficulty, as

$$1.\ \frac{\sqrt{245}+\sqrt{75}}{\sqrt{5}-\sqrt{3}}+\frac{\sqrt{245}-\sqrt{75}}{\sqrt{5}+\sqrt{3}}=\frac{7\sqrt{5}+5\sqrt{3}}{\sqrt{5}-\sqrt{3}}+\frac{7\sqrt{5}-5\sqrt{3}}{\sqrt{5}+\sqrt{3}}$$

$$=\frac{50+12\sqrt{15}}{2}+\frac{50-12\sqrt{15}}{2}$$

$$=\frac{100}{2}=50.$$

XXXVIII.

1. The expressions $2^{\frac{1}{2}}$, $3^{\frac{1}{3}}$, $4^{\frac{1}{4}}$, $5^{\frac{1}{5}}$, $6^{\frac{1}{6}}$, are respectively equivalent to $(2^{30})^{\frac{1}{60}}$, $(3^{20})^{\frac{1}{60}}$, $(4^{15})^{\frac{1}{60}}$, $(5^{12})^{\frac{1}{60}}$, $(6^{10})^{\frac{1}{60}}$; and by expanding the powers 2^{30}, 3^{20}, 4^{15}, 5^{12}, 6^{10}, their numerical values will be known and the given expressions can be arranged in the order of magnitude.

$$2.\ \frac{2}{\sqrt{3}}=\frac{2}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}=\frac{2\sqrt{3}}{3}.$$

$$\frac{2\sqrt{3}}{\sqrt{2}}=\frac{2\sqrt{3}}{\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{2\sqrt{6}}{2}=\sqrt{6}.$$

$$\left(\frac{3}{5}\right)^{\frac{1}{2}}=(\cdot 6)^{\frac{1}{2}}=\sqrt{\cdot 6}.$$

$$\left(\frac{3}{2}\right)^{\frac{3}{2}}=\left(\frac{3^3}{2^3}\right)^{\frac{1}{2}}=\left(\frac{27}{8}\right)^{\frac{1}{2}}=(3\cdot 375)^{\frac{1}{2}}.$$

$$\left(\frac{2}{3}\right)^{\frac{2}{3}}=\left(\frac{2^2}{3^2}\right)^{\frac{1}{3}}=\left(\frac{4}{3^2}\times\frac{3}{3}\right)^{\frac{1}{3}}=\left(\frac{12}{3^3}\right)^{\frac{1}{3}}=\frac{(12)^{\frac{1}{3}}}{3}=\frac{\sqrt[3]{12}}{3}.$$

$$8^{-\frac{3}{2}}=\frac{1}{8^{\frac{3}{2}}}=\frac{1}{16\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}=\frac{\sqrt{2}}{32}.$$

$$7^{\frac{1}{6}}=\{7^{\frac{1}{2}}\}^{\frac{1}{3}}\text{ or }\{7^{\frac{1}{3}}\}^{\frac{1}{2}}.$$

$$(\cdot\dot{0}3\dot{7})^{\frac{2}{3}}=\left\{\frac{37}{999}\right\}^{\frac{2}{3}}=\left\{\frac{1}{27}\right\}^{\frac{2}{3}}=\frac{1}{(3^3)^{\frac{2}{3}}}=\frac{1}{3^2}=\frac{1}{9}.$$

The extractions and divisions in each example are to be carried out to four places of decimals.

3. $\sqrt[3]{4}$ is greater or less than $\sqrt[4]{5}$.

According as $4^{\frac{1}{3}}$,, ,, $5^{\frac{1}{4}}$

,, 4^4 ,, ,, 5^3, raising each to the 12th power.

,, 256 ,, ,, 125,

but 256 is greater than 125,

∴ also $\sqrt[3]{4}$ is greater than $\sqrt[4]{5}$.

Similarly $2^{\frac{1}{2}}$ is less than $3^{\frac{1}{3}}$.

$3\sqrt[3]{2}$ is greater than $2\sqrt{3}$.

$(5^2)^{\frac{1}{3}}$ is less than $(5^3)^{\frac{1}{2}}$,

and because $(5^3)^{\frac{1}{2}}=5^{\frac{3}{2}}=5^{1\cdot 5}$,

and $5^{2^{\frac{1}{3}}}=5^{1.259}\ldots$

therefore $(5^3)^{\frac{1}{2}}$ is greater than $5^{2^{\frac{1}{3}}}$.

$\sqrt{7}$ is greater than $2\sqrt[4]{3}$.

$\frac{1}{2}$ is less than $\left(\frac{2}{3}\right)^{\frac{3}{2}}$

$\frac{\sqrt[3]{3}}{\sqrt{2}}$ is greater than $\frac{\sqrt{3}}{\sqrt[3]{5}}$.

4. Here $\sqrt{10}+\sqrt{7}$ is greater or less than $\sqrt{19}+\sqrt{3}$.
According as $17+2\sqrt{70}$,, ,, $22+2\sqrt{57}$, by squaring each side.
,, $2\sqrt{70}$,, ,, $5+2\sqrt{57}$, subtracting 17.
,, 280 ,, ,, $25+20\sqrt{57}+228$,
but 280 is obviously less than $25+20\sqrt{57}+228$,
$\therefore$ $\sqrt{10}+\sqrt{7}$ is less than $\sqrt{19}+\sqrt{3}$.

(2) $\sqrt{2}+\sqrt{7}$ is greater than $\sqrt{3}+\sqrt{5}$.
(3) $\sqrt{5}+\sqrt{14}$ is greater than $\sqrt{3}+3\sqrt{2}$.
(4) $\sqrt{6}-\sqrt{5}$ is greater than $\sqrt{8}-\sqrt{7}$.
(5) $2^{\frac{1}{3}}+5^{\frac{1}{3}}$ is less than 3.

5. If every two of the three lines be greater than the third, a triangle can be formed with the three lines.

6. It is obvious that if the sides of the equilateral triangle, the square, the pentagon, be given in magnitude, the areas of these regular figures can be determined, and consequently the surfaces of each of the five regular solids.

The surface of the tetrahedron is equal to the area of four equal and equilateral triangles.

The surface of the hexahedron is the area of six equal squares.

The surface of the octahedron is equal to the areas of eight equal and equilateral triangles.

The surface of the dodecahedron is equal to the area of twelve equal equilateral and equiangular pentagons.

The surface of the icosahedron is equal to the area of twenty equal and equilateral triangles.

A sphere can be inscribed within, and another circumscribed about, each of the five regular solids, in the same manner as a circle can be inscribed within and another circumscribed about each of the plane regular figures, the equilateral triangle, the square, the pentagon.

Suppose a sphere described about a regular tetrahedron, the surface of the sphere will pass through the four points of the trihedral angles of the tetrahedron. If lines be supposed to be drawn from these points to the centre of the circumscribing sphere, these lines will be each equal to the radius of the sphere. Hence the tetrahedron may be conceived to be divided into four equal triangular pyramids, whose common vertices are at the centre of the sphere, the three edges of each, radii of the sphere, and their four bases the equilateral triangles which make up the surface of the tetrahedron.

Since the content of any pyramid is equal to one-third of the content of a prism on the same base and of the same altitude, the contents of the four equal pyramids can be found, and the content, or volume, of the tetrahedron determined.

In a similar manner it may be shown that:—

The volume of the hexahedron is equal to the volumes of six equal pyramids whose bases are equal squares.

The volume of the octahedron is equal to the volumes of eight equal pyramids whose bases are equal equilateral triangles.

The volume of the dodecahedron is equal to the volumes of twelve equal pyramids whose bases are equal equilateral and equiangular pentagons.

The volume of the icosahedron is equal to the volumes of twenty equal pyramids whose bases are equal equilateral triangles.

XXXIX.

1. $x-x^{-1}=\frac{2b}{a}$, $x+x^{-1}=\frac{2(a^2+b^2)^{\frac{1}{2}}}{a}$.

3. Since $c=b(1-b^2)^{\frac{1}{2}}+b(1-a^2)^{\frac{1}{2}}$;

$$\therefore\ c-a(1-b^2)^{\frac{1}{2}}=b(1-a^2)^{\frac{1}{2}},$$
$$c^2-2ac(1-b^2)^{\frac{1}{2}}+a^2-a^2b^2=b^2-b^2a^2,$$
$$c^2+a^2-b^2=2ac(1-b^2)^{\frac{1}{2}},$$
$$c^4+a^4+b^4+2a^2c^2-2b^2c^2-2a^2b^2=4a^2c^2-4a^2b^2c^2;$$
$$\therefore\ 4a^2b^2c^2=2a^2c^2+2b^2c^2+2a^2b^2-a^4-b^4-c^4.$$

See Section IV., XIII., Ex. 10, p. 32.

5. Since $x(a^2-y^2)^{\frac{1}{2}}+y(a^2-x^2)^{\frac{1}{2}}=a^2$,

$x(a^2-y^2)^{\frac{1}{2}}=a^2-y(a^2-x^2)^{\frac{1}{2}}$, by squaring each side,

then $x^2a^2-x^2y^2=a^4-2a^2y(a^2-x^2)^{\frac{1}{2}}+a^2y^2-x^2y^2$,

and $2y(a^2-x^2)^{\frac{1}{2}}=a^2-(x^2-y^2)$, squaring again;

$\therefore\ 4y^2a^2-4y^2x^2=a^4-2a^2(x^2-y^2)+(x^2-y^2)^2$,

whence $a^4-2a^2(x^2+y^2)+(x^2+y^2)=0$

$\therefore\ a^2-(x^2+y^2)=0$ and $x^2+y^2=a^2$.

6. Since $x^{\frac{1}{2}}+y^{\frac{1}{2}}=a^{\frac{1}{2}}$, $x^{\frac{1}{2}}+y^{\frac{1}{2}}-a^{\frac{1}{2}}=0$,

and $x+y+a+2x^{\frac{1}{2}}y^{\frac{1}{2}}-2a^{\frac{1}{2}}x^{\frac{1}{2}}-2a^{\frac{1}{2}}y^{\frac{1}{2}}=0$

$$x+y+a=2(a^{\frac{1}{2}}x^{\frac{1}{2}}+a^{\frac{1}{2}}y^{\frac{1}{2}}-x^{\frac{1}{2}}y^{\frac{1}{2}})$$
$$(x+y+a)^2=4(ax+ay+xy+2ax^{\frac{1}{2}}y^{\frac{1}{2}}-2xa^{\frac{1}{2}}y^{\frac{1}{2}}-2ya^{\frac{1}{2}}x^{\frac{1}{2}})$$
$$=4\left\{\frac{(x+y-a)^2}{4}+a(x+y)+a(a-x-y)+a(x+y-a\right\}$$
$$=(x+y-a)^2+xa(x+y)=2(x^2+y^2+a^2).$$

The truth of this also may be verified by numerical examples, as, for instance, when $(16)^{\frac{1}{2}}+(25)^{\frac{1}{2}}=(81)^{\frac{1}{2}}$.

8. If $(a+b^{\frac{1}{2}}+c^{\frac{1}{2}}+d^{\frac{1}{2}})^{\frac{1}{2}}=x^{\frac{1}{2}}+y^{\frac{1}{2}}+z^{\frac{1}{2}}$,

Then $a+b^{\frac{1}{2}}+c^{\frac{1}{2}}+d^{\frac{1}{2}}=x+y+z+2(xy)^{\frac{1}{2}}+2(xz)^{\frac{1}{2}}+2(yz)^{\frac{1}{2}}$,

and $x+y+z=a$, $2(xy)^{\frac{1}{2}}=b^{\frac{1}{2}}$, $2(xz)^{\frac{1}{2}}=c^{\frac{1}{2}}$, $2(yz)^{\frac{1}{2}}=d^{\frac{1}{2}}$.

Hence $2(xy)^{\frac{1}{2}}.2(xz)^{\frac{1}{2}}.2(yz)^{\frac{1}{2}}=b^{\frac{1}{2}}c^{\frac{1}{2}}d^{\frac{1}{2}}$, or $8xyz=(bcd)^{\frac{1}{2}}$, and $x+y+z=a$,

$\therefore\ 2a(bcd)^{\frac{1}{2}}=16xyz(x+y+z)$.

But $4xy=b$, $4xz=c$, $4yz=d$, $\therefore\ bc+bd+cd=16(x^2yz+y^2xz+z^2xy)=16xyz(x+y+z)$.

Hence, therefore, $2a(bcd)^{\frac{1}{2}}=bc+bd+cd$.

9. Let $\{r+(r^2+q^3)^{\frac{1}{2}}\}^{\frac{1}{3}}=a, \{r-(r^2+q^3)^{\frac{1}{2}}\}^{\frac{1}{3}}=b$;

then $x^3=(a+b)^3=a^3+b^3+3ab(a+b)=2r-3qx$;

$\therefore\ x^3+3qx-2r=0$.

10. Find y and z in terms of x, and substitute these values in $x^{-1}+y^{-1}+z^{-1}=1$, whence x is known, and y, z by substitution. Lastly, substitute these values of x, y, x in $ax^2+by^2+cz^2$.

XL.

4. Since $a(b-c)^2-c(b+c)^2=c$;

$$\therefore\ \frac{a}{c}=\frac{(b+c)^2}{(b-c)^2}, \text{ and } \frac{a^{\frac{1}{2}}}{c^{\frac{1}{2}}}=\frac{b+c}{b-c}.$$

Hence $\frac{2b}{b-c}=\frac{a^{\frac{1}{2}}+c^{\frac{1}{2}}}{c^{\frac{1}{2}}}$, and $\frac{2c}{b-c}=\frac{a^{\frac{1}{2}}-c^{\frac{1}{2}}}{c^{\frac{1}{2}}}$; $\therefore\ \frac{b}{c}=\frac{a^{\frac{1}{2}}+c^{\frac{1}{2}}}{a^{\frac{1}{2}}-c^{\frac{1}{2}}}$, and $\frac{c}{b}\cdot\frac{a^{\frac{1}{2}}+c^{\frac{1}{2}}}{a^{\frac{1}{2}}-c^{\frac{1}{2}}}=1$.

5. In the verification, when $x^2=ab$ is found,

$$\frac{(a+x)(b+x)}{(a-x)(b-x)}=\frac{2ab+(a+b)a^{\frac{1}{2}}b^{\frac{1}{2}}}{2ab-(a+b)a^{\frac{1}{2}}b^{\frac{1}{2}}}=\frac{2a^{\frac{1}{2}}b^{\frac{1}{2}}+a+b}{2a^{\frac{1}{2}}b^{\frac{1}{2}}-(a+b)}=\frac{(a^{\frac{1}{2}}+b^{\frac{1}{2}})^2}{-(a^{\frac{1}{2}}-b^{\frac{1}{2}})^2};$$

$$\therefore\left\{\frac{(a+x)(b+x)}{(a-x)(b-x)}\right\}^2=\left\{\frac{(a^{\frac{1}{2}}+b^{\frac{1}{2}})^2}{-(a^{\frac{1}{2}}-b^{\frac{1}{2}})^2}\right\}^2=\left\{\frac{a^{\frac{1}{2}}+b^{\frac{1}{2}}}{a^{\frac{1}{2}}-b^{\frac{1}{2}}}\right\}^4.$$

6. Express the products of the factors in the numerators of the two fractions, as the sums of two squares, and $1+\frac{(a-b)^2}{(ab-1)^2}=1+\frac{(c-d)^2}{(cd+1)^2}$.

7. Suppose $\frac{1+ax+by}{\{(1+a^2+b^2)(1+x^2+y^2)\}^{\frac{1}{2}}}=\pm 1$,

then $(1+ax+by)^2=(1+a^2+b^2)(1+x^2+y^2)$, whence $(a-x)^2+(b-y)^2+(ay-bx)^2=0$.

8. The quotient is $\surd(1-x^2)\surd(1-y^2)-xy$.

9. From $xy+xz+yz=1$, $x=\frac{1+yz}{z+y}$, and $1+x^2=\frac{(y+z)^2+(1-yz)^2}{(z+y)^2}$;

$$\text{also } (z+y)^2=\frac{(1+z)^2(1+y^2)}{1+x^2},\ \therefore\ x\left\{\frac{(1+z^2)(1+y^2)}{1+x^2}\right\}^{\frac{1}{2}}=x(z+y).$$

$$\text{Similarly } y\left\{\frac{(1+z^2)(1+x^2)}{1+y^2}\right\}^{\frac{1}{2}}=y(x+z),\text{ and } z\left\{\frac{(1+x^2)(1+y^2)}{1+z^2}\right\}^{\frac{1}{2}}=z(x+y).$$

10. Since $x^2+y^2+z^2+2xyz=1$, $2xyz+z^2=1-x^2-y^2$, and adding x^2y^2 to these equals, $x^2y^2+2xyz+z^2=1-x^2-y^2+x^2y^2$, or $(1-x^2)(1-y^2)=(z+xy)^2$.

Similarly $(1-y^2)(1-z^2)=(x+yz)^2$ and $(1-x^2)(1-z^2)=(y+xz)^2$.

XLI.

1. See Art. 15, p. 16. 2. See Art. 16, p. 17.

5. The former is the greater or the less, according as x^2 is greater or less than ab.

6. Here $a^{\frac{3}{2}}+a^{\frac{1}{2}}b$ is greater or less than $ab^{\frac{1}{2}}+b^{\frac{3}{2}}$, according as $a^{\frac{1}{2}}(a+b)$ is greater or less than $b^{\frac{1}{2}}(a+b)$.

7. Since $a(1-b^2)^{\frac{1}{2}}+b(1-a^2)^{\frac{1}{2}}<1$, then $a(1-b^2)^{\frac{1}{2}}<1-b(1-a^2)^{\frac{1}{2}}$,

and $a^2-a^2b^2<1-2b(1-a^2)^{\frac{1}{2}}+b^2-a^2b$, or $2b(1-a^2)^{\frac{1}{2}}<(b^2-a^2)+1$,

also $4b^2-4a^2b^2<(b^2-a^2)^2+2(b^2-a^2)+1$, whence $(b^2+a^2)^2-2(b^2+a^2)+1>0$;

$\therefore\ b^2+a^2-1>0$ and $a^2+b^2>1$.

8. Let $a>b$, $b>c$, then $a^2-2ab+b^2>0$, and $a^2+2ab+b^2>4ab$;

$\therefore\ a+b>2(ab)^{\frac{1}{2}}$, similarly $b+c>2(bc)^{\frac{1}{2}}$, $a+c>2(ac)^{\frac{1}{2}}$;

$\therefore\ 2(a+b+c)>2\{(ab)^{\frac{1}{2}}+(bc)^{\frac{1}{2}}+(ac)^{\frac{1}{2}}\}$, and $a+b+c>(ab)^{\frac{1}{2}}+(bc)^{\frac{1}{2}}+(ac)^{\frac{1}{2}}$.

In a similar way may be shewn that $a+b+c+d>4(abcd)^{\frac{1}{4}}$.

For $a+b>2(ab)^{\frac{1}{2}}$, $c+d>2(cd)^{\frac{1}{2}}$;

$\therefore\ a+b+c+d>2\{(ab)^{\frac{1}{2}}+(cd)^{\frac{1}{2}}\}>4\{(ab)^{\frac{1}{2}}(cd)^{\frac{1}{2}}\}^{\frac{1}{2}}>4(abcd)^{\frac{1}{4}}$.

9. Since $x=a+b$, and $y=c+d$,

$\therefore\ xy=(a+b)(c+d)=(a^{\frac{1}{2}}c^{\frac{1}{2}}+b^{\frac{1}{2}}d^{\frac{1}{2}})^2+(a^{\frac{1}{2}}d^{\frac{1}{2}}-b^{\frac{1}{2}}c^{\frac{1}{2}})^2$;

consequently $xy>(a^{\frac{1}{2}}c^{\frac{1}{2}}+b^{\frac{1}{2}}d^{\frac{1}{2}})^2$, and $x^{\frac{1}{2}}y^{\frac{1}{2}}>a^{\frac{1}{2}}c^{\frac{1}{2}}+b^{\frac{1}{2}}d^{\frac{1}{2}}$.

10. If the sum of first and third expressions be taken to be greater than the second, the sum of the squares of the first and third will also be greater than the square of the second. From this inequality may be deduced that

$2x^2y^2+2x^2z^2+2y^2z^2-x^4-y^4-z^4$ is greater than zero;

and as this expression is homogeneous, like expressions may be deduced by assuming the sum of the first and second greater than the third, and the sum of the second and third greater than the first. See Section IV., XIII. Ex. 10, p. 32.

12. If $\dfrac{ab(x^2-y^2)+xy(a^2-b^2)}{(a^2+b^2)(x^2+y^2)}<\dfrac{1}{2}$,

Then $(a^2+b^2)(x^2+y^2)>2ab(x^2-y^2)+2xy(a^2-b^2)$,

and $(a^2-2ab+b^2)x^2+(a^2+2ab+b^2)y^2>2xy(a^2-b^2)$;

$\therefore\ (a-b)^2x^2-2(a-b)(a+b)xy+(a+b)^2y^2>0.$

$\therefore\ (a-b)x-(a+b)y>0$; $(a-b)x>(a+b)y$; and $\dfrac{x}{y}>\dfrac{a+b}{x-b}$.

XLII.

4. $$\frac{x^{\frac{m+n}{n}}-x^{\frac{n+m}{m}}}{x^{\frac{2m-n}{n}}-x^{\frac{2n-m}{m}}}=\frac{x^{\frac{m}{n}+1}-x^{\frac{n}{m}+1}}{x^{\frac{2m}{n}-1}-x^{\frac{2n}{m}-1}}=\frac{x\left\{x^{\frac{m}{n}}-x^{\frac{n}{m}}\right\}}{\frac{1}{x}\left\{x^{\frac{2m}{n}}-x^{\frac{2n}{m}}\right\}}=\frac{x^2}{x^{\frac{m}{n}}+x^{\frac{n}{m}}}.$$

5. $\dfrac{a\pm b}{a^{\frac{1}{n}}\pm b^{\frac{1}{n}}}=\dfrac{(a^{\frac{1}{n}})^n\pm(b^{\frac{1}{n}})^n}{(a^{\frac{1}{n}})\pm(b^{\frac{1}{n}})}=\dfrac{x^n\pm y^n}{x\pm y}$, which is of the forms Section IV., Art. 13, p. 22.

$\dfrac{a^{mn}\pm b^{mn}}{a^m\pm b^n}=\dfrac{(a^m)^n\pm(b^n)^m}{(a^m)\pm(b^n)}=\dfrac{x^n\pm y^m}{x\pm y}$, not of the required forms, except when $m=n$.

$\dfrac{a^m\pm b^n}{a^p\pm b^q}$. If $\dfrac{m}{p}=\dfrac{n}{q}=r$; then $\dfrac{a^m\pm b^n}{a^p\pm b^q}=\dfrac{x^r\pm y^r}{x\pm y}$.

6. $a^n-b^{\frac{n}{m}}=a^n-(b^{\frac{1}{m}})^n=a^n-x^n.$ $\qquad a^m-b=a^m-(b^{\frac{1}{m}})^m=a^m-x^m.$

Here a^n-x^n and a^m-x^m have a common divisor when m and n are odd or even numbers.

7. Let $a^m-b^n=(a^m)-(b^n)=x-y$;

then $a^{\frac{1}{n}}-b^{\frac{1}{m}}=(a^m)^{\frac{1}{mn}}-(b^n)^{\frac{1}{mn}}=x^{\frac{1}{mn}}-y^{\frac{1}{mn}}$; and the given expressions are reduced to the forms $a-b$ and $a^{\frac{1}{n}}-b^{\frac{1}{n}}$.

9. Since $x^z=z^x$, $\therefore\ z=x^{\frac{z}{x}}$, $\therefore\ \dfrac{z}{x}=x^{\frac{z}{x}-1}$, and $\dfrac{x}{z}=x^{1-\frac{z}{x}}$,

$$\therefore\ \left\{\frac{x}{z}\right\}^{\frac{x}{z}}=\left\{x^{1-\frac{z}{x}}\right\}^{\frac{x}{z}}=x^{\frac{x}{z}-1}$$

When $x=2z$, then $2^2=2z$, $\therefore\ z=2$.

10. Since $x=\left\{\dfrac{a+b}{a-b}\right\}^{\frac{2mn}{n-m}}$, $\therefore\ x^{\frac{1}{m}}=\left\{\dfrac{a+b}{a-b}\right\}^{\frac{2n}{n-m}}$, and $x^{\frac{1}{n}}=\left\{\dfrac{a+b}{a-b}\right\}^{\frac{2m}{n-m}}$.

Then $$\frac{a^2-b^2}{2(a^2+b^2)}\cdot\left\{x^{\frac{1}{m}}+x^{\frac{1}{n}}\right\}=\frac{a^2-b^2}{2(a^2+b^2)}\cdot\left\{\left(\frac{a+b}{a-b}\right)^{\frac{2n}{n-m}}+\left(\frac{a+b}{a-b}\right)^{\frac{2m}{n-m}}\right\};$$

$$=\frac{a^2+b^2}{2(a^2+b^2)}\cdot\left\{\frac{a+b}{a-b}\right\}^{\frac{2m}{n-m}}\cdot\left\{1+\left(\frac{a+b}{a-b}\right)^{\frac{2(n-m)}{n-m}}\right\}$$

$$=\frac{a^2-b^2}{2(a^2+b^2)}\cdot\left\{\frac{a+b}{a-b}\right\}^{\frac{2m}{n-m}}\cdot\frac{2(a^2+b^2)}{(a-b)^2}$$

$$=\frac{a^2-b^2}{(a-b)^2}\cdot\left\{\frac{a+b}{a-b}\right\}^{\frac{2m}{n-m}}=\frac{a+b}{a-b}\cdot\left\{\frac{a+b}{a-b}\right\}^{\frac{2m}{n-m}}=\left\{\frac{a+b}{a-b}\right\}^{\frac{n+m}{n-m}}.$$

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The portion of Newton's Principia which is here presented to the Student contains the solution of the principal problem in celestial Mechanics, and must be interesting to all, even those who do not intend to follow out the more complicated problems of the Lunar and Planetary Theories.

The study of the geometrical methods employed by Newton cannot be too strongly recommended to a student who intends to pursue Mathematics, whether Pure or Applied, to the higher branches; for he will, under this training, be less likely to work in the dark when he uses more intricate machinery.

I have endeavoured in this work to explain how several of the results obtained in the Differential and Integral Calculus can be represented in a geometrical form; and I have shown how, in a large class of problems, the geometrical methods are at least as good an 'open sesame' as the Differential Calculus.

In this, the third edition, I have given Solutions or Hints for the solutions of all the problems, in order that a student may, unaided by a tutor, have all the advantages which the book supplies.

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ELEMENTARY ALGEBRA,
WITH BRIEF NOTICES OF ITS HISTORY.

SECTION VII.

SIMPLE EQUATIONS.

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EQUATIONS.

The investigation of the properties of equations and the methods of the solution constitute one of the most important portions of the science of Algebra. An equation has been defined to be the equality of two algebraical expressions, denoting some relation between the known and unknown quantity or quantities involved in the equation. The expressions "known quantities" and "unknown quantities" are terms employed in equations; the former denoting those which are supposed to be known or assignable, and the latter those which are to be determined. The unknown quantities are always dependent for their values on the known quantities involved in the equations, and the solution of equations is simply confined to determine "the unknown quantities" in terms of "the known quantities."

By means of the notations assumed to designate known and unknown quantities, it is always possible to distinguish what is given from what is required, or in other words the data and quæsita of a problem.

By the solution of an equation or a set of equations, is meant the process whereby the value or values of the unknown quantity or quantities can be determined in terms of the known quantities.

An equation may involve one unknown quantity, or more than one; and all equations may be separated into two general classes, *determinate equations* and *indeterminate equations;* the former class including equations which admit of a limited number of values of the unknown quantities, and the latter an unlimited number.

Equations both determinate and indeterminate may not admit of any intelligible solution. This arises from some inconsistent relations either in the data or in the quæsita, or in both of them, and will always become manifest in the results of the solution.

Every single equation expresses a condition—some specific relation between the known and unknown quantities, and the number of given independent equations may be *equal to*, *greater than*, or *less than* the number of the unknown quantities involved.

When the number of given independent equations is equal to the number of the unknown quantities involved in the equations, the equations are *determinate*, or limited to certain single values of the unknown quantities. If there be two independent equations, each involving two unknown quantities, in this case two unknown quantities are to be determined subject to two independent conditions. Between the two given equations one of the unknown quantities can be eliminated, and a third equation is found involving only one unknown quantity, which can be determined in terms of the known quantities. The other unknown quantity is also determined by substituting in either of the given equations the value of that already determined. Hence it appears that two independent equations involving two unknown quantities are sufficient to determine the values of the two unknown quantities.

In the same manner, if there be three independent equations involving three unknown quantities; the first unknown quantity can be eliminated between the first and second equations, also between the first and third or second and third equations; and thus are derived two equations, each involving two unknown quantities, which case has been considered. Hence it is manifest that three independent equations involving three unknown quantities, are sufficient to determine the three unknown quantities.

When the number of equations is *greater than* the number of unknown quantities; if some of the equations can be derived from others so that the number of independent equations is really equal to the number of unknown quantities, the equations admit of a definite solution. If, however, all the equations are *independent*, they may express incompatible conditions, and every set of the equations, equal to the number of unknown quantities involved, will give different sets of solutions.

If the number of independent equations be less than the number of unknown quantities, the given conditions are insufficient to determine the unknown quantities, and they will therefore be *indeterminate.*

Equations may also be divided into different orders. An equation of the first, second, third, fourth, &c., order, is said to be of one, two, three, four, &c., dimensions, according as the highest power of the unknown quantity or quantities involved is the first, second, third, fourth, &c., power.

Equations of the first order, which involve when reduced only the first power of the unknown quantity, are named simple equations.

Equations which involve the second power of the unknown quantity, with or without the first power, are named quadratic equations.

Equations which involve the third power, with or without inferior powers of the unknown quantity or quantities, are named cubic equations.

Equations which involve the fourth power of the unknown quantity or quantities, with or without inferior powers, are named biquadratic equations.

And equations which involve higher powers of the unknown quantity, are named according to the index of the highest power of the unknown quantity involved in the equations.

Any process, whereby any quantity common to two or more equations, can be made to disappear in the resulting equation or equations, is called "the elimination of that quantity."

The elimination of quantities from equations is an important process, essential in the solution of all equations which involve more than one unknown quantity.

When there are two independent equations with two unknown quantities, x and y; if one of them, x, be eliminated, the resulting equation will involve only y, the other unknown quantity.

When there are three independent equations with three unknown quantities, x, y, z; any one of them, x, can be eliminated between the first and second, the first and third, and between the second and third equations; and each of the three resulting equations involves y and z. As the values of x, y, z in the given equations are *determinate*, it follows that the values of y and z are the same in each of these last three equations, and any one of these equations is derivable from the other two, and consequently that there are only two independent equations resulting from the elimination.

Also from the two equations involving y and z, if y, one of them, be eliminated, the final equation will involve z, one unknown quantity. And this final equation may involve the first, second, third, &c., powers of z, depending on the powers of x, y, z, the unknown quantities in the original equations.

If there be four independent equations, involving four unknown quantities, by the successive elimination of one of them, a final equation will at length be obtained involving only one unknown quantity.

When two equations involve two unknown quantities, x and y; if when y is changed into x and x into y in the two equations, they retain the same form, they are called *simultaneous equations*, from the mutual interchange of the two unknown quantities.

Also when there are three equations and three unknown quantities, x, y, z; if after the mutual interchange of these quantities in the three equations, they retain the same form, they also are simultaneous equations.

The value of the unknown quantity in an equation is named the *root* of the equation. To solve an equation is to determine its root or roots, if there be more than one. It is always possible to ascertain whether the root found by any process is the true value of the unknown quantity, as this quantity when substituted in the given equation for the unknown quantity always reduces the equation to an identity, or to the form $0=0$, if all the terms are brought to one side of the equation. And this value of the unknown quantity is said "to satisfy the equation," as 3 satisfies the equation $5x+7=22$; for when 3 is substituted for x, both sides of the equation become identical in value, thus $15+7=22$ or $22=22$. And when 3 is substituted for x in the equation $x^2-5x+6=0$, it becomes $9-15+6=0$ or $0=0$.

And it is important to remark that this is the criterion by which it is known that the true root of an equation has been determined. In problems proposed for solution by means of equations the first step is to consider the nature of the question, and to discriminate accurately what is given, and what is required to be found, and to reject every condition or circumstance which has no necessary connection or relation to the result required.

The next step is to assume letters to denote the data and quæsita of the problem, taking care to avoid ambiguity or contradiction in the assumptions.

After this designation of the known and unknown quantities, the problem must, with the aid of symbols of operation, be translated into algebraical language; that is, the conditions of the problem must be denoted by equations, making no distinction between the known and unknown quantities; and when this is effected it will then appear whether the problem is determinate or indeterminate.

Having formed the equations, the last process consists in their solution. It will be found that some methods are more operose and tedious than others. The most direct and simple methods of solution are to be preferred when such are possible; but these are not always subject to any general rule, but rather depend on the skill which is acquired by experience.

In the solution of equations involving more than one unknown quantity, it will sometimes be found that they are so involved that they cannot easily be separated by any of the ordinary methods. In such cases some artifice is generally possible, by means of which the equations may become simplified and their solution rendered practicable. It is impossible to lay down any rules for the employment of artifices in the solution of equations. The manner in which the unknown quantities are involved only can suggest what artifices are possible for rendering the equations more easy of solution. Instead of assuming single letters to denote unknown quantities, it will be found convenient in some cases to assume for them the sum and difference of the two unknown quantities, in others the product and the quotient, &c.

All homogeneous equations of two unknown quantities may be simplified by assuming for one of them some unknown multiple of the other, and making the substitution in the two equations. By this means both of the unknown quantities can be eliminated, and the resulting equation will involve only the quantity which denotes the unknown multiple.

The same method can be extended to equations involving three or more than three unknown quantities.

It has been pointed out that equations proposed for solution may be deducible from, or contradictory to, one another. In the former case, in the attempted solution, some quantity may be found equal to itself, and in the latter some absurdity will follow, as a greater quantity equal to a less. And this dependency or inconsistency of equations is not always obvious until the identity or absurdity appears in the final result of the operation.

SIMPLE EQUATIONS.

Art. 1. Simple equations, or equations of the first degree, may involve one, two, three, or more than three unknown quantities.

If a simple equation involve only one unknown quantity, the equation admits of only one value of that quantity.

The solution of all equations in general will be found to depend on the axioms stated on the eighth page of Section IV.

In the application of the second and third axioms in the solution of equations, it will be seen that if an equal quantity be added to or subtracted from each side of an equation, the process is the same as simply transposing or removing the quantity from one side to the other side of the equation with its algebraical sign changed; as the addition of $+a$ to each side of the equation $mx-a=b$, is $mx-a+a=b+a$; and is equivalent to the removal of $-a$ from one side to $+a$ on the other, as $mx=b+a$. And the subtraction of $+a$ from each side of the equation $mx+a=b$, is $mx+a-b=b-a$, and is equivalent to the removal of $+a$ from one side to $-a$ on the other, as $mx=b-a$. Hence the second and third axioms are united in the general rule, that any quantity may be transposed from one side to the other side of an equation by changing its sign. By the application of this rule the terms involving the unknown and known quantities can be separated and placed together, the former on one side and the latter on the other side of the equation; and when the equation is reduced to the form $mx=c$, by the seventh axiom the unknown quantity is determined to be $x=\frac{c}{m}$.

And by this axiom an equation may be simplified if every term of it contain the same common factor which does not involve any unknown quantity.

Also by the sixth axiom an equation may be cleared from fractions by multiplying each side of the equation by the denominators of the fractions, or by their least common multiple, as the equation $\frac{x}{a}+\frac{x}{b}=c$ becomes $bx+ax=abc$ by multiplying each side by a and b, the denominators of the fractions.

And further, the signs of all the terms of an equation may be changed, as the process is equivalent to multiplying the terms on each side by -1.

It is also obvious that if the same quantity with the same sign be found on both sides of an equation, it may be removed from each side.

If a term involving the unknown quantity in an equation be affected with a fractional index, by means of the eleventh axiom the fractional index can be reduced to an integral index, and the separation of the known and unknown quantities become possible, as if $(mx+b)^{\frac{1}{2}}=c$, by raising each side of the equation to the second power, it becomes $mx+b=c^2$, and the known quantity b can be separated from x the unknown quantity.

2. *To solve the general equations* $a_1x+b_1y=c_1$, $a_2x+b_2y=c_2$, *involving two unknown quantities, or to find the values of* x *and* y *which satisfy both equations.*

When there are given two independent equations of the first degree which simultaneously involve two unknown quantities x and y, it is obvious that, in order to determine the value of one of them, there must first be deduced from the two given equations an equation which shall contain only one of the unknown quantities.

There are three methods by which the values of x and y may be determined from these equations, $a_1x+b_1y=c_1$, $a_2x+b_2y=c_2$.

First method: By finding the value of one of the unknown quantities in terms of the other and known quantities, from one equation, and substituting it in the other equation.

Let the value of x be found from the first equation $a_1x+b_1y=c_1$.

By transposing b_1y, $a_1x=c_1-b_1y_1$ and $x=\frac{c_1-b_1y}{a_1}$, then substituting this value of x in the second equation, $a_2x+b_2y=c_2$, it becomes $\frac{a_2(c_1-b_1y)}{a_1}+b_2y=c_2$, clearing the equation from fractions, $a_2c_1-a_2b_1y+a_1b_2y=a_1c_2$, and by transposition $(a_1b_2-a_2b_1)y=a_1c_2-a_2c_1$, and $\therefore\ y=\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$, and the value of x is found by substituting this value of y in either of the two given equations.

$$\therefore\ x=\frac{c_1-b_1y}{a_1}=\frac{c_1}{a_1}-\frac{b_1y}{a_1}=\frac{c_1}{a_1}-\frac{b_1}{a_1}\cdot\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1},$$

$$\text{or } x=\frac{b_2c_1-b_1c_2}{a_1b_2-a_2b_1}.$$

And these values of x and y, when substituted in the given equations, satisfy both equations by reducing each to an identity.

$$\text{Thus } a_1x+b_1y=\frac{a_1(b_2c_1-b_1c_2)}{a_1b_2-a_2b_1}+\frac{b_1(a_1c_2-a_2c_1)}{a_1b_2-a_2b_1}$$

$$=\frac{a_1b_2c_1-a_1b_1c_2+a_1b_1c_2-a_2b_1c_1}{a_1b_2-a_2b_1}$$

$$=\frac{(a_1b_2-a_2b_1)c_1}{a_1b_2-a_2b_1}=c_1.$$

$$\text{And } a_2x+b_2y=\frac{a_2(b_2c_1-b_1c_2)}{a_1b_2-a_2b_1}+\frac{b_2(a_1c_2-a_2c_1)}{a_1b_2-a_2b_1}$$

$$=\frac{a_2b_2c_1-a_2b_1c_2+a_1b_2c_2-a_2b_2c_1}{c_1b_2-a_2b_1}$$

$$=\frac{(a_1b_2-a_2b_1)c_2}{a_1b_2-a_2b_1}=c_2.$$

Second method: By equating the values of the same unknown quantity from each equation.

From the first equation, $a_1x+b_1y=c_1$, $x=\frac{c_1-b_1y}{a_1}$,

From the second, $a_2x+b_2y=c_2$, $x=\frac{c_2-b_2y}{a_2}$.

Next equating these values of x,

$$\therefore\ \frac{c_1-b_1y}{a_1}=\frac{c_2-b_2y}{a_2}.$$

Clearing the equation from fractions, $a_2c_1-a_2b_1y=a_1c_2-a_1b_2y$, and $(a_1b_2-a_2b_1)y=a_1c_2-a_2c_1$ by transposition,

$$\therefore\ y=\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1},$$

and the value of x can be found as before,

$$x=\frac{b_2c_1-b_1c_2}{a_1b_2-a_2b_1}.$$

Third method: By equalising the coefficients of one of the unknown quantities in each equation, and then eliminating it by addition or subtraction of the equations.

In the given equations $a_1x+b_1y=c_1$, $a_2x+b_2y=c_2$ if a_1, a_2, the coefficients of x, be prime to each other, multiply the first equation by a_2, and the second equation by a_1,

then $a_1a_2x+a_2b_1y=a_2c_1$, and $a_1a_2x+a_1b_2y=a_1c_2$.

Next add or subtract these equations according as the signs of the coefficients of x are either one positive and the other negative, or both positive or both negative, and the resulting equation is

$$(a_2b_1-a_1b_2)y=a_2c_1-a_1c_2,$$

$$\therefore\ y=\frac{a_2c_1-a_1c_2}{a_2b_1-a_1b},\ \text{ and }\ x=\frac{b_1c_2-b_2c_1}{a_2b_1-a_1b_2},\ \text{as before.}$$

It may be remarked that the signs of the terms in the values of x and y appear with signs contrary to the terms in the values found by the other methods. This circumstance arises from subtracting the second equation from the first. If the first had been subtracted from the second, the signs would not have been changed.

The values of x and y are of the same value in both instances, as a fraction remains unchanged in value when the signs of every term of the numerator and denominator are changed.

Also, if the known quantities involved in the equations be positive or negative integers or fractions, the values of x and y in general will be either integers or fractions positive or negative.*

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* The values of x and y in two simple equations when reduced to the form

It is possible, however, that the relation of the constants in the equations to each other may be such that the values of x and y may assume the forms of $\frac{0}{m}$, $\frac{m}{0}$, or $\frac{0}{0}$.

If the values of one of the unknown quantities assume the form of $\frac{0}{0}$, and that of the other $\frac{m}{0}$:

Let the value of x assume the form $\frac{0}{0}$, then from $x=\frac{b_2c_1-b_1c_2}{a_1b_2-a_2b_1}$,

$$b_2c_1-b_1c_2=0, \text{ and } a_1b_2-a_2b_1=0,$$

$$\therefore\ b_2c_1=b_1c_2, \text{ and } \frac{c_1}{c_2}=\frac{b_1}{b_2};$$

$$\text{also } a_1b_2=a_2b_1, \text{ and } \frac{a_1}{a_2}=\frac{b_1}{b_2}. \text{ Hence } \frac{c_1}{c_2}=\frac{a_1}{a_2}.$$

Next let the value of y assume the form $\frac{m}{0}$, then from $y=\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}$,

$$a_1b_2-a_2b_1=0, \text{ and } a_1c_2-a_2c_1=m,$$

$$\therefore\ a_1b_2=a_2b_1, \text{ and } \frac{a_1}{a_2}=\frac{b_1}{b_2}; \text{ also } a_1c_2=m+a_2c_1, \text{ and } \frac{a_1}{a_2}=\frac{m}{a_2c_2}+\frac{c_1}{c_2}.$$

But from the former $\frac{a_1}{a_2}=\frac{c_1}{c_2}$, $\therefore\ \frac{c_1}{c_2}=\frac{m}{a_2c_2}+\frac{c_1}{c_2}$, which is impossible, if m have a definite value, therefore the given equations are incompatible, and do not admit of any definite values of x and y.

If, however, the values of x and y both assume the form of $\frac{0}{0}$, then $a_1c_2-a_2c_1=0$, $a_1b_2-a_2b_1=0$, and $b_2c_1-b_1c_2=0$,

$$\therefore\ \frac{a_1}{a_2}=\frac{b_1}{b_2},\ \frac{a_1}{a_2}=\frac{c_1}{c_2},\ \frac{b_1}{b_2}=\frac{c_1}{c_2}, \text{ if } \frac{a_1}{a_2}=n, \text{ then } \frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}=n,$$

$$\text{and } a_1=na_2,\ b_1=nb_2,\ c_1=nc_2,$$

which shews that the co-efficients of one equation are respectively equimultiples of the corresponding coefficients of the other. Hence it follows that one of the equations may be deduced from the other, and the two equations do not express two independent conditions.*

$a_1x+b_1y=c_1$, $a_2x+b_2y=c_2$, can be found by substituting the particular for the general values of the constants in the values of x and y, as, for example,

To find the values of x and y in the equations $5x+8y=31$, $2x-3y=0$.

$$\text{Generally } x=\frac{b_2c_1-b_1c_2}{a_1b_2-a_2b_1},\quad y=\frac{a_1c_2-a_2c_1}{a_1b_2-a_2b_1}.$$

Here $a_1=5$, $b_1=8$, $c_1=31$; $a_2=2$, $b_2=-3$, $c_2=0$.

$$\therefore\ x=\frac{-93-0}{-15-16}=\frac{-93}{-31}=+3,$$

$$y=\frac{0-62}{-15-16}=\frac{-62}{-31}=+2.$$

$\therefore$ 3 and 2 are the values of x and y in the equations $5x+8y=31$, $2x-3y=0$.

* The subject of indeterminate equations arises from the fact of some problems involving more unknown quantities than independent conditions to determine them.

And in this case there is only *one independent equation* involving two unknown quantities. Hence the equation is indeterminate, and admits of an indefinite number of solutions, as it is evident that for every value of x in the equation there will always be some corresponding value of y.

If there be three equations involving two unknown quantities, and if one of the equations can be deduced from either of the other two, then these three equations become only two independent equations involving two unknown quantities.

If one of the equations be deducible from either or both of the other two, the three equations can be reduced to one equation having two unknown quantities, and is therefore indeterminate.

If, however, all the three equations are independent of each other, they may express conditions incompatible with each other, or be contradictory. But if the equations be taken two and two at a time, the three sets may admit each of different solutions with definite values of the unknown quantities.

3. *To solve the general equations involving three unknown quantities,* $a_1x+b_1y+c_1z=d_1$ (1), $a_2x+b_2y+c_2z=d_2$ (2), $a_3x+b_3y+c_3z=d_3$ (3); *or to find the values of* x, y, z *which will satisfy the equations.*

Here any one of the three unknown quantities can be eliminated between any two of the three equations.

First, eliminate x between the first and second equations.

Multiply (1) by a_2, then $a_1a_2x+a_2b_1y+a_2c_1z=a_2d_1$,

Multiply (2) by a_1, $a_1a_2x+a_1b_2y+a_1c_2z=a_1d_2$,

$\therefore$ $(a_2b_1-a_1b_2)y+(a_2c_1-a_1c_2)z=a_2d_1-a_1d_2$ by subtraction,

or $py+qz=r$, putting p, q, r for the coefficients.

Next, eliminate x between the first and third equations.

Multiply (1) by a_3, then $a_1a_3x+a_3b_1y+a_3c_1z=a_3d_1$,

The particular problems of Diophantus are characterised by their simplicity and neatness of solution, and the peculiar skill of the artifices he employs, and they afford considerable scope for exercising the ingenuity and sharpening the invention of the student.

Bachet and Fermat, in the seventeenth century, made some additions to the knowledge of this subject as left by Diophantus. Bachet was the first who gave a complete method of solving all indeterminate problems of the first degree. Fermat devised some methods for the solution of indeterminate problems above the first degree. Diophantus solved a problem which is equivalent to that of finding x so that $x^2\pm1$ shall be a square number. Fermat extended it to that of finding x so that $ax^2\pm b$ shall be a square number, a and b being some constant numbers.

In the second part of the third English edition of Euler's Algebra, the student will find the subject of indeterminate problems very amply treated, containing some generalisations of the Diophantine problems involving higher powers than the first of the unknown quantities.

In the additions of M. de la Grange at the end of the volume will be found the method of applying continued fractions to the solution of indeterminate problems.

Multiply (3) by a_1, $a_1a_3x+a_1b_3y+a_1c_3z=a_1d_3$,

$\therefore (a_3b_1-a_1b_3)y+(a_3c_1-a_1c_3)z=a_3d_1-a_1d_3$, by subtraction,

or $p'y+q'z=r'$, putting p', q', r', for the coefficients.

Thirdly, eliminate x between the second and third equations.

Multiply (2) by a_3, then $a_2a_3x+a_3b_2y+a_3c_2z=a_3d_2$,

Multiply (3) by a_2, then $a_2a_3x+a_2b_3y+a_2c_3z=a_2d_3$,

$\therefore (a_3b_2-a_2b_3)y+(a_3c_2-a_2c_3)z=a_3d_2-a_2d_3$, by subtraction,

or $p''y+q''z=r''$, putting p'', q'', r'' for the coefficients.

Hence by the elimination of x are obtained the three equations,

$$py+yz=r,\ p'y+q'z=r',\ p''y+q''z=r'',$$

each involving two of the unknown quantities. But since the given equations are determinate, the values of y and z in these equations have the same values, and any one of the equations may be derived from the other two. Hence, there are only two independent equations from which the values of y and z can be determined.

If the two equations $py+qz=r$, $p'y'+q'z=r'$ be taken,

$$\text{then } z=\frac{p'r-pr'}{p'q-pq'} \text{ and } y=\frac{q'r-qr'}{pq'-p'q},$$

and by substituting these values of y and z in any one (the first of the given equations), the value of x will also be determined,

$$\therefore z=\frac{(a_2d_1-a_1d_2)(a_3b_1-a_1b_3)-(a_3d_1-a_1d_3)(a_2b_1-a_1b_2)}{(a_2c_1-a_1c_2)(a_3b_1-a_1b_3)-(a_3c_1-a_1c_3)(a_2b_1-a_1b_2)},$$

$$y=\frac{(a_3c_1-a_1c_3)(a_2d_1-a_1d_2)-(a_2c_1-a_1c_2)(a_3d_1-a_1d_3)}{(a_2b_1-a_1b_2)(a_3c_1-a_1c_3)-(a_3b_1-a_1b_3)(a_2c_1-a_1c_2)},$$

$$x=\frac{(b_1c_3-b_3c_1)(c_2d_1-c_1d_2)-(b_1c_2-b_2c_1)(c_3d_1-c_1d_3)}{(a_1c_2-a_2c_1)(b_1c_3-b_3c_1)-(a_1c_3-a_3c_1)(b_1c_2-b_2c_1)}.$$

$$\text{Or } z=\frac{(a_3b_2-a_2b_3)d_1+(a_1b_3-a_3b_1)d_2+(a_2b_1-a_1b_2)d_3}{(a_3b_2-a_2b_3)c_1+(a_1b_3-a_3b_1)c_2+(a_2b_1-a_1b_2)c_3},$$

$$y=\frac{(a_3c_2-a_2c_3)d_1+(a_1c_3-a_3c_1)d_2+(a_2c_1-a_1c_2)d_3}{(a_3b_2-a_2b_3)b_1+(a_1c_3-a_3c_1)b_2+(a_2c_1-a_1b\ )b_3},$$

$$x=\frac{(b_2c_3-b_3c_2)d_1+(b_3c_1-b_1c_3)d_2+(b_1c_2-b_2c_1)d_3}{(b_2c_3-b_3c_2)a_1+(b_3c_1-c_2b_1)a_2+(b_1c_2-b_2c_1)a_3}.$$

This method may be extended for the solution of four independent equations involving four unknown quantities. And generally, in a similar way, whatever may be the number of unknown quantities, their values can always be found, provided there be given as many independent equations as there are unknown quantities involved in them.

The values of x, y, z in any set of three equations of the first degree can be determined by substituting the values of the coefficients in the given equations corresponding to those in the general equation.

If each of the values of x, y, z in the solution of three equations, assumes the form of $\frac{0}{0}$, then, as in Art. 2, it may be shewn that $\frac{a_2}{a_1}=\frac{b_2}{b_1}=\frac{c_2}{c_1}=\frac{d_2}{d_1}=m$, suppose: then $a_2=ma_1$, $b_2=mb_1$, $c_2=mc_1$, $d_2=md_1$,

and $\frac{a_3}{a_2}=\frac{b_3}{b_2}=\frac{c_3}{c_2}=\frac{d_3}{d_2}=n$, suppose: then $a_3=na_2$, $b_3=nb_2$, $c_3=nc_2$, $d_3=nd_2$,

therefore the coefficients of the second equation are respectively equimultiples of those of the first, and those of the third equimultiples of those of the second. Hence it follows that the second and third equations are derivable from the first, and there is only one independent equation involving three unknown quantities. The values of x, y, z are therefore indeterminate.

This conclusion is deduced on the supposition that all the coefficients involved are positive or negative numbers. If any of them be zero, it is possible that although the general forms of the values of the unknown quantities may assume the form $\frac{0}{0}$, the real values of the unknown quantities may not be indeterminate.

If any of the values of the unknown quantities assume the form of $\frac{m}{0}$, this will indicate that the equations are inconsistent with one another.

If one of three equations involving three unknown quantities be derivable from one or both of the remaining two; then there are only two independent equations involving three unknown quantities; and the values of x, y, z are indeterminate.

4. *To solve the equations*

$$a_1x+b_1y+c_1z=d_1,\quad a_2x+b_2y+c_2z=d_2,\quad a_3x+b_3y+c_3z=d_3,$$

by the method of indeterminate multipliers.

Multiply the second and third equations by p, q respectively.

Then the three equations are

$$a_1x+b_1y+c_1z=d_1,\quad pa_2x+pb_2y+pc_2z=pd_2,\quad qa_3x+qb_3y+qc_3z=qd_3,$$

and their sum is

$$(a_1+pa_2+qa_3)x+(b_1+pb_2+qb_3)y+(c_1+pc_2+qc_3)z=d_1+pd_2+qd_3.$$

Suppose the multipliers p, q, such that they render the coefficients of y and z equal to zero, then $b_1+pb_2+qb_3=0$, $c_1+pc_2+qc_3=0$.

$$\text{Whence } p=\frac{b_3c_1-b_1c_3}{b_2c_3-b_3c_2},\ q=\frac{b_1c_2-b_2c_1}{b_2c_3-b_3c_2},$$

$$\text{and } (a_1+pa_2+qa_3)x=d_1+pd_2+qd_3,$$

$$\therefore\ x=\frac{d_1+pd_2+qd_3}{a_1+pa_2+qa_3},$$

and when the values of p and q are substituted, the value of x is determined.

By a similar process the value of y is determined by making the coefficients of x and z equal to zero; and z is also determined by making the coefficients of x and y equal to zero.[*]

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[*] Example. $x+2y+z=12$, $2x-5y+3z=2$, $5x+4y-2z=7$.

Multiply the second equation by p, the third by q, and add the three equations,

5. *To solve the indeterminate equations $ax+by=c$, and $ax-by=c$; limiting the solution to positive integers.*

Since the solution is to be restricted to positive integral values of x and y, the most simple mode of solution is to determine the values of x and y in terms of a third unknown quantity, and then to ascertain within what limits the third unknown quantity must be assumed in order to determine the integral values of x and y.

As x and y are to be both positive integers, all expressions involving x and y must be integral, as also the sum, difference, and product of two such expressions.

Let the equation $ax+by=c$ be taken.

Here $x=\frac{c-by}{a}$, which is required to be an integer.

Let $\frac{c-by}{a}$ be reduced, so that the coefficient of y may be unity.

Let $\frac{c-by}{a}=m-ny+\frac{m'-n'y}{a}$, which is also an integer,

$\therefore$ $\frac{m'-n'y}{a}$ must be an integer, in which n' must be made greater or less by unity than a, if it be not so.

Also $\frac{ay}{a}$ is an integer, $\therefore \frac{m'+(a-n')y}{a}=\frac{y+m'}{a}$, an integer.

Suppose $\frac{y+m'}{a}=p$, some unknown integer.

Then $y=ap-m'$, and $x=n-bp$, suppose, by substituting the value of y in the given equation, $ax+by=c$.

From $x=n-bp$, in order that x may be a positive integer, p *must not be greater than* $\frac{n}{b}$.

From $y=ap-m'$, in order that y may be a positive integer, p *must not be less than* $\frac{m}{a}$.

Hence the number of solutions in positive integers can be known.

By proceeding in the same manner to solve $ax-by=c$, it will be found that the values of x and y will assume the forms of

$$x=ap\pm m,\ y=bp\pm n.$$

$\therefore$ $(1+2p+5q)x+(2-5p+4q)y+(1+3p-2q)z=12+2p+7q$.

Equate to zero the coefficients of y and z, in order to find the values of p and q.

$2-5p+4q=0$, $1+3p-2q=0$. Here $p=-4$, and $q=-\frac{11}{2}=-5\frac{1}{2}$.

$$\therefore\ x=\frac{12+2p+7q}{1+2p+5q}=\frac{12-8-38\frac{1}{2}}{1-8-27\frac{1}{2}}=\frac{8-77}{-14-55}=\frac{-69}{-69}=+1.$$

By a similar process may be found $y=3$, and $z=5$.

From $x = ap \pm m$, in order that x may be positive, p must not be *less than* $\frac{\pm m}{a}$.

From $y = bp \pm n$, in order that y may be positive, p *must not be less than* $\frac{\pm x}{b}$; and the greater of these two values of p will give the least positive and integral values of x and y, and the number of solutions in positive integers is unlimited.*

6. *To solve the general independent equation* $ax + by + cz = d$, *limiting the solution to positive and integral values of* x, y, z.

By transposing one of the terms, as cz, the equation becomes $ax + by = d - cz$. Now x and y cannot be less than unity, and $d - cz$ must be a positive integer; it follows that z must be less than $\frac{d}{c}$. Let

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* Find all the positive integral values of x and y in the equations $2x + 3y = 35$, and $2x - 3y = 35$.

First, here $2x = 35 - 3y$, $\therefore x = \frac{35 - 3y}{2} = 17 - y - \frac{y - 1}{2} =$ an integer,

$\therefore \frac{y - 1}{2} =$ an integer $= p$, suppose; then $y = 2p + 1$,

and $x = \frac{35 - 3y}{2} = \frac{35 - 6p - 3}{2} = 16 - 3p$.

In order that x may be a positive integer, from $x = 16 - 3p$, it is obvious that p must not be *greater than* 5.

And in order that y may be a positive integer, from $y = 2p + 1$, p must not be *less than* 0.

Hence p may be 0, 1, 2, 3, 4, 5; and the equation admits of the following six different solutions in positive integers.

When $p =$	0,	1,	2,	3,	4,	5.
$x = 16 - 3p =$	16,	13,	10,	7,	4,	1.
$y = 2p + 1 =$	1,	3,	5,	7,	9,	11.

Next, $2x - 3y = 35$, and $x = \frac{35 + 3y}{2} = 17 + y + \frac{y + 1}{2} =$ an integer;

$\therefore \frac{y + 1}{2} =$ an integer $= p$, suppose; then $y = 2p - 1$,

and $x = \frac{35 + 3y}{2} = \frac{35 + 6p - 3}{2} = 3p + 16$.

In order that x may be a positive integer, from $x = 3p + 16$, p must not be less than -5.

In order that y may be a positive integer, from $y = 2p - 1$, p must not be less than $+1$.

And that both x and y may be positive integers, it is obvious that p must not be less than $+1$, and may be any number whatever. Hence the equation $2x - 3y = 35$ admits of an unlimited number of solutions in positive integers, the least of which is obtained when $p = +1$, then $x = 19$, and $y = 1$, and the series of solutions may be exhibited to any extent.

When $p =$	1,	2,	3,	4,	5,	6,	7,	8,	9,	10, &c.
$x = 3p + 16 =$	19,	22,	25,	28,	31,	34,	37,	40,	43,	46, &c.
$y = 2p - 1 =$	1,	3,	5,	7,	9,	11,	13,	15,	17,	19, &c.

$a-cz=d'$, then $ax+by=d'$, an equation, the solution of which has been determined in Art. 2.

But as d' may admit of more values than one, there may be several equations involving x and y, corresponding to the number of values of d'. And for every value of z there will be corresponding values of x and y.

The given equation may be of the three following forms:—

$$ax+by+cz=d,\quad ax+by-cz=d,\quad ax-by-cz=d,$$

$$\text{or } ax+by=d-cz,\quad ax-cz=d-by,\quad ax-by=d+cz.$$

In equations of the first form, the number of solutions in positive integers is limited.

In equations of the second and third form, the number of solutions in positive integers is unlimited.*

7. *To solve the general independent equations*

$$a_1x+b_1y+c_1z=d_1,\quad a_2x+b_2y+c_2z=d_2,$$

involving three unknown quantities, limiting the solution to positive integers.

First eliminate z between the two given equations, and the result is $(a_1c_2-a_2c_1)x+(b_1c_2-b_2c_1)y=c_2d_1-c_1d_2$, an equation of the form $ax\pm by=c$, according as $b_1c_2-b_2c_1$ is positive or negative.

The equations $ax+by=c$ and $ax-by=c$ have been considered in Art. 2, and when the values of x and y have been found, the corresponding values of z may be determined by the substitution of the values of x and y in either of the given equations.

If the elimination of z from the two given equations give an equation of the form $ax+by=c$, the number of solutions in positive integers is limited; but if it gives an equation of the form $ax-by=c$, the number of solutions in positive integers is unlimited.

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* Ex. Solve the equation $5x+6y+20z=187$ in positive integers.

$$\text{Here } 5x+6y=187-20z.$$

But x and y cannot be each less than unity, and z must not be greater than 9.

Hence z may be 1, 2, 3, 4, 5, 6, 7, 8, and these 8 values of y will give 8 different equations for determining the values of x and y corresponding to the 8 values of z.

When these 8 values of z are successively substituted in the given equation, the values of x and y determined in each case, will be those which correspond to the several values of z.

In the first case, when $z=1$, $5x+6y=167$, and this equation being solved, $x=31-6p$, from which it appears that p must not be greater than 5.

And $y=5p+2$, from which it also appears that p must not be less than 0.

Hence this equation $5x+6y=167$ admits of six solutions, when $z=1$.

$p=$	0,	1,	2,	3,	4,	5.
$x=31-6p=$	31,	25,	19,	13,	7,	1.
$y=5p+2=$	2,	7,	12,	17,	22,	27.
$z=$	1,	1,	1,	1,	1,	1,

The rest of the solutions can be found in the same manner, when z is 2, 3, 4, 5, 6, 7, or 8, respectively.

EXERCISES.

EQUATIONS OF THE FIRST DEGREE.

I.

Express generally in words at length, and solve each of the following equations:—

1. $x=14-6x$
2. $2x+x=46+2x$.
3. $2x+11=7x-14$.
4. $3x+14=9x+22$.
5. $8x-5=13-7x$.
6. $3x-7=5x+17$.
7. $5x-9=7x+15$.
8. $x+7=3x-9$.
9. $3x-7=4-5x$.
10. $2x-3x-2=0$.
11. $5x-7-2x=10x+8-2x$.
12. $x-10+8=70-x$.

II.

Solve the following equations:—

1. $3{\cdot}75x+{\cdot}5=2{\cdot}25+8$.
2. ${\cdot}5x-2={\cdot}25x+{\cdot}2x-1$.
3. ${\cdot}5x+{\cdot}6x-{\cdot}8={\cdot}75x+{\cdot}25$.
4. ${\cdot}15x+1{\cdot}575-{\cdot}875x={\cdot}0625$.
5. ${\cdot}15x+1{\cdot}2-{\cdot}875x+{\cdot}375={\cdot}0625x$.
6. $1{\cdot}2x-\frac{1}{6}({\cdot}18x-{\cdot}05)={\cdot}4x+8{\cdot}9$.
7. ${\cdot}84x-7{\cdot}6-2{\cdot}2x-10=0$.

III.

Solve the following equations, and verify the correctness of the results in each case:—

1. $(x-3)-3(x-5)+5(x-7)=0$.
2. $2(x-1)+3(x-2)+4(x-3)=0$.
3. $(x+4)(x+2)+(x+3)(x-3)=2(x+3)(x+2)$.
4. $(x-3)-(3-x)(x+1)=(x-3)(1+x)+(3-x)$.
5. $(4x+2)x+(4x+1)x=(2x+1)(4x-2)$.
6. $(x-5)(x-3)-(x-5)-(x+7)(x-2)=0$.
7. $24x-12(48-x)=504$.
8. $(x-4)+3(x-1)(x-2)=3(x^2-2x-4)$.
9. $(x-1)(x-2)=x^2$.
10. $(x-9)(x-7)(x-1)=(x-2)(x-4)(x-11)$.
11. $(x-3)(x-4)=x^2-5x-8$.
12. $(x+1)(x+2)(x+3)=(x-1)(x-2)(x-3)+3(4x-1)(x+1)$.

IV.

1. $\frac{x}{7}+\frac{x}{3}=30$.
2. $\frac{x}{5}+3=\frac{2x}{3}-1$.
3. $\frac{7}{2}-\frac{2}{x}=\frac{2}{x}-\frac{7}{2}$.

4. $\frac{x}{2}+\frac{x}{3}+\frac{x}{4}=1.$ 5. $\frac{x-1}{2}+\frac{x-2}{3}+\frac{x-3}{4}=1.$

6. $\frac{x-1}{2}+\frac{2x-1}{3}+\frac{3x-1}{4}=1.$ 7. $\frac{2x-3}{2}+\frac{3x-4}{3}+\frac{4x-5}{4}=1.$

8. $\frac{1}{x}+\frac{1}{2x}+\frac{1}{3x}=\frac{7}{3}.$ 9. $\frac{3x}{4}+\frac{7x}{15}+\frac{11x}{6}-366=0.$

10. $\frac{x+1}{2}+\frac{x+2}{3}=\frac{x-3}{4}+\frac{x-4}{6}+1.$ 11. $\frac{x-2}{2}-\frac{7x-12}{3}=\frac{x-11}{5}-4.$

12. $\frac{3x-4}{5}-\frac{2x-7}{9}=\frac{11}{15}.$ 13. $3x-\frac{x-4}{4}-4=\frac{5x+14}{3}-\frac{1}{12}.$

14. $\frac{2x+5}{3}+\frac{2x-5}{5}=\frac{3x+1}{4}+\frac{3x-1}{7}.$ 15. $\frac{3x-20}{50}-\frac{4x-27}{15}+1=\frac{143}{150}.$

16. $\frac{7x+3}{8}-\frac{3x+1}{7}=\frac{9x-13}{4}-\frac{249-9x}{14}.$ 17. $3+\frac{3-x}{3}=\frac{x^2}{3(3-x)}.$

18. $\frac{7}{9}(7x-9)+\frac{9}{7}(9x-7)=1.$ 19. $\frac{x+1}{7}+x(x-2)=(x-1)^2.$

V.

1. $\frac{x+1}{3}+\frac{5-2x}{4}-\frac{2+5x}{2}+\frac{10-2x}{6}=0.$

2. $\frac{x-2}{3}-\frac{2}{3}\cdot\frac{3x-2}{4}=\frac{1}{2}\left\{\frac{62}{15}-\frac{5x+8}{2}\right\}.$

3. $\frac{7x+23}{11}-\frac{5x-2(x-1)}{7}-\frac{3x-7(8x+4)}{2}-2=0.$

4. $\frac{3x-1}{5}-\frac{13-x}{2}=\frac{72}{3}-\frac{11(x-3)}{6}.$

5. $\frac{1}{7}\left\{x-\frac{1}{2}\right\}+\frac{1}{5}\left\{x+\frac{2}{3}\right\}=\frac{43}{30}.$

6. $\frac{7x+8}{8}-\frac{9x-12}{16}=\frac{9x+1}{10}-\frac{29-8x}{20}.$

7. $\frac{x+3}{7}-\frac{1}{3}\left\{x-\frac{1}{5}(3x-5)+1\right\}=\frac{x}{12}.$

8. $\frac{x}{2}-\frac{x-2}{3}=\frac{1}{4}\left\{x-\frac{2}{3}\left(\frac{5}{2}-x\right)\right\}-\frac{x-5}{3}.$

9. $\frac{1}{2}\left\{\frac{2x}{3}+4\right\}-\frac{7\frac{1}{2}-x}{3}=\frac{x}{2}\left\{\frac{6}{x}-1\right\}.$

10. $\frac{7x-13\frac{1}{2}}{11}-\frac{2}{3}\cdot\frac{x-15}{7}=\frac{15}{14}\left\{x-\frac{1}{2}\right\}.$

11. $\frac{4x-17}{9}-\frac{3\frac{2}{3}-22x}{33}=x-\frac{6}{x}\left\{1-\frac{x^2}{54}\right\}.$

12. $\frac{x-4\frac{2}{3}}{3}-\frac{2x-3\frac{2}{3}}{4}=\frac{3}{2}\left\{x-\frac{x-1\frac{1}{2}}{2}\right\}+\frac{4x}{3}\cdot\left\{x-3-\frac{(x-1)(x-2)}{x}\right\}.$

VI.

1. $\frac{x-1}{x+1}=1+\frac{1}{x}.$ 2. $\frac{2x-1}{2x-2}=\frac{14x-21}{14x-26}.$ 3. $\frac{42x}{3x-2}=\frac{35x}{12x-3}.$

4. $\frac{x+1}{x-1}+\frac{x+2}{x-2}=\frac{2(x+3)}{x-3}$. 5. $\frac{x+2}{x+7}+\frac{2-x}{x+5}=\frac{3x+17}{x^2+12x+35}$.

6. $\frac{x-2}{x-3}-\frac{x-3}{x-4}=\frac{x-5}{x-6}-\frac{x-6}{x-7}$. 7. $\frac{6x+13}{15}-\frac{3x+5}{5x-25}=\frac{2x}{5}$.

8. $\frac{2x}{15}+\frac{3x+50}{30}=\frac{12x-70}{190}$. 9. $\frac{x+1}{x^2-1}-\frac{x-1}{x^2+1}=\frac{a}{x^2+1}$.

10. $\frac{102x-80}{16}-10x=0$. 11. $\frac{4(x+3)-3(2x-4)}{2(2x-1)-3(3x-2)}=10$.

12. $\frac{4x^3+4x^2+8x+1}{2x^2+2x+3}=\frac{2x^2+2x+1}{x+1}$.

13. $\frac{3+10x}{5x-7}=9-\frac{7x+17}{x+1}$. 14. $\frac{1}{x-1}-\frac{1}{x}=\frac{1}{x}-\frac{1}{x+2}$.

15. $\frac{x^2+2x+2}{x+1}-\frac{x^2+4x+6}{x+2}=\frac{x^2+6x+12}{x+3}-\frac{x^2+8x+20}{x+4}$.

16. $\frac{14-12x}{3x+1}+\frac{4x+2}{2x+7}=\frac{10-9x}{x+1}+\frac{7x-5}{x+2}$.

VII.

1. $ax+b=c-x$. 2. $ax-bx=a+bx-b$. 3. $ax-bx=cx-d$.

4. $(x-a)=(b-a)x$. 5. $(a+b)(a-x)=a(b-x)$.

6. $a(b^2-ax)=ab^2-b(a^2-bx)$. 7. $(ax+c)(bx+d)=(p-ax)(q-bx)$.

8. $(a+x)(b+x)=(c+x)(d+x)$.

9. $(x-a)(x-b)-(x+a)(x+b)+ab=o$.

10. $\frac{x}{a}+\frac{x}{b}=1$. 11. $\frac{a}{x}+\frac{b}{x}=1$. 12. $\frac{a}{bx}-\frac{b}{ax}=a^2-b^2$.

13. $\frac{bx}{a}-\frac{d}{c}=\frac{a}{b}-\frac{cx}{d}$. 14. $\frac{a}{x}=\frac{b}{c-x}$. 15. $\frac{ac}{bx}-\frac{bc}{ax}=a+b$.

16. $\frac{a-bx}{a}-\frac{b-ax}{b}=1$. 17. $\frac{a+bx}{b}=\frac{c+dx}{d}$. 18. $\frac{1}{ax}+\frac{1}{ab}+\frac{1}{bx}=0$.

19. $\frac{ax+b}{c}-\frac{cx+d}{c}=\frac{a}{b}$. 20. $\frac{x+a}{b}+3c=\frac{x+b}{a}+c$.

VIII.

1. $\frac{x+a}{x+b}=\frac{1+a}{1+b}$. 2. $\frac{x-a-b}{x-c-d}=\frac{a+b}{c+d}$.

3. $\frac{(x+a)(x+b)}{x+a+b}=\frac{(x+c)(x+d)}{x+c+d}$. 4. $\frac{ax}{a-b}+\frac{bx}{b-c}=\frac{d}{a-c}$.

5. $\frac{b}{x}=\frac{c}{a-x}$. 6. $\frac{1}{a(x-b)}+\frac{1}{b(x-c)}=\frac{1}{a(x-c)}$.

7. $\frac{1}{x+a+b}=\frac{1}{c}+\frac{1}{a}+\frac{1}{b}$. 8. $\frac{x-a}{3}+\frac{2x-3b}{5}-\frac{a-x}{2}=10a+11b$.

9. $\frac{ax+b}{c}+\frac{bx+c}{a}+\frac{cx+a}{b}=0$. 10. $\frac{ax-b}{c}+\frac{bx-c}{a}+\frac{cx-a}{b}=1$.

11. $\frac{b^2-c^2}{x-a}+\frac{c^2-a^2}{x-b}+\frac{a^2-b^2}{x-c}=0.$ 12. $\frac{2x+4a+b}{a+b}=\frac{4x+a+2b}{a-b}.$

13. $\frac{x-a}{a^2-b^2}=\frac{x-b}{c(a+b)}+\frac{x-c}{c(a-b)}.$ 14. $\frac{mx-(a+b)}{nx-(c+d)}=\frac{mx-(a+c)}{nx-(b+d)}.$

15. $\frac{x+a}{m}+\frac{x+mb}{n}=\frac{mx+a}{n}-\frac{x+b}{m}.$

16. $\frac{x}{a^2+c^2}+\frac{x}{b^2+d^2}=\frac{a^2+b^2+c^2+d^2}{(ab-cd)^2+(ad+bc)^2}.$

17. $\frac{a(c-d)}{x+a}+\frac{d(a-b)}{x+d}=\frac{b(c-d)}{x+b}+\frac{c(a-b)}{x+c}.$

18. $(a+x)^3+(b+x)^3+(c+x)^3=3(a+x)(b+x)(c+x).$

19. $(x+3)^3-3(x+2)^3+3(x+1)^3-x^3=x+15.$

20. $\frac{1\ \ 1}{a\ +b}\,\frac{1}{+c}\,\frac{1}{+x}=\frac{1}{d}.$ 21. $\frac{(x+a)(x+b)}{x+a+b}=\frac{(x+c)(x+d)}{x+c+d}.$

IX.

1. $x^{\frac{1}{2}}+2=(x+16)^{\frac{1}{2}}.$ 2. $(5x)^{\frac{1}{2}}+2=(5x+10)^{\frac{1}{2}}.$

3. $(x+7)^{\frac{1}{2}}+(9x+7)^{\frac{1}{2}}=(32x)^{\frac{1}{2}}.$ 4. $\{(x-a)^2+2ab+b^2\}^{\frac{1}{2}}=x-a+b.$

5. $\{(x-1)(x-2)\}^{\frac{1}{2}}+\{(x-3)(x-4)\}^{\frac{1}{2}}=2^{\frac{1}{2}}.$

6. $\{a^2+x(x^2+a^2)^{\frac{1}{2}}\}^{\frac{1}{2}}=a+x.$ 7. $\frac{(x-1)^{\frac{3}{2}}}{4(x-1)^{\frac{1}{2}}}+\frac{x}{2}=\frac{11}{16}.$

8. $(9+x)^{\frac{1}{2}}+x^{\frac{1}{2}}=\frac{45}{(x+9)^{\frac{1}{2}}}.$ 9. $\frac{2x-a}{(x^2-ax+a^2)^{\frac{1}{2}}}=\frac{2x-b}{(x^2-bx+b^2)^{\frac{1}{2}}}.$

X.

1. Find the two numbers whose sum shall be 100, and whose difference shall be 10.

2. What number is that whose third part exceeds the seventh part by 20?

3. What value of x will make the difference between $(2x+4)(3x+6)$ and $(3x-2)(2x-8)$ equal to 96?

4. What number is that to which, if 8 be added, one fourth of the sum is equal to 29?

5. The difference between a certain number and its defect from 17, is equal to 3 times its defect from 21; find the number.

6. Divide 240 into two parts, such that one-fourth of one of them added to one-tenth of the other shall be equal to 36.

7. Divide the number 144 into three such parts, that one half of the first, one-third of the second, and one-fourth of the third, shall be equal to one another.

8. Find a number which, when trebled and diminished by 3, exceeds the sum of its fifth and seventh parts by 90.

9. Find a number such that if $\frac{3}{8}$ of it be subtracted from 20, and $\frac{5}{11}$ of the remainder from $\frac{1}{4}$ of the original number, 12 times the second remainder shall be half of the original number.

10. What fraction is that from which if $\frac{7}{8}$ be subtracted, the result is the same as if it had been multiplied by $\frac{8}{15}$?

11. Find a number, the sum of whose fifth and sixth parts exceeds the difference between its fourth and seventh parts by 109.

12. What number is that, to the half of whose excess above 3 if a third of the excess of 90 above the required number be added, the sum is 30?

13. Divide the number a into three parts, which shall be respectively equimultiples of the numbers m, n, p.

14. There is a number consisting of six digits; the digit on the left-hand side is 1; if this digit be removed and placed on the right-hand side, the new number is three times the original number.

15. Divide a number a into three parts so that the second may be m times the first and the third n times the second.

16. What number divided by 7, 8, 9 respectively leaves the remainders 1, 2, 3, so that the sum of the three quotients shall be equal to 570?

XI.

1. If A be a years old, and B, b years at any given epoch; when will A be n times as old as B?

2. A is 40 years older than B, and in four years A will be three times the age of B; what are their respective ages?

3. Thirty years ago a man was three times as old as his son, whose present age is 45; how old is his father?

4. Two years ago, A's age was three times B's age; eighteen years hence, A's age will be twice B's; find their present ages.

5. In the year 1830 A's age was 50 and B's 35; give the date at which A is twice as old as B.

6. A and B shoot by turns at a target. A puts 7 bullets out of 12 into the bull's-eye, and B puts in 9 out of 12; between them they put in 32 bullets; how many shots did each fire?

XII.

1. A sets out from Cambridge to London (51½ miles) at the rate of 8 miles an hour, and B sets out at the same time from London to Cambridge at the rate of 9 miles an hour; at what distance from each place will they meet?

2. The two sides of a street are of equal length, but one side contains 20 and the other 25 houses, the width of the front of any house on one side being 5 feet greater than that of any house on the other; find the length of the street.

3. A starts from a place C and travels to another D at the rate of six miles an hour; two hours afterwards B starts also, and going ten miles an hour reaches D four hours before A; find the distance between C and D.

4. Of a certain dynasty, one-third of the kings are of the same name, one-fourth of another, one-eighth of a third, and one-twelfth of a fourth, and there are five besides; how many are there of each name?

5. A farmer buys a sheep for p pounds, and sells b of them at a gain of m per cent.; at what rate ought he to sell the remainder so as to gain $2m$ per cent. on the whole?

If in the result, $-b$ be substituted for $+b$, state the question to which the result so modified will be the solution.

6. At a certain election, 1,255 persons recorded their votes for the two candidates, and the successful candidate was elected by a majority of 125 votes; how many persons voted for each candidate?

7. If a garrison of 1,000 men having provisions for 30 days be reinforced after 10 days, and then the provisions were exhausted in 5 days; find the number of men in the reinforcement.

XIII.

1. A and B have the same incomes. Given that the sum of money originally invested between them both is £49,000, of which A invested his share in the 3½ per cents. at 84, and B in the 4 per cents. at par; find the *sums* each invested, and their common income.

2. An income is derived from the proceeds of £4,550 at a certain rate per cent., and of £5,420 at one per cent. more than the former, and the whole annual income is £453; determine the rates.

3. If £1,000 be invested, partly in 3 per cent. stock at $79\frac{7}{8}$, and partly in 5 per cent. at $106\frac{1}{2}$; how much must be invested in each kind of stock that the whole income derived may be £40?

4. On a certain day the 3 per cent. consols were ten per cent. higher than on a certain other day, and a person investing in them on the latter day received 6s. 8d. per cent. more annual interest than a person investing in them on the former day. What was the price of consols on each day?

5. The incomes of A and B are in the ratio of 3 to 2, and their expenditures are in the ratio of 5 to 3, and each saves £1,000 a year; find their incomes.

6. Divide p pounds among A, B, C, so that A may have a pounds more, and B, b pounds less than C.

7. In a piece of mechanism there are two wheels, one of which has 15 and the other 36 teeth which work into each other; find what number of revolutions each wheel must make before the same teeth come in contact.

8. A purse contains a certain number of sovereigns, three times as many shillings, and five times as many pence, and the whole sum is £280; find how many sovereigns, shillings, and pence it contains.

9. How many sovereigns and half-crowns are equal to £10, so

that the number of half-crowns may be equal to twice the number of sovereigns?

10. An article is bought and then sold so as to gain 5 per cent. If it had been bought at 5 per cent. less and sold for one shilling less, 10 per cent. would have been gained; find the cost price.

11. If from a sum of money there be taken £50 more than the half, then from the remainder £30 more than the fifth, next from the second remainder £20 more than the fourth part, and at last only £10 remains; what was the original sum?

12. A person after paying the poor rate and the income tax of seven pence in the pound, had £486 remaining; the poor rate amounted to £22 10s. more than the income tax; find his original income, and the number of pence per pound in the poor rate.

XIV.

1. If one agent can produce an effect e_1 in time t_1, and another agent the effect e_2 in time t_2, in what time can both together produce the effect e_3?

2. If a men or b boys can dig m acres of ground in n days, shew that the number of boys whose assistance will be required to enable $(a-p)$ men to dig $(m+p)$ acres in $(n-p)$ days is equal to

$$\frac{pb}{a}\left(1+\frac{n+m}{n-p}+\frac{a}{m}\right).$$

3. A and B could do a piece of work in 10 days, B and C in 15 days, A and C in 25 days; all work 4 days, when A leaves; B and C work on 5 days, when B leaves; in how many days will C finish the work?

4. A hare is 80 of her leaps before a greyhound. The greyhound takes two leaps for every three of the hare; but one of the greyhound's leaps is equal to two of the hare's. How many leaps will the hare have taken before she is overtaken by the greyhound?

5. The rent of an estate in one year was greater by 8 per cent. than it was in the preceding year. If the increased rent was £1,890, what was the rent in the year before? What would have been the result if the rent had been less by 8 per cent.?

6. A cistern is supplied by the pipes A and B, and may be emptied by the pipe C; when A alone is open, it is filled in 5 hours; when A and C are open, it is filled in $7\frac{1}{2}$ hours; and when B and C are open it is filled in 3 hours. In what time will the cistern be filled when A and B are open, and C closed?

7. A cistern has three pipes A, B, C. A will fill it in m hours, and B in n hours, and C will empty it in c hours. The cistern being empty, the pipe A is opened at a certain hour, the pipe B at an interval of one hour, and the pipe C at an interval of two hours after the first pipe was opened. In what times will the cistern be either full or empty?

8. In the attempt to draw up a regiment in a solid square, it was found that, placing a certain number of men in the side of the square, there were 39 men over; but on extending the side of the square by one man, 50 men were wanted to complete the square. How many men did the regiment contain?

9. If a cistern be filled by each of four pipes singly in a, b, c, d hours respectively; in what time will the four pipes together fill the cistern?

10. Explain the rule which is called "the chain rule," and exemplify its use.

XV.

1. Explain what is meant by the root or roots of an equation, and instance it in the equation $4x+2=3x+8$.

2. State the advantages of denoting the data of a problem by the letters a, b, c, &c., and the quæsita by the letters z, y, x, &c.

3. Explain the nature of the following equations, and give correct answers:—

$$(1)\ \frac{3x}{5}+6=\frac{15x-1}{25}. \qquad (2)\ \frac{x-2}{4}=\frac{3x-2}{12}-\frac{1}{3}.$$

$$(3)\ \frac{1}{(2x-1)(2x-5)}=\frac{1}{2x-5}-\frac{1}{2x-1}.$$

4. If $x-a=\frac{1}{2}(x-b)=\frac{1}{3}(x-c)=\frac{1}{4}(x-d)$ and $x=a+b+c+d$, then shall $\frac{1}{7}a=\frac{1}{4}b=c=-\frac{1}{2}d$.

5. If $(b-c)(bc+x)p=(c-a)(ca+x)q=(a-b)(ab+x)r$, obtain an equation which does not contain x.

6. If $a+b+c=0$, find the value x in order that

$$\frac{a}{a-x}+\frac{b}{b-x}+\frac{c}{c-x}=0.$$

7. Solve the equations $\dfrac{a\{m-(x-n)\}}{b}=\dfrac{b\{n-(x-m)\}}{a}$ and $\dfrac{3\{5-(x-4)\}}{71}=\dfrac{71\{4-(x-5)\}}{3}$; and shew that if in the former of these a and b be equal, x may have any value whatever; but that if a and b be unequal, x has only one and the same value, whatever unequal values a and b may have.

8. If the equations $a_1x+b_1=a_2x+b_2=a_3x+b_3$ be simultaneous, shew that x may be expressed in a symmetrical form:

$$x=\frac{a_1b_1(a_2-a_3)+a_2b_2(a_3-a_1)+a_3b_3(a_1-a_2)}{(a_1-a_2)(a_2-a_3)(a_3-a_1)}.$$

9. If $(x-a)(x+b)(x-c)+(x+a)(x-b)(x+c)+(x+a)(x+b)(x-c)$

$$=3(x-a)(x-b)(x-c),$$

then shall $x=\dfrac{ab+ac+bc}{a+b+c}$.

10. In the product of $(x+a)(x+b)(x+c)$, the coefficient of x^2 vanishes, and in the product of $(x-a)(x+b)(x+c)$, the coefficient of x vanishes,

and the coefficient of x in the former is that of x^2 in the latter; find *a* in terms of b and c.

XVI.

Solve the following equations, and express in words at length the first four of them:

1. $2x+3y=32,\ 3x+2y=33.$ 2. $4x+5y=41,\ 4y-5x=4.$
3. $12x+13y=37,\ 17x-19y=15.$
4. $87x-72y=78,\ 78x+14y=934.$
5. $x-y=y+1,\ x+2=2(y-2).$ 6. $3x-2y=1=3y-4x.$
7. $(x+5)(y+7)=(x+1)(y-9),\ 2x+10=3y+1.$
8. $2y+3x=16xy,\ 3y+2x=14xy.$
9. $\frac{x}{7}+\frac{y}{9}=11,\ \frac{x}{9}+\frac{y}{2}=10.$ 10. $\frac{3}{x}+\frac{4}{y}=17,\ \frac{4}{x}-\frac{1}{y}=10.$
11. $\frac{3}{x}+\frac{4}{y}=1=\frac{5}{x}-\frac{4}{y}.$ 12. $\frac{1}{3x}+\frac{1}{5y}=8,\ \frac{1}{9x}-\frac{1}{10y}=1.$

XVII.

1. $\cdot 08x=\cdot 21y=\cdot 33,\ \cdot 12x+\cdot 7y=3\cdot 54.$
2. $x+y=18\cdot 73,\ \cdot 56x+13\cdot 421y=763\cdot 4.$
3. $2x+\cdot 4y=1\cdot 2,\ 3\cdot 4x-\cdot 02y=\cdot 01.$
4. $\cdot 3x+\cdot 125y=3x-y,\ 3x-\cdot 5y=2\cdot 25y-3y.$
5. $\cdot 1y+\cdot 125x=y-x,\ y-\cdot 5x=\cdot 75xy-3x.$
6. $4x+8y=2\cdot 4,\ 10\cdot 2x-\cdot 06y=\cdot 03.$

XVIII.

1. $x=9-\frac{y}{2},\ y=11+\frac{x}{3}.$ 2. $\frac{y}{x+1}=3,\ \frac{x}{y+1}-\frac{1}{4}.$
3. $\frac{x-2}{3}+\frac{y-3}{4}=\frac{3}{2},\ \frac{y-2}{3}-\frac{x-3}{4}=\frac{2}{3}.$
4. $\frac{x}{3}+3y=39,\ \frac{y}{3}+3x=31.$ 5. $7x+\frac{y}{7}=99,\ 7y+\frac{x}{7}=51.$
6. $\frac{x+y}{2}=10,\ \frac{2x+3y}{4}=12.$ 7. $\frac{y+x}{5}+\frac{y}{3}=8=\frac{2x-y}{4}+27.$
8. $\frac{3x-y}{3}-\frac{2x+y+1}{5},\ 8-\frac{x-y}{5}=6.$
9. $\frac{y}{3}+\frac{2y-3x}{6}=8,\ \frac{7x-3y}{2}-x=11.$
10. $\frac{x+2y}{7}+\frac{2-2y}{5}=2,\ \frac{x-4y+1}{3}-\frac{x+6y-1}{12}=1.$
11. $\frac{x}{2}-12=\frac{y}{4}+8,\ \frac{x+y}{5}+\frac{x}{3}-8=\frac{2y-x}{4}+27.$
12. $2x+3y-\frac{3}{4}=18,\ 3x+2y-\frac{4}{3}=15.$

XIX.

1. $\frac{x-y}{x-4y}=\frac{3}{2},\ \frac{x+y}{x+8y}=\frac{11}{14}.$ 2. $\frac{x-y}{x+y}=19,\ \frac{2x+y-1}{x+2y+1}=\frac{30}{31}.$

3. $\frac{2+x}{2+y}=\frac{3}{4},\ \frac{2-x}{2-y}=\frac{4}{3}.$

4. $\frac{24x-2y-x^2}{23-x}=\frac{2x-19}{2},\ \frac{xy-17y-3}{x-18}=\frac{3y+17}{3}.$

5. $\frac{2x+y}{2}-\frac{9x-7}{8}=\frac{3y+9}{4}-\frac{4x+5y}{16},\ \frac{5x-3}{3}-\frac{3x-19}{2}=4-\frac{3y-x}{3}.$

6. $\frac{7+x}{5}-\frac{2x-y}{4}=3y-5,\ \frac{5x-y}{2}+\frac{4x-3}{6}=18-5x.$

7. $\frac{4y-3x+3\frac{3}{4}}{16}-\frac{7y-2x+2\frac{1}{2}}{18}+\frac{1403}{\cdot 576}=7y-11x=0.$

8. $\frac{7x+6}{11}+\frac{4y-9}{3}=3x-\frac{13-x}{2}-\frac{3y-x}{5},\ \frac{3x+4}{2y+3}=\frac{5}{3}.$

9. $\frac{4x}{3}+8y=16,\ \frac{2}{3}+\frac{3(x+y)}{2}=\frac{3x}{2}+\frac{2}{3}\left\{x+\frac{13}{4}-y\right\}.$

10. $\frac{4x+3y-3}{5}-1=\frac{4y+3x-2}{5}-\frac{18-x}{3},\ \frac{3x+y}{7}=\frac{3x-5}{3}+2.$

11. $x-\frac{2y-x}{23}=20-\frac{5y-2x}{2},\ y+\frac{y-3}{18}=30-\frac{73-3y}{3}.$

12. $\frac{3x+2y}{5}-\frac{5x-\frac{3}{4}y+1}{3}=x+\frac{y-2x}{10},\ \frac{12x+6y-3}{6x+12y-1}=1.$

XX.

1. $mx+y-a,\ x-my=b.$ 2. $ax-by=a^2,\ bx-ay=b^2.$

3. $(a-b)x+(c-d)y=p+q,\ (a+b)x-(c+d)y=p-q.$

4. $ax-by=m,\ cx+dy=n.$ 5. $ax+by=c,\ px=qy.$

6. $a(x-y)+b(x+y)+cx-cy=d,\ x-y=1.$

7. $(a+b)x+(b+c)y=d,\ (b-c)x-(c-b)y=e.$

8. $ax+by=xy=bx-ay.$ 9. $axy=c(bx+ay),\ bxy=c(ax-by).$

10. $\frac{x}{a}+\frac{y}{b}=c,\ \frac{x}{a'}+\frac{y}{b'}=c'.$ 11. $\frac{a}{x}+\frac{b}{y}=c,\ \frac{a'}{x}+\frac{b'}{y}=c''.$

12. $ax+by=a-b,\ \frac{x+a}{y+b}=\frac{b}{a}.$ 13. $\frac{x}{a}+\frac{y}{b}=1,\ \frac{x}{3a}+\frac{y}{6b}=\frac{2}{3}.$

14. $\frac{a}{2x}-\frac{b}{3y}=c,\ \frac{a}{2y}+\frac{b}{3x}=d.$ 15. $x\left\{\frac{a}{x}+\frac{b}{y}\right\}=c,\ y\left\{\frac{a'}{x}-\frac{b'}{y}\right\}=c'$

XXI.

1. Find two numbers such that the sum of twice the first and the second is equal to 17, and the sum of three times the second and the first is equal to 19.

2. Find two numbers such that half of the first and three-fourths

of the second together are equal to the difference of three times the first and the second, and this difference is equal to 11.

3. Two digits, which form a number, change places on the addition of 9, and the sum of these two numbers is 33; find the number.

4. There are two numbers, each consisting of three figures. If unity be added to the sum of the two numbers the amount is 1,000. If the smaller number be written as a decimal fraction after the greater, the result is six times the number composed of the greater number written as a decimal fraction after the smaller; find the two numbers.

5. If the less of two numbers be divided by the greater, the quotient is ·21 and the remainder is ·04162. If the greater be divided by the less the quotient is 4 and the remainder ·742; find the numbers.

6. Find two numbers, such that if $\frac{2}{3}$ of the less be added to $\frac{1}{4}$ of the greater, the sum will be 7; but if $\frac{1}{3}$ of the greater be taken from the less, the remainder will be 2.

7. Divide the number 208 into two parts such that the sum of one-quarter of the greater and one-third of the less part when increased by 4 shall be equal to four times the difference of the parts.

8. There is a fraction such that if 5 be added to its numerator the value of the fraction is doubled, and if 4 be taken from its denominator the value of the fraction is trebled; find the fraction.

9. Find a fraction such that when its numerator is diminished by unity, two-thirds of it is equal to 9; and when the denominator is diminished by unity, one-seventh of this shall be equal to 2.

10. There is a fraction such that when its denominator is increased by one the fraction becomes one, but when the numerator is increased by 1 the fraction becomes $\frac{1}{2}$; find the fraction.

11. What fraction is that whose numerator being doubled and the denominator increased by 7, the value of the fraction becomes $\frac{2}{3}$, but if the denominator be doubled and the numerator increased by 2, the value becomes $\frac{3}{5}$.

12. Solve the equations $a_1x+b_1y=c_1$, $a_2x+b_2y=c_2$. What form do the values of x and y take when $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$, and what does it indicate?

XXII.

1. A sum of money consists of half-crowns and sixpences, and is worth as many shillings as there are pieces of money, also the square of the number of sixpences is 36 times the number of half-crowns; find the sum of money.

2. On a division of the House of Commons, if the number of members for the motion had been increased by 40 from the other side, the question would have been carried by 3 to 2; but if those against the motion had received sixty of the other party, the motion would

have been lost by 2 to 1. Did the motion succeed, and how many members voted on the question?

3. A certain number of men do a piece of work in a certain number of days; had there been 4 men more they would have finished the work in 8 days, but three times the number of men would have finished the work in two days less than half the time employed; required the number of men actually employed.

4. Show how £12 can be paid by means of guineas and moidores, so that the total number of coins which pass in the transaction shall be 16.

5. *A*'s annual income is half of *B*'s, and *B* spends £60 a year more than *A*. At the end of two years *A* has saved £200, and *B* £600. What are their yearly incomes?

6. A pound of tea and three pounds of sugar cost six shillings; but if sugar were to rise 50 per cent., and tea 10 per cent., they would cost seven shillings. What were the prices of tea and sugar?

7. Two plugs are opened in the bottom of a cistern containing 192 gallons of water; after three hours one of them becomes stopped, and the cistern is emptied by the other in eleven hours; had six hours occurred before the stoppage it would only have required six hours more to empty it. How many gallons will each plug-hole discharge in an hour, supposing the discharge uniform?

XXIII.

1. $x+y=5,\ x+z=4,\ y+z=8.$
2. $x+y-z=8,\ x+z-y=9,\ y+z-x=10.$
3. $2x+3y=13,\ 5x+4z=26,\ 4y+3z=24.$
4. $x+100=y+z,\ y+100=2x+2z,\ z+100=3x+3y.$
5. $x-y-z=6,\ 3y-x-z=12,\ 72-y-z=24.$
6. $x+y-z=8x+3y-6z=3z-4x-y=1.$
7. $5(2x+3y)=11(x+y),\ 3(x+z)=5(x-z),\ 14(10y-3z)$ $=37(4y-2z).$
8. $3x+2y-z=20,\ 2x+3y+6z=70,\ x-y+6z=41.$
9. $x+2y+3z=17,\ z+2x+3y=12,\ y+2z+3x=13.$
10. $2^2x+3^2y+4^2z=70,\ 5^2x+6^2y+7^2z=224,\ 8^2x+9^2y+10^2z=526.$
11. $2x+3y-8z+35=0,\ 7x-4y+z-8=0,\ 12x-5y-3z+10=0$
12. $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=62,\ \frac{x}{3}+\frac{y}{4}+\frac{z}{5}=47,\ \frac{x}{4}+\frac{y}{5}+\frac{z}{6}=38.$
13. $\frac{y}{2}+\frac{x}{3}+\frac{z}{4}=10,\ y+z+x=29,\ y+2x+3z=62.$
14. $y+\frac{x}{2}=41,\ x+\frac{z}{4}=20\frac{1}{2},\ y+\frac{z}{y}=34.$
15. $\frac{x}{3}+\frac{y}{4}+\frac{z}{5}=5\frac{2}{3},\ \frac{y}{3}+\frac{x}{4}+\frac{z}{10}=5\frac{1}{2},\ \frac{z}{3}+\frac{x}{5}+\frac{y}{6}=5\frac{11}{15}.$

16. $\frac{2}{x}+\frac{1}{y}=\frac{3}{2}$, $\frac{3}{z}-\frac{2}{y}=2$, $\frac{1}{x}+\frac{1}{z}=\frac{4}{3}$.

17. $\frac{xy}{x+2y}=\frac{2}{5}$, $\frac{yz}{2y+3z}=\frac{6}{13}$, $\frac{xz}{3x+4z}=\frac{3}{13}$.

XXIV.

1. $ax+y=c$, $by+z=a$, $cz+x=b$.
2. $c(x+y)=az$, $c(x+z)=by$, $x+y+z=c$.
3. $a(x+ \;=bz$, $b(y+c)=az$, $bcx+acy+abz=0$.
4. $ay+bx=c$, $bz+cy=a$, $cx+az=b$.
5. $x+y+z=a$, $bx+cy+dz=0$, $b^2x+c^2y+d^2z=0$.
6. $x+z=y$, $(a+b)x+(b+c)z=(a+c)y$, $abx+bcz=1+acy$.
7. $x+y+z=0$, $(a+b)x+(a+c)y+(b+c)z=0$, $abx+acy+bcz=1$.
8. $ax+by+cz=a^2$, $bx+cy+az=b^2$, $cx+ay+bz=c^2$.
9. $(b+c)x+(a+c)y+(a+b)z=q$, $x+y+z=p$, $bcx+acy+abz=r$.
10. $ax+by+cz=p$, $a^2x+b^2y+c^2z=q$, $a^3x+b^3y+c^3z=r$.
11. $ax+(b+c)y+(b-c)z=b$, $bx+(c+a)y+(c-a)z=c$,
$cx+(a+b)y+(a-b)z=a$.
12. $ax+cy+bz=cx+by+az=bx+ay+cz=a^3+b^3+c^3-3abc$.
13. $xy=a(x+y)$, $xz=b(x+z)$, $zy=c(z+y)$.
14. $\frac{a}{x}-\frac{b}{y}=\frac{b}{y}-\frac{c}{z}=\frac{c}{z}-\frac{a}{x}$, $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1$.
15. $xy=m(ay+bx)$, $yz=n(cz+dy)$, $xz=p(ez+fx)$.
16. $ax+cy+bz=a^2+2bc$, $cx+by+az=b^2+2ca$, $bx+ay+cz=c^2+2ab$.
17. $\frac{x}{b}+\frac{y}{a}=\frac{ab}{c^2}$, $\frac{x}{c}+\frac{z}{a}=\frac{ac}{b^2}$, $\frac{y}{c}+\frac{z}{b}=\frac{bc}{a^2}$.
18. $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=\frac{x}{b}+\frac{y}{c}+\frac{z}{a}=\frac{x}{c}+\frac{y}{a}+\frac{z}{b}=1$.
19. $\frac{x+y-z}{b+c}=\frac{y+z-x}{c+a}=\frac{z+x-y}{a+b}=a+b+c$.
20. $\frac{ax+by}{cz}=\frac{ax+cz}{by}=\frac{by+cz}{ax}=x+y+z$.
21. $axyz+ayz+cxz+bxy=o$, $bxyz+cyz+bzx+axy=o$,
$cxyz+byz+axz+cxy=o$.

XXV.

Eliminate x, y, z from the following equations:—

1. $x=a(y+z)$, $y=b(x+z)$, $z=c(x+y)$.
2. $(b+c)x+(c+a)y+(a+b)z=0$, $(b-c)x+(c-a)y+(a-b)z=0$,
and $x^{-1}+y^{-1}+z^{-1}=0$.
3. $ax+by+cz=bx+cy+az=cx+ay+bz=1$, when $x^2+y^2+z^2=p^2$.
4. $mn(x-\alpha)=ln(y-\beta)=lm(z-\gamma)$; $m'n'(x'-x)=l'n'(y'-y)$
$=l'm'(z'-z)$.
5. $bx+cy+az=ax+ay+bz=ab+bc+ac$, $x+y+z=a+b+c$.

XXVI.

1. Determine three numbers such that their sum shall be 9, the sum of the first, twice the second, and three times the third shall be 22, and the sum of the first, four times the second, and nine times the third shall be 58.

2. The differences between the first and second, and the second and third digits of a certain number are equal. If the number be divided by the sum of the digits in the units and tens place, the quotient is 107; but if 396 be subtracted from the number its digits will be inverted. What is the number?

3. A certain number of three digits exceeds the sum of the digits by 180. If the digits be reversed it exceeds the sum by 378; but if divided by the sum of the digits, the quotient is 14 and the remainder is 11. Find the number.

4. A number consisting of three digits is equal to 42 times the sum of the middle and left-hand digits, the sum of the digits is equal to 9, and the right-hand digit is twice the sum of the other two. What is the number?

5. The sum of three numbers is $(p+1)(q+1)n$, the sum of the two larger is equal to p times the least, and the sum of the two smaller is q times the greatest. Find the numbers.

6. Divide 27 into four parts such that the first increased by 2, the second diminished by 2, the third multiplied by 2, and the fourth divided by 2, shall give equal results.

7. To find four numbers such that the sum of the first, second, and third shall be 13; the sum of the first, second, and fourth, 15; the sum of the first, third, and fourth, 18; and, lastly, the sum of the second, third, and fourth, 20.

XXVII.

1. A and B together possess £120, B and C £150, and C and A £140. What does each possess?

2. A person has 200 coins consisting of guineas, half-sovereigns, and half-crowns; the sums of money in guineas, half-sovereigns, and half-crowns are as 14, 8, 3. Find the numbers of the different coins.

3. Handel's last oratorio, "Jephtha," was composed in the eighteenth century. The first and last digits in the number expressing its date are the same, and if the digits be inverted the number will be diminished by 180. Find the date.

4. A sum of money which consists of shillings, half-crowns, and sixpences, is worth as many shillings as there are pieces of money; it is also worth as many half-crowns as there are shillings; and the number of sixpences is one more than the number of shillings. Find the number of each of the coins.

5. If three persons, *A*, *B*, *C*, can separately perform a work in the times t_1, t_2, t_3 respectively; in what time they can do it jointly? and conversely, if they can do it jointly in time t, find in what times they can do it separately. Shew also that $(t_1+t_2+t_3)\div t$ can never be less than 9, if t be fixed, and when it is equal to 9, $t_1=t_2=t_3$.

6. *A* and *B* can complete a work in a days, *A* and *C* in b days, and *B* can do p times as much as *C* in a given time; shew that *B* and *C* can complete the work in $\dfrac{(p-1)ab}{(p+1)(b-a)}$ days, and that *A* must do $\dfrac{b-pa}{a-b}$ times as much as *C*. Why cannot p be greater than unity, and a greater than b simultaneously, and consequently this result negative?

7. The effect m is produced by the action of the agents *A* and *B* together in the time a, and by *B* and *C* together in the time b, and by *C* and *A* together in the time c; in what time can each separately, and all together produce the same effect?

8. To complete a certain work *A* requires m times as long a time as *B* and *C* together, *B* requires n times as long as *A* and *C* together, and *C* requires p times as long as *A* and *B* together. Explain the nature of the results which the problem admits. Show that $\dfrac{1}{m+1}+\dfrac{1}{n+1}+\dfrac{1}{p+1}=1$.

XXVIII.

1. Determine the relations which must subsist between the coefficients of three equations involving three unknown quantities in order that one of the equations may be derived from the other two.

2. Explain the meaning of the equations

$$a_1x+b_1y+c_1z=a_2x+b_2y+c_2z=a_3x+b_3y+c_3z=1$$

when the values of x, y, or z take the form of $\dfrac{0}{0}$. Also shew that they are equivalent to two independent equations when

$$\frac{a_1-a_2}{a_2-a_3}=\frac{b_1-b_2}{b_2-b_3}=\frac{c_1-c_2}{c_2-c_3}.$$

3. If $ax+c_1y+b_1z=0$, $c_1x+by+a_1z=0$, $b_1x+a_1y+cz=0$,

shew that $aa_1^2+bb_1^2+cc_1^2=abc+2a_1b_1c_1$,

and $(ab-c_1^2)(ca-b_1^2)x^2=(bc-a_1^2(ab-c_1^2)y^2=(bc-a_1^2)(ac-b_1^2)z^2$.

4. Find the condition which must exist in order that the equations $a_1x+b_1y=c_1$, $a_2x+b_2y=c_2$, $a_3x+b_3y=c_3$, may be consistent; and shew that under this condition the equations $a_1x+b_1y+c_1z=d_1$, $a_2x+b_2y+c_2z=d_2$, $a_3x+b_3y+c_3z=d_3$, are inconsistent.

5. Explain the method of indeterminate multipliers for the solution of linear equations, and reduce the equations $\dfrac{a_1x+b_1y+c_1z+d_1}{m_1}=\dfrac{a_2x+b_2y+c_2z+d_2}{m_2}=\dfrac{a_3x+b_3y+c_3z+d_3}{m_3}$ to the form

$$\frac{x+\alpha}{p}=\frac{y+\beta}{q}=\frac{z+\gamma}{r}.$$

6. If $a = a_1x + a_2y + a_3z$, $b = b_1x + b_2y + b_3z$, $a_1^2 + a_2^2 + a_3^2 = 0$
$b_1^2 + b_2^2 + b_3^2 = 0$, $a_1b_1 + a_2b_2 + a_3b_3 = 0$;

Then shall $\frac{a}{b} = \frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$.

7. Solve the equations $x + y + z = 1$, $ax + by + cz = -(a + b + c)$,
$a^2x + b^2y + c^2z = (a + b + c)^2$;
and shew that $a^4x + b^4y + c^4z = (a^2x + b^2y + c^2z)^2$.

8. Give a solution of the equations $3x + 4y + 5z = 10$,
$4x + 5y + 6z = 12$, $5x + 6y + 7z = 14$,
and point out the meaning of the result.

9. Shew that the following three equations are inconsistent, $27x + 3y + 2z = 40$, $x - y + z = 3$, $2x + 4y - 3z = 12$.

10. Prove that the equations $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$, $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1$, $\frac{a^3}{x} + \frac{b^3}{y} + \frac{c^3}{z} = 0$, are equivalent to only two independent equations if $a + b + c = 0$.

11. The system of equations $x(m + n)^2 + yn^2 + zm^2 = a$,
$xn^2 + y(n + l)^2 + zl^2 = b$, $xm^2 + yl^2 + z(l + m)^2 = c$, is incompatible, if $mn + nl + lm = 0$, unless $la + mb + nc = 0$, and is then indeterminate.

XXIX.

1. Apply the method of indeterminate multipliers to find the value of x from the equations,
$y + z + u = 14$, $2x + 3z + 2u = 33$, $x + y + 2u = 18$, $3x + 2y + 2z = 20$.

2. Given $ax + by + cz + dv = h$, $a^2x + b^2y + c^2z + d^2v = h^2$, $a^3x + b^3y + c^3z + d^3v = h^3$, $a^4x + b^4y + c^4z + d^4v = h$, find x, y, z, v.

3. Shew that the five equations, $2(x_1 - x_2) = (1 + \sqrt{5})(x_3 - x_5)$, $2(x_2 - x_3) = (1 + \sqrt{5})(x_4 - x_1)$, $2(x_3 - x_4) = (1 + \sqrt{5})(x_5 - x_2)$, $2(x_4 - x_5) = (1 + \sqrt{5})(x_1 - x_3)$, $2(x_5 - x_1) = (1 + \sqrt{5})(x_2 - x_4)$, are equivalent only to two independent relations between x_1, x_2, x_3, x_4, and x_5.

4. Eliminate u, x, y, z between the equations
$y + z + u = ax$, $z + u + x = by$, $u + x + y = cz$, $x + y + z = du$.

5. If $A + B + C = D + 2d$. $B + C + D = A + 2a$, $C + D + A = B + 2b$, $D + A + B = C + 2c$, and $A + B + C + D = 2x$.
then shall $x = \frac{1}{3}\{a + b + c + d\}$.

6. If $y = \frac{2}{2 - x}$, $z = \frac{2}{2 - y}$, $u = \frac{2}{2 - z}$, $t = \frac{2}{2 - u}$, find the relation between t and x.

7. What are the conditions necessary in order that an equation, or a series of equations, may give definite values for the unknown quantities contained in them?

8. How many equations must be given in order to eliminate 1, 2, 3, . . . n unknown quantities between them? Explain when and how such elimination is possible.

9. If there be $m+n$ unknown quantities and only m independent equations; then n of the unknown quantities can be considered as perfectly arbitrary and independent in value.

XXX.

Find the solutions in positive integers of the equations :—

1. $2x-3y=1$. 2. $7x-5y=1$. 3. $35x-19y=1$.
4. $19x-117y=11$. 5. $17x-19y=42$. 6. $13x-17y=54$.
7. $13x+7y=141$. 8. $9x+13y=200$. 9. $11x+13y=400$.
10. $9x+13y=2000$. 11. $7x+9y=653$. 12. $11x+5y=100$.
13. $30x+43y=335$. 14. $11x+17y=248$. 15. $32x+11y=182$.
16. $17x+23y=200$. 17. $3079x+2711y=37819000$.

XXXI.

Find the solutions of the following equations in positive integers :—

1. $3x+7y+17z=100$. 2. $6x-4y+7z=190$.
3. $17x+23y+3z=200$. 4. $17x+19y+21z=400$.
5. $12x+15y+20z=1001$. 6. $7x+9y+23z=999$.
7. $4x+5y-14z=49$. 8. $5x+7y+11z=224$.

XXXII.

Determine the solutions in positive integers of the following equations :—

1. $x+2y+3z=20$ and $4x+5y+6z=47$.
2. $6x+7y+4z=122$ and $11x+8y-6z=145$.
3. $4x+5y+7z=560$ and $9x+25y-49z=250$.
4. $3x+8y+9z=656$ and $x+5y+4z=272$.

XXXIII.

1. A certain number when divided by 5 and by 4 leaves a remainder of unity in each case, but when divided by 3 it leaves no remainder; what is the number?

2. What number is that which when divided by 4 leaves a remainder 3, and when divided by 9 leaves a remainder 4?

3. Determine the least number which being divided by 3, 5, 7, the remainders shall be 2, 4, 6 respectively.

4. Find a number which when divided by 6, 5, 4, 3, 2, shall leave the remainders 5, 4, 3, 2, 1 respectively.

5. Can an integral number x be found such that when 4, 5, 6 are respectively taken from x, $2x$, $3x$, and the remainders divided by 7, 8, 9, the quotients shall be integral? If possible, shew that the

number of solutions is indefinite; if not possible, explain the reason.

6. Divide 196 into two parts such that the greater divided by 7 gives a remainder 5, and the less divided by 5 gives a remainder 1.

7. Shew that $(x-a)(y-b)+(y-a)(x-b)=0$ is an indeterminate equation such that for each value of x there is one, and only one value of y, and trace the changes in the value of y, as x changes from a to b.

8. In the equation $ax \pm by = \pm c$, if a and b have a common divisor, which is not also a divisor of c, the solution of the equation is impossible in integers.

9. Shew that the number of solutions in positive integers of the equation $x+y+z=n+z$ is $\frac{1}{2}n(n+1)$.

XXXIV.

1. The sum of two numbers is 78, one of them is divisible by 5 and the other by 3; how many pairs of numbers satisfy these conditions?

2. Divide 1,000 into two parts, such that one shall be divisible by 7 and the other by 11.

3. Find all the numbers of two digits which exceed six times the left-hand digit by 38.

4. There is a number of two digits which, if its digits be reversed, becomes less by 1 than its half. What is the number?

5. Find two fractions whose denominators are prime to each other, (1) when their sum, (2) when their difference is equal to $\frac{32}{45}$.

6. Show that the indeterminate equation $49x+63y=491$, can have no solution in which both x and y are integers.

7. Find what multiples of 355 and of 452 differ from one another by any given number.

XXXV.

1. Find the number of different ways in which a substance of a ton weight may be weighed by weights of 9lb. and 14lb.

2. Show that £20 can be paid in seven different ways in half-guineas and half-crowns.

3. In how many ways may a bill of £1,000 be paid in half-sovereigns and half-crowns?

4. *A*, having only one pound notes, owes 12s. to *B*, who has only seven shilling pieces. What is the least number which *A* and *B* must respectively interchange so that the debt may be discharged?

5. Supposing a sovereign be worth 25 francs, and a napoleon worth 20 francs, find the simplest way of paying a debt of 28 shillings by giving English and receiving French gold.

6. Find a sum consisting of x pounds and y shillings, the double of which shall be y pounds and x shillings.

7. *A* has to pay *B* the sum of 19s. 6d., having only 12 half-crowns and 10 shillings, while *B* has 4 half-crowns and 8 shillings. In how many ways can the payment be effected?

8. If the 3 per cent. consols are at 64 and the 4 per cents. at 84 in how many ways can £3,480 sterling be invested in these stocks? Which is most advantageous? Can there be an equal number of hundreds of each stock?

9. If capital be invested in the 4 per cents at 96, and in the 5 per cents at 105, how much must be invested in each stock so as to realise $4\frac{1}{2}$ per cent. on the capital invested?

XXXVI.

1. Determine three fractions with prime denominators whose sum is $1\frac{23}{30}$.

2. Find three fractions whose denominators shall be 6, 9, 18, whose sum shall be $2\frac{2}{3}$, and the difference of the first and second equal to that of the second and third.

3. In 150 coins consisting of sovereigns, half-crowns, and shillings, if the sum represented by the half-crowns and shillings be equal to that represented by the sovereigns; find the numbers of the coins.

4. Find the number of ways in which the sum of five pounds can be paid in exactly 50 coins, consisting of half-crowns, florins, and fourpenny pieces.

5. Find in how many different ways the sum of £6 16s. can be paid in guineas, crowns, and shillings, so that the number of coins used shall be exactly 16; and in each case determine the number of guineas, crowns, and shillings respectively.

6. In how many ways can a year be made up of 12 months of 28, 30, and 31 days?

7. Four jewellers possessing respectively 8 rubies, 10 sapphires, 100 pearls, and 5 diamonds, presented each from his own stock, one apiece to the rest in token of regard and gratification at meeting; and they thus become owners of stock of precisely equal value; tell what were the prices of their gems respectively. (*Vija-Ganita*, ch. iv.)

8. The horses belonging to four persons respectively are 5, 3, 6, and 8; the camels appertaining to them are 2, 7, 4, and 1; their mules are 8, 2, 1, and 3; and the oxen owned by them are 7, 1, 2, and 1; all are equally rich; tell the rates of the horses and the rest. (*Vija-Ganita*, ch. vi.)

RESULTS, HINTS, ETC., FOR THE EXERCISES ON EQUATIONS OF THE FIRST DEGREE.

I.

1. $x=2$. 2. $x=46$. 3. $x=5$. 4. $x=-\frac{4}{5}$. 5. $x=\frac{4}{5}$.
6. $x=12$. 7. $x=-12$. 8. $x=8$. 9. $x=\frac{11}{5}$. 10. $x=-2$.
11. $x=-3$. 12. $x=36$.

II.

1. $x=2\cdot6$. 2. $x=20$. 3. $x=3$. 4. $x=2\cdot086\dot{2}$.
5. $x=-2$. 6. $x=11\cdot636\ldots$ 7. $x=-12\cdot9411\ldots$

III.

1. $x=7\frac{1}{3}$. 2. $x=2\frac{4}{7}$. 3. $x=-3\frac{1}{6}$. 4. $x=3$. 5. $x=-\frac{2}{3}$.
6. $x=2\frac{4}{7}$. 7. $x=30$. 8. $x=7$. 9. $x=\frac{2}{3}$. 10. $x=-5$.
11. $x=10$. 12. $x=1\frac{2}{3}$.

IV.

1. $x=30$. 2. $x=8\frac{4}{7}$. 3. $x=\frac{4}{7}$. 4. $x=\frac{11}{13}$. 5. $x=2\frac{2}{13}$.
6. $x=1\frac{8}{23}$. 7. $x=1\frac{23}{36}$. 8. $x=\frac{11}{14}$. 9. $x=120$. 10. $x=-3\frac{4}{5}$.
11. $x=4\frac{4}{61}$. 12. $x=2$. 13. $x=7$. 14. $x=5$. 15. $x=7$.
16. $x=8\frac{26}{137}$. 17. $x=2\frac{1}{3}$. 18. $x=\frac{1871}{1874}$. 19. $x=6$.

V.

1. $x=\frac{3}{4}$. 2. $x=1\frac{4}{35}$. 3. $x=-\frac{2174}{4113}$. 4. $x=12\frac{14}{44}$. 5. $x=4$.
6. $x=3\frac{11}{76}$. 7. $x=9\frac{61}{111}$. 8. $x=7$. 9. $x=3$. 10. $x=1\frac{121}{490}$.
11. $x=-3$. 12. $x=\frac{44}{65}$.

VI.

1. $x=-\frac{1}{3}$. 2. $x=4$. 3. $x=\frac{9}{37}$. 4. $x=1\frac{2}{3}$. 5. $x=7$.

6. The equation may be reduced to the form

$$1+\frac{1}{x-3}-\left\{1+\frac{1}{x-4}\right\}=1+\frac{1}{x-6}-\left\{1+\frac{1}{x-7}\right\}.$$

$$\text{or }\frac{1}{x-3}-\frac{1}{x-4}=\frac{1}{x-6}-\frac{1}{x-7},\text{ and }\frac{-1}{(x-3)(x-4)}=\frac{-1}{(x-6)(x-7)}.$$

7. $x=20$. 8. $x=7\frac{2}{37}$. 9. $x=\dfrac{a}{a-2}$. 10. $x=-1\frac{11}{25}$.

11. $x=1\frac{1}{3}$. 12. $x=2$. 13. $x=\frac{5}{64}$. 14. $x=2$.

15. The equation is $x+1+\dfrac{1}{x+1}-x-2-\dfrac{2}{x+2}=x+3+\dfrac{3}{x+3}-x+4-\dfrac{4}{x+4}$.

16. The equation is $-4+\dfrac{18}{3x+1}+2-\dfrac{12}{2x+7}=-9+\dfrac{19}{x+1}+7-\dfrac{19}{x+2}$.

Whence $\dfrac{114}{(3x+1)(2x+7)}=\dfrac{19}{(x+1)(x+2)}$, and $\dfrac{6}{(3x+1)(2x+7)}=\dfrac{1}{(x+1)(x+2)}$.

VII.

1. $x=\dfrac{a-b}{a-1}$. 2. $x=\dfrac{a-b}{a-2b}$. 3. $x=\dfrac{d}{b+c-a}$.

4. $x=\frac{a}{a-b+1}$. 5. $x=\frac{a^2}{b}$. 6. $x=\frac{a^2b}{a^2+b^2}$.

7. $x=\frac{pq-cd}{(d+q)a+(c+p)b}$. 8. $x=\frac{cd-ab}{a+b-c-d}$. 9. $x=\frac{-ab}{2(a+b)}$.

10. $x=\frac{ab}{a+b}$. 11. $x=a+b$. 12. $x=\frac{1}{ab}$.

13. $x=\frac{ad}{bc}$. 14. $x=\frac{ac}{a+b}$. 15. $x=\frac{(a-b)c}{ab}$.

16. $x=\frac{ab}{a^2-b^2}$.

17. The equation is an identity if $ab=bc$; if not, it is impossible.

18. $x=\frac{-1}{a+b}$. 19. $x=\frac{ac+bd-b^2}{(a-c)b}$. 20. $x=\frac{-(a^2+2abc-b^2)}{a-b}$.

VIII.

1. $x=1$.

2. $(a+b)x=(c+d)x$, an identity if $a+b=c+d$, if not, what explanation can be given?

3. $\frac{x^2+(a+b)x+ab}{x+a+b}=\frac{x^2+(c+d)x+cd}{x+c+d}$,

$$x+\frac{ab}{x+a+b}=x+\frac{cd}{x+c+d^2}, \therefore x=\frac{(a+b)cd-(c+d)ab}{ab-cd}.$$

4. $x=\frac{(a-b)(b-c)d}{(a-c).\{a(b-c)+b(a-b)\}}$. 5. $x=\frac{ab}{b+c}$. 6. $x=\frac{b(a+c-b)}{a}$.

7. $x=\frac{-(a^2b+ab^2+a^2c+b^2c+abc)}{ab+ac+bc}$. 8. $x=\frac{355a+348b}{37}$.

9. $x=\frac{-(ab^2+bc^2+ca^2)}{a^2b+b^2c+c^2a}$. 10. $x=\frac{ab^2+bc^2+ca^2+abc}{a^2b+b^2c+c^2a}$.

11. $x=\frac{(b^2-c^2)bc+(c^2-a^2)ac+(a^2-b^2)ab}{(b-c)bc+(c-a)ac+(a-b)ab}=a+b+c$.

12. $x=\frac{3a^2-6ab-3b^2}{2(a+3b)}$. 13. $x=\frac{(a-b+c)b}{2a-c}$. 14. $x=\frac{a+b+c+d}{m+n}$.

15. $x=\frac{(m^2+n)b-(m-n)a}{m^2-m-2n}$. 16. $x=1$. 17. $x=\frac{cd-ab}{a+b-c-d}$.

18. The equation when reduced and simplified becomes

$$a^3+b^3+c^3-3abc=-3\{a^2+b^2+c^2-ab-ac-bc\}x, \text{ and}$$

$$x=-\frac{a^3+b^3+c^3-3abc}{3(a^2+b^2+c^2-ab-ac-bc)}=-\frac{a+b+c}{3}.$$

19. $x=3$. 20. $=\frac{bd-ab}{(a-d)bc}$.

IX.

1. $x=9$. 2. $x=\frac{9}{10}$. 3. $x=2$. 4. $x=2a$. 5. $x=3$.

6. $x=-\frac{3a}{4}$. 7. $x=-\frac{131}{108}$. 8. $x=20\frac{1}{4}$. 9. $x=\frac{ab}{a+b}$.

X.

1. Let x denote one number, then $x+10$ is the other, and

$x+x+10=100$ $\therefore$ $2x=90$, and $x=45$ the less, and $x+10=55$ the greater.

Otherwise, if x denote the larger number, then $x-10$ denotes the smaller, and $x+x-10=100$, $\therefore$ $2x=110$, and $x=55$ the greater, and $x-10=45$ the less.

2. $x=105$. 3. $x=3$. 4. $x=108$. 5. $x=60$.

6. The parts are 80 and 160.

7. Here $\frac{1}{2}$ of the first part $=\frac{1}{3}$ of the second $=\frac{1}{4}$ of the third. Let $2x$ denote th first part, then $3x$ is the second part, and $4x$ the third part $\therefore 2x+3x+4x=144$.

8. $x=90$. 9. The original number is 24. 10. The fraction is $\frac{16}{3}$.

11. The number is 120. 12. $x=9$. 13. Let mx, nx, px, be the parts.

14. Let x denote the number when the first digit is removed, then $100000+x$ expresses the number; but $10x+1=3$ times the number $=300000+3x$.

$$\therefore 7x=299999, \text{ and } x=42857 \therefore \text{ the number is } 142857.$$

15. The parts are $\frac{a}{1+m+mn}$, $\frac{ma}{1+m+mn}$, $\frac{na}{1+m+mn}$.

XI.

1. If A be supposed older than B, from the birth of B they advance equally in age, and it is obvious that A may become x times as old as B some years *after* the given epoch, or some years *before* that epoch. Suppose A becomes m times as old as B in x years *after* the epoch,

$$\text{Then } a+x=n(b+x), \text{ and } (n-1)x=a-nb, \therefore x=\frac{a-nb}{n-1};$$

and x will be *positive* or *negative* according as $a-nb$ is positive or negative; the former refers to time *after*, the latter to time *before* the epoch.

2. $x+44=3(x+4)$, $x=16$, $x+40=56$. 3. The father is 75 years of age.

4. A's age 62, B's 22.

5. $x=-20$, indicating time past, A was therefore twice as old as B in 1830 - 20 or in 1810.

6. One fired 14 shots, the other 18.

XII.

1. They meet when A has travelled $29\frac{9}{11}$ miles from Cambridge, and when B has travelled $26\frac{2}{11}$ miles from London.

2. 500 feet. 3. 90 miles.

4. 8 of the first dynasty, 6 of the second, 3 of the third, and 2 of the fourth.

6. 565 voted for one, 690 for the other.

7. After 10 days, 20 days' provisions are left. If x be the reinforcement, then 20 days' provisions gives $5(x+1000)$ rations; but the 20 days' provisions would have given to 1000 men 20×1000 rations. $\therefore 5(x+1000)=20 \times 1000$.

XIII.

1. A invested £24,000 in the $3\frac{1}{2}$ per cents., and B 25,000 in the 4 per cents.

2. Let x the rate on £4,550, then $\frac{4550x}{100}+\frac{5420(x+1)}{100}=453$.

3. Let x be the sum invested in the 3 per cents., then $\frac{3x}{79\frac{7}{8}}+\frac{5(1000-x)}{106\frac{1}{2}}=40$.

4. Consols were at $81\frac{9}{11}$ and at 90.

5. Let $3x$, $2x$ denote their incomes, then $\frac{3x-1000}{2x-1000}=\frac{5}{3}$, and the incomes were £6,000 and £4,000.

6. Let $x=B$'s share, then A's share is $\frac{p+2a+b}{3}$; B's $\frac{p-a+b}{3}$; C's $\frac{p-a-3b}{3}$.

8. 240 sovereigns, 720 shillings, 960 pence.

9. £8 and 10 half-crowns. 10. Cost price 6s. 8d.

11. $x=$ £2,200, the original sum. 12. £540 original income.

XIV.

1. In the time $\frac{e_3 t_1 t_2}{e_1 t_2 + e_2 t_1}$. The problem may be extonded to 3, 4, &c., agents.

2. Here a men or b boys dig m acres in n days. $\therefore$ 1 man digs $\frac{m}{na}$, and 1 boy digs $\frac{m}{nb}$ acres in 1 day. $\therefore$ $(a-p)$ men dig $\frac{m(x-p)(n-p)}{na}$ acres in $(n-p)$ days. Suppose x boys are wanted, then x boys dig $\frac{m(n-p)x}{nb}$ acres in $m(n-p)$ days, and the whole is $(m+p)$ acres.

$$\therefore \frac{m(a-p)(n-p)}{na}+\frac{m(m-p)x}{nb}=m+p, \text{ from which } x \text{ can be found.}$$

3. C will finish the portion left in 76 days. 4. The hare takes 240 leaps.

7. Let the cistern contain a gallons, and suppose x hours the time required when pipe A is opened. Then in x hours pipe A pours in $\frac{ax}{m}$ gallons, pipe B pours in $\frac{a(x-1)}{n}$ gallons, and pipe C pours out $\frac{a(x-2)}{p}$ gallons.

$$\text{If the cistern become full, then } \frac{ax}{m}+\frac{a(x-1)}{n}-\frac{a(x-2)}{p}=a.$$

$$\text{If the cistern become empty, then } \frac{ax}{m}+\frac{a(n-1)}{n}-\frac{a(x-2)}{p}=0,$$

$$\text{whence } x=\frac{m(n+1)p-2mn}{(m+n)p-mn} \text{ in the former case,}$$

$$\text{and } x=\frac{mp-2mn}{(m+n)p-mn} \text{ in the latter case,}$$

under the condition that in both cases $(m+p)$ is greater than mn; and in the second also that p is greater than $2n$.

8. 1975 men. 9. If x be the number of hours, $x=\frac{abcd}{abc+abd+acd+bcd}$.

10. The chain rule is applicable to questions of equivalents, as the comparison of the standards of moneys, weights, and measures of different nations, and in almost all transactions of international commerce where equivalents are considered.

Questions of this nature may always be most conveniently solved by aid of this rule.

For instance. If there be three different kinds of merchandise, such that
a_1 units of the first kind is equivalent to b_2 of the second kind,
b_1 ,, second ,, ,, c_2 ,, third ,,
c_1 ,, third ,, ,, a_2 ,, first ,,
and if p be the value of an unit of a_1, a_2; q of b_1, b_2; r of c_1, c_2;
then $a_1 b_1 c_1 pqr = a_2 b_2 c_2 pqr$, and therefore $a_1 b_1 c_1 = a_2 b_2 c_2$.

Hence, if any five of these six quantities be given, the sixth may be found by a simple equation.

The number of equivalents may be extended to 4, 5, or to any number whatever.

The following example will illustrate the rule :—

If 3 pounds of tea be worth 4 pounds of coffee, and 15 pounds of coffee be equal to 37 pounds of sugar, how many pounds of sugar are equal in value to 180 pounds of tea.

Here 3lb. tea=4lb. coffee,
15lb. coffee=37lb. sugar,
Let $x$lb. sugar=180lb. tea.

Then $3\times 15\times x=4\times 37\times 180$.

$$\therefore x=\frac{4\times 37\times 180}{3\times 15}.$$
$$=592\text{lb. sugar.}$$

XV.

1. See page 3. 2. See page 1.

4. Here $x-a=\frac{1}{2}(x-b)$, $x-a=\frac{1}{3}(x-c)$, $x-a=\frac{1}{4}(x-d)$. By substituting the value of x, $b+c+d=\frac{1}{2}(a+c+d)$, $b+c+d=\frac{1}{3}(a+b+d)$, $b+c+d=\frac{1}{4}(a+b+c)$, from these equations the relations may be found.

5. $\dfrac{(b-c)bcp-(c-a)acq}{(c-a)acq-(a-b)abr}=\dfrac{(c-a)q+(b-c)p}{(a-b)r-(c-a)q}$.

6. $x=\dfrac{3abc}{2(ab+ac+bc)}$. 10. $a=\dfrac{b(c-1)-c}{1+b+c}$.

XVI.

1. $x=7$, $y=6$. 2. $x=3\frac{21}{41}$, $y=5\frac{15}{41}$. 3. $x=2$, $y=1$. 4. $x=10$, $y=11$.

5. The equations are when reduced, $x-2y=1$, $x-2y=-6$. Explain.

6. $x=5$, $y=7$. 7. $x=-3$, $y=1$. 8. $x=\frac{1}{2}$, $y=\frac{1}{4}$. 9. $x=74\frac{18}{67}$, $y=3\frac{23}{67}$.

10. $x=\frac{1}{3}$, $y=\frac{1}{2}$. 11. $x=4$, $y=16$. 12. $x=\frac{7}{18}$, $y=\frac{1}{10}$.

XVII.

1. $x=12$, $y=3$. 2. $x=39\cdot812\ldots$, $y=58\cdot542\ldots$ 3. $x=\cdot02$, $y=2\cdot9$.

4. The reduced equations are $\cdot125y=\cdot3x$, $3x=-\cdot25y$. Explain them.

5. $x=4$, $y=5$. 6. $x=\frac{8}{1705}$, $y=\frac{911}{341}$.

XVIII.

1. $x=25$, $y=12$. 2. $x=4$, $y=15$. 3. $x=4\frac{12}{25}$, $y=5\frac{8}{25}$.

4. $x=9$, $y=12$. 5. $x=14$, $y=7$. 6. $x=12$, $y=8$.

7. $x=-25\frac{13}{15}$, $y=24\frac{13}{15}$. 8. $x=-77$, $y=-87$. 9. $x=21\frac{1}{11}$, $y=27\frac{9}{11}$.

10. $x=6$, $y=\frac{1}{2}$. 11. $x=60$, $y=40$. 12. $x=\frac{71}{10}$, $y=\frac{383}{60}$.

XIX.

1. Incompatible equations. From (1) $10y=x$, from (2) $3x=74y$.

2. $x=\frac{610}{461}$, $y=-\frac{244}{461}$. 3. $x=-4$, $y=-2$.

4. $x=21$, $y=20$. 5. $x=12\frac{3}{13}$, $y=-2\frac{6}{13}$.

6. The equations when reduced are $6x+55y=328$, $49x-3y=111$.

7. $x=8\frac{13}{42}$, $y=13\frac{1}{2}$.

8. The reduced equations are $911x-478y=1695$, $9x-10y=3$.

9. $x=\frac{49}{37}$, $y=\frac{41}{37}$. 10. $x=4\frac{2}{14}$, $y=58\frac{1}{14}$.

11. $x=105$, $y=7\frac{24}{113}$. 12. $x=-\frac{31}{75}$, $y=-2\frac{14}{75}$.

XX.

1. $x=\dfrac{b+ma}{m^2+1}$, $y=\dfrac{a-mb}{m^2+1}$. 2. $x=\dfrac{a^2+ab+b^2}{a+b}$, $y=\dfrac{ab}{a+b}$.

3. $x=\dfrac{pc+dq}{ac-bd}$, $y=\dfrac{aq+bp}{ac-bd}$. 4. $x=\dfrac{md+nb}{ad+bc}$, $y=\dfrac{na-mc}{ad+bc}$.

5. $x=\dfrac{qc}{aq+bp}$, $y=\dfrac{pc}{aq+bp}$. 6. $x=\dfrac{b+d-a-c}{2b}$, $y=\dfrac{-(a+b+c-d)}{2b}$.

7. $x=\dfrac{(b-c)d-(b+c)e}{(b-c)(a-c)}$, $y=\dfrac{(b-c)d-(a+b)e}{(b-c)(c-a)}$.

8. The equations may be reduced to $\dfrac{a}{y}+\dfrac{b}{x}=1$, and $\dfrac{b}{y}-\dfrac{a}{x}=1$,

and $x=\dfrac{b^2+a^2}{b-a}$, $y=\dfrac{b^2+a^2}{b+a}$.

9. $x=\frac{(a^2+b^2)c}{a^2-b^2}$, $y=\frac{(a^2+b^2)c}{2ab}$. 10. $x=\frac{aa'(bc-b'c')}{a'b-ab'}$, $y=\frac{bb'(ac-a'c')}{ab'-a'b}$.

11. $x=\frac{ab'-a'b}{b'c-bc'}$, $y=\frac{a'b-ab'}{a'c-ac'}$.

12. $x=\frac{-(a-b)(a+b-1)}{2a}$, $y=\frac{(a-b)(a+b+1)}{2b}$.

13. $x=3a$, $y=-2b$.

14. $x=\frac{9a^2+4b^2}{6(3ac+2bd)}$, $y=\frac{9a^2+4b^2}{6(3ad-2bc)}$.

15. The equations when reduced become $(a-c)y=bx$, and $a'y=(b'+c')x$. Explain the meaning of them.

XXI.

1. The numbers are $6\frac{7}{8}$ and $4\frac{1}{5}$.
2. The numbers are 33 and 22.
3. Let x be the digit in the units, y in the tens place, then $10y+x+9=10x+y$, and $x+y=33$, $x=1$, $y=2$.
4. The two numbers are 857 and 842.
5. The numbers required are 5·678 and 1·234.
6. If x, y, denote the numbers, $x=7$, $y=6$.
7. The parts are $145\frac{3}{29}$, and $62\frac{26}{29}$.
8. The fraction is $\frac{5}{8}$.
9. The fraction is $\frac{408}{30}$.
10. If $\frac{x}{y}$ be the fraction, the equations $\frac{x}{y+1}=1$, $\frac{x+1}{y}=\frac{1}{2}$ give $x=-3$, $y=-4$.
11. The fraction is $\frac{4}{5}$.

XXII.

1. The sum is 16 shillings.
2. 375 members voted, 185 for, 190 against the motion.
3. Let x denote the number of men, y the days, and w the work.

Then x men do w in y days, and ∴ 1 man does $\frac{w}{xy}$ in 1 day,

and $x+4$ „ 8 „ „ $\frac{w}{8(x+4)}$ „

also $3x$ „ $\frac{1}{2}y-2$ „ „ $\frac{w}{3x(\frac{1}{2}y-2)}$ „

$$\text{Hence } \frac{w}{xy}=\frac{w}{8(x+4)}=\frac{w}{3x(\frac{1}{2}y-2)}.$$

From these equations $x=8$, $y=12$.

4. If 2 guineas be received, and 14 moidores be given, the difference is £12.
5. A's yearly income is £260, and B's £520.
6. The price of tea was 5s. and of sugar 4d. per pound.
7. One discharges 8 gallons, the other 12 gallons each hour.

XXIII.

1. The equations are $x+y=5$ (1), $x+z=4$ (2), $y+z=8$ (3).

First eliminate x between equations (1) and (2).

$$\begin{array}{c} x+y=5 \\ x+z=4 \\ \hline y-z=1 \text{ (4)}. \end{array}$$

Next eliminate x between (3) and (4),

$$\begin{aligned} y+z&=8 \quad (3) \\ y-z&=1 \quad (4) \\ \hline 2z&=7 \end{aligned}$$

$$\therefore\ z=\tfrac{7}{2},$$

$$\text{and } y=8-z=8-\tfrac{7}{2}=\tfrac{9}{2},$$

$$\text{also } x=5-y=5-\tfrac{9}{2}=\tfrac{1}{2}.$$

2. By adding the first and second equations, $2x=17$; the first and third, $2y=18$; the second and third, $2z=19$.

3. Eliminate x between first and second equations, and $15y-8z=13$. Next eliminate y between this and the third equation, and $z=4$, $y=3$, $x=2$.

4. $x=\frac{600}{11}$, $y=\frac{-300}{11}$, $z=\frac{-200}{11}$.

5. $x=3$, $y=3$, $z=12$. 6. $x=9$, $y=3$, $z=4$.

7. The first equation gives $x=4y$, the second $x=4z$, and the third $y=4z$. Explain these results.

8. $x=5$, $y=6$, $z=7$. 9. $x=1$, $y=2$, $z=4$.

10. $x=37\frac{1}{3}$, $y=-55\frac{7}{9}$, $z=26\frac{1}{3}$.

11. From these equations eliminate z, and the resulting equations are $58x-29y=29$, and $33x-17y=14$.

12. $x=24$, $y=60$, $z=120$. 13. $x=9$, $y=8$, $z=12$.

14. $x=18$, $y=32$, $z=10$. 15. $x=2$, $y=12$, $z=10$.

16. $x=1$, $y=-2$, $z=3$.

17. Invert the equations and simplify, and $x=\frac{32}{37}$, $y=\frac{78}{47}$, $z=\frac{12}{7}$.

XXIV.

1. $ax+y=c$ (1), $by+z=a$ (2), $cz+x=b$ (3).

Eliminate y between (1) and (2),

$$\begin{aligned} abx+by&=bc, \text{ multiplying (1) by } b \\ by+z&=a \quad (2) \\ \hline \therefore\ abx-z&=bc-a \quad (4) \text{ by subtraction.} \end{aligned}$$

Next eliminate z between (3) and (4),

$$\begin{aligned} cz+x&=b \quad (3) \\ abcx-cz&=bc^2-ac, \text{ multiplying (4) by } c. \\ \hline (abc+1)x&=bc^2-ac+b, \text{ by addition.} \end{aligned}$$

$$\therefore\ x=\frac{(c^2+1)b-ac}{abc+1}.$$

$$y=c-ax=\frac{(a^2+1)c-ab}{abc+1}.$$

$$z=a-by=\frac{(b^2+1)a-bc}{abc+1}.$$

2. $z=\frac{c^2}{a+c}$, $y=\frac{c^2}{b+c}$, $x=\frac{(a+c)(b+c)c-(a+b+2c)c^2}{(a+b)(b+c)}$.

3. $x=\frac{a^2b^2c-b^2c^2a-a^2c^2}{a^2b^2+b^2c+a^2c}$, $y=\frac{a^2c^2b-b^2a^2c-b^2c^2}{a^2b^2+b^2c+a^2c}$, $z=\frac{a^2c^2b+b^2c^2a}{a^2b^2+b^2c+a^2c}$.

4. $x=\frac{b^2+c^2-a^2}{2bc}$, $y=\frac{a^2+c^2-b^2}{2ac}$, $z=\frac{a^2+b^2-c^2}{2ab}$.

5. $x=\frac{-acd}{(b-d)(c-d)}$, $y=\frac{-abd}{(b+d)(c-d)}$, $z=\frac{abc}{(b-d)(c-d)}$.

6. $z=\frac{1}{(a+b)(c-a)}$, $y=\frac{1}{2a(c-b)}$, $x=\frac{(a+c)(b-a)}{2a(c-b)(b+a)(c-a)}$.

7. $z=\dfrac{1}{(a-b)(a-c)}$, $y=\dfrac{-1}{(a-b)(b-c)}$, $x=\dfrac{1}{(a-c)(b-c)}$.

8. Eliminate x, then $z=\dfrac{a^2b^2+b^2c^2+a^2c^2-a^3b-b^3c-c^3a}{a^3+b^3+c^3-3abc}$, and similarly x and y may be found.

9. Eliminate x, and $z=\dfrac{c^2p-cq+r}{(a-c)(b-c)}$.

10. Eliminate x, and $z=\dfrac{abp-(a+b)q+r}{(a-c)(c-b)c}$.

11. By adding the three equations and dividing by $a+b+c$, $x+2y=1$.

12. Here $ax+cy+bz=cx+by+az$, and $(a-c)x+(c-b)y+(b-a)z=0$.

$$cx+by+az=bx+ay+cz, \text{ and } (c-b)x+(b-a)y+(a-c)z=0.$$

By eliminating z, $y=\dfrac{(a-c)^2-(c-b)(b-a)}{(b-a)^2-(c-b)(a-c)}\cdot x=\dfrac{(a^2+b^2+c^2-ab-ac-bc)x}{a^2+b^2+c^2-ab-ac-bc}=x.$

By eliminating y, $z=\dfrac{(c-b)^2-(a-c)(b-a)}{(b-a)^2-(a-c)(c-b)}\cdot x=x.$ Hence $x=y=z$.

But $ax+cy+bz=a^3+b^3+c^3-3abc$, $bx+ay+cz=a^3+b^3+c^3-3abc$,

$$cx+by+az=a^3+b^3+c^3-3abc.$$

Adding these equations $(a+b+c)(x+y+z)=3(a^3+b^3+c^3-3abc)$, and $x+y+z=3(a^2+b^2+c^2-ab-ac-bc)=3x$, $\therefore$ $x=a^2+b^2+c^2-ab-ac-bc=y=z$.

13. These equations may be reduced to

$\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{a}$, $\dfrac{1}{x}+\dfrac{1}{z}=\dfrac{1}{b}$, $\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{1}{c}$. Then taking the sum of the equations,

$$\begin{array}{l}\dfrac{2}{x}+\dfrac{2}{y}+\dfrac{2}{z}=\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}\\ \dfrac{2}{x}+\dfrac{2}{y}\qquad=\dfrac{2}{a}\\ \hline \end{array}$$

$$\therefore\ \frac{2}{z}=\frac{1}{b}+\frac{1}{c}-\frac{1}{a}=\frac{ab+ac-bc}{abc}.\quad \therefore\ z=\frac{2abc}{ab+ac-bc}.$$

$$\text{Similarly}\quad \frac{2}{y}=\frac{1}{a}+\frac{1}{c}-\frac{1}{b}=\frac{ab+bc-ac}{abc}.\qquad y=\frac{2abc}{ab+bc-ac}.$$

$$\frac{2}{x}=\frac{1}{b}+\frac{1}{c}-\frac{1}{a}=\frac{ab+bc-ab}{abc}.\qquad z=\frac{2abc}{ac+bc-ab}.$$

14. $x=3a$, $y=3b$, $z=3c$.

15. The equations may be reduced to $\dfrac{a}{x}+\dfrac{b}{y}=\dfrac{1}{m}$, $\dfrac{c}{y}+\dfrac{d}{z}=\dfrac{1}{n}$, $\dfrac{e}{x}+\dfrac{f}{z}=\dfrac{1}{p}$.

16. Add the equations, and $(a+b+c)(x+y+z)=(a+b+c)^2$.

17. $x=\dfrac{bc}{2}\left\{\dfrac{ab}{c^2}+\dfrac{ac}{b^2}-\dfrac{bc}{a^2}\right\}$, $y=\dfrac{ac}{2}\left\{\dfrac{ab}{c^2}+\dfrac{bc}{a^2}-\dfrac{ac}{b^2}\right\}$, $z=\dfrac{ab}{2}\left\{\dfrac{bc}{a^2}+\dfrac{ac}{b^2}-\dfrac{ab}{c^2}\right\}$.

18. Eliminating x, and then from the resulting equations is found the value of $z=\dfrac{\{a^2b^2+a^2c^2+b^2c^2-abc(a+b+c)\}abc}{a^3b^3+a^3c^3+b^3c^3-3a^2b^2c^2}$, and in a similar form the values of x and y. By adding the given equations $x+y+z=\dfrac{3abc}{ab+ac+bc}$.

19. $x+y-z=(b+c)(a+b+c)$, $y+z-x=(c+a)(a+b+c)$, $z+x-y=(a+b)(a+b+c)$, and adding these equations $x+y+z=2(a+b+c)^2$.

20. Here $ax+by=cz(x+y+z)$, $ax+cz=by(x+y+z)$, $by+cz=ax(x+y+z)$.

Hence $(x+y+z-2)(ax+by+cz)=0$, $\therefore$ $x+y+z=2$, $ax+by+cz=0$.

$$x=\frac{2bc}{ab+ac+bc},\quad y=\frac{2ac}{ab+ac+bc},\quad z=\frac{2ab}{ab+ac-bc}.$$

21. Dividing each equation by xyz, there results,

$$\frac{1}{x}+\frac{c}{y}+\frac{b}{x}=-a,\quad \frac{c}{x}+\frac{1}{y}+\frac{a}{z}=-b,\quad \frac{b}{x}+\frac{a}{y}+\frac{1}{z}=-c.$$

From which may be found

$$y=\frac{(bc-a)^2-(c^2-1)(b^2-1)}{(bc-a)(c-ab)-(b-ac)(b^2-1)},\quad x=\frac{(bc-a)^2-(c^2-1)(b^2-1)}{(bc-a)(b-ac)-(c^2-1)(c-ab)},$$

$$z=\frac{(bc-a)^2-(c^2-1)(b^2-1)}{(bc-a)(b-ac)-(c^2-1)(c-ab)}.$$

XXV.

1. Add ax, by, cz to each equation respectively.

Then $(1+a)x=a(x+y+z)$, $(1+b)y=b(x+y+z)$, $(1+c)z=c(x+y+z)$,

$$\therefore\ \frac{1+a}{1+b}\cdot\frac{x}{y}=\frac{a}{b},\quad \frac{1+b}{1+c}\cdot\frac{y}{z}=\frac{b}{c},\quad \frac{1+a}{1+c}\cdot\frac{x}{z}=\frac{a}{c},\ \text{and}\ y=\frac{b}{a}\cdot\frac{(1+a)x}{1+b},\quad z=\frac{c}{a}\cdot\frac{(1+a)x}{1+c}.$$

$$\therefore\ y+z=\frac{b}{a}\cdot\frac{(1+a)x}{1+b}+\frac{c}{a}\cdot\frac{(1+a)x}{1+c},\ \text{and}\ \frac{y+z}{x}=\frac{b}{a}\cdot\frac{1+a}{1+b}+\frac{c}{a}\cdot\frac{1+a}{1+c},$$

$$\text{but}\ \frac{y+z}{x}=\frac{1}{a}\ \therefore\ \frac{1}{a}=\frac{b}{a}\cdot\frac{1+a}{1+b}+\frac{c}{a}\cdot\frac{1+a}{1+c},$$

$$\text{whence}\ (1+a)\{(1+b)c+(1+c)b\}=(1+b)(1+c).$$

Otherwise, here $\frac{1}{a}=\frac{y+z}{x}$ add 1 to each side, and $\frac{1+a}{a}=\frac{x+y+z}{x}$, similarly,

$$\frac{1+b}{b}=\frac{x+y+z}{y},\quad \frac{1+c}{c}=\frac{x+y+z}{z};\ \therefore\ \frac{a}{1+a}+\frac{b}{1+b}+\frac{c}{1+c}=1.$$

2. From the first two equations $y=\frac{(b^2-ac)z}{c^2-ab}$, $x=\frac{(a^2-bc)z}{c^2-ba}$. Substitute these values of x and y in $x^{-1}+y^{-1}+z^{-1}=0$, and $\frac{c^2-ab}{(a^2-bc)z}+\frac{c^2-ab}{(b^2-ac)z}+\frac{1}{z}=0$,

$$\therefore\ \frac{1}{a^2-bc}+\frac{1}{b^2-ac}+\frac{1}{c^2-ab}=0\ \text{is the resulting equation.}$$

3. Find the values of x, y, z from the first three equations and substitute them in the fourth.

4. $$\frac{(ml'-lm')x}{(nl'-ln')x}=\frac{ll'(y'-\beta)+(ml'\alpha-m'lx')}{ll'(z'-\gamma)+(l'n\alpha-ln'x')}.$$

5. Find the values of x, y, z from the three equations, and substitute those values in the third, and reduce the expression to its simplest form.

XXVI.

1. If x, y, z denote the numbers, then $x=1$, $y=3$, $z=5$.
2. Let x, y, z denote the digits in the place of hundreds, tens, units,

then $100z+10y+x$ is the number. $x-y=y-z$, $\frac{100z+10y+x}{x+y}=107$,

$$100z+10y+x-396=100x+10y+z.$$

3. The number is 193. 4. The number is 126.
5. $x=(p+1)n$, $z=(q+1)n$, $y=(pq-1)n$. 6. The parts are 4, 8, 3, 12.
7. Let x, y, z, u, the four parts $x=2$, $y=4$, $z=7$, $u=9$.

XXVII.

1. *A* has £55; *B*, £65; *C*, £85.
2. Let x, y, z denote the guineas, half-sovereigns, and half-crowns respectively,

then $x+y+z=200$, $\frac{21x}{10y}=\frac{14}{8}$, $\frac{21x}{2\frac{1}{2}z}=\frac{14}{3}$. $x=50$, $y=60$, $z=90$.

3. The date is 1751.

4. 8 shillings, 3 half-crowns, and 9 sixpences.

5. The problem gives $\frac{1}{t}=\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}$, and therefore

$$\frac{t_1+t_2+t_3}{t}=\left\{\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}\right\}.\{t_1+t_2+t_3\}=3+\left\{\frac{t_1}{t_2}+\frac{t_2}{t_1}\right\}+\left\{\frac{t_1}{t_3}+\frac{t_3}{t_1}\right\}+\left\{\frac{t_2}{t_3}+\frac{t_3}{t_2}\right\}.$$

But $\frac{t_1}{t_2}+\frac{t_2}{t_1}$ is greater than 2. $\therefore\ \frac{t_1+t_2+t_3}{t}$ is greater than 9.

And if $\frac{t_1+t_2+t_3}{t}=9$, then $\frac{t_1}{t_2}+\frac{t_2}{t_1}=2$, and $\left\{\frac{t_1}{t_2}-\frac{t_2}{t_1}\right\}^2=0$, &c. $\therefore\ t_1=t_2=t_3$.

7. Here A, B, effect m in a units of time, $\therefore\ \frac{m}{a}$ in an unit of time.

„ B, C, „ b „ $\therefore\ \frac{m}{b}$ „

„ C, A, „ c „ $\frac{m}{c}$ „

$\therefore\ A$, B, C together do $\frac{m}{2}\left\{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right\}$ in an unit of time.

Let A, B, C together effect m in x units, they do $\frac{m}{x}$ in one unit,

$\therefore\ \frac{m}{x}=\frac{m}{z}\left\{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right\}$, and $x=\frac{2abc}{ab+ac+bc}$.

Next suppose A does m in x units, B in y, and C in z,

then A, B, C do $\frac{m}{x}$, $\frac{m}{y}$, $\frac{m}{z}$ respectively in an unit.

Then $\frac{m}{x}+\frac{m}{y}=\frac{m}{a}$, $\frac{m}{y}+\frac{m}{z}=\frac{m}{b}$, $\frac{m}{x}+\frac{m}{z}=\frac{m}{c}$,

and $\frac{1}{x}+\frac{1}{y}=\frac{1}{a}$, $\frac{1}{y}+\frac{1}{z}=\frac{1}{b}$, $\frac{1}{x}+\frac{1}{z}=\frac{1}{c}$,

whence $x=\frac{2abc}{ac+bc-ab}$, $y=\frac{2abc}{ab+bc-ac}$, $z=\frac{2abc}{ac+ab-bc}$.

8. Suppose A completes the work in x days, B in y days, and C in z days,

Then $\frac{1}{y}+\frac{1}{z}=\frac{m}{x}$, $\frac{1}{x}+\frac{1}{z}=\frac{n}{y}$, $\frac{1}{x}+\frac{1}{y}=\frac{p}{z}$,

and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{m+1}{x}$, $\therefore\ \frac{1}{m+1}=\frac{yz}{xy+xz+yz}$. Similarly $\frac{1}{n+1}=\frac{xz}{xy+xz+yz}$,

and $\frac{1}{p+1}=\frac{xy}{xy+xz+yz}$, $\therefore\ \frac{1}{m+1}+\frac{1}{n+1}+\frac{1}{p+1}=\frac{xy+xz+yz}{xy+xz+yz}=1$.

XXVIII.

1. This may most readily be effected by the method of indeterminate multipliers.

3. By eliminating z from the first and second equations, $\frac{ax+c_1y}{c_1x+by}=\frac{b_1}{c_1}$,

and z between the second and third equations, $\frac{c_1x+by}{b_1x+a_1y}=\frac{a_1}{c_1}$.

Next eliminate x and y from these two equations,

and $\frac{aa_1-b_1c_1}{cc_1-a_1b_1}=\frac{bb_1-a_1c_1}{a_1^2-bc}$, from this may be deduced;

$$aa_1^2+bb_1^2+cc_1^2=abc+2a_1b_1c_1.$$

5. See Art. 4, p. 11.

7. First eliminate x between the first and second equations, and between the second and third; next eliminate y between the resulting equations, and the value of z is determined: then find the values of x and y. Having found the values o x, y, z, then $a^4x+b^4y+c^4z$ may be readily shewn equal to $(a^2x+b^2y+c^2z)^2$.

XXIX.

4. From the first and second equation, $y=\frac{(a+1)x}{b+1}$:

from the second and third $z=\frac{(b+1)y}{c+1}=\frac{(a+1)x}{c+1}$:

from the third and fourth, $u=\frac{(c+1)z}{d+1}=\frac{(a+1)x}{d+1}$.

Substitute in the first equation,

$$\therefore\ \frac{(a+1)x}{b+1}+\frac{(a+1)x}{c+1}+\frac{(a+1)x}{d+1}=ax, \text{ and } \frac{1}{b+1}+\frac{1}{c+1}+\frac{1}{d+1}=\frac{a}{a+1}.$$

Otherwise, since $y+z+u=ax$, add x to each side, then $x+y+z+u=(1+a)x$.

And $\frac{x}{x+y+z+u}=\frac{1}{1+a}$; similarly for the rest of the equations; then taking the sum, there results $\frac{1}{1+a}+\frac{1}{1+b}+\frac{1}{1+c}+\frac{1}{1+d}=1$.

6. By successive substitutions of y, z, u, in terms of x, it will be found that $x=t$.

7. See pp. 1, 2.

XXX.

See the two examples solved in the note, page 13.

The general values of x and y in terms of a third unknown quantity, p, are here given, from which the numerical values of x and y can be found.

1. $x=3p-2$, $y=2p-1$. Here p must not be less than unity.
2. $x=5p-2$, $y=7p-3$.
3. $x=19p+6$, $y=3p-19$.
4. $x=117p+56$, $y=19p+9$.
5. $x=19p-2$, $y=17p-4$.
6. $x=17p+12$, $y=13p+6$.
7. $x=7p-1$, $y=22-13p$.
8. $x=28-13p$, $y=9p-4$.
9. $x=34-13p$, $y=11p+2$.
10. $x=228-13p$, $y=9p-4$.
11. $x=92-7p$, $y=7p+1$.
12. $x=5p$, $y=20-11p$.
13. $x=43p+4$, $y=5-30p$.
14. $x=21-17p$, $y=11p+1$.
15. $y-34-11p$, $x=11p-6$.
16. $x=17p+5$, $y=5-23p$.
17. $x=7000$, $y=6000$ is one solution of the equation. Any number can readily be found when one is known.

XXXI.

1. $3x+7y+17z=100$, then $3x+7y=100-17z$. In order that $100-17z$ may be positive, z must not be greater than 5.

If z be taken equal to 1, 2, 3, 4, 5, then will arise the following five different equations: $3x+7y=83$, $3x+7y=66$, $3x+7y=49$, $3x+7y=32$, $3x+7y=15$.

And from these equations the values of x and y may be determined.

2. $6x-4y+7z=190$, $6x-4y=190-7z$. In order that $190-7z$ may be a positive integer, z must be an even number less than 27. And by substituting for z, 1, 2, 3, &c., there will be found 13 values of $190-7z$, to each of which $6x-4y$ may be made equal. Each of these equations will give an unlimited number of solutions.

XXXII.

1. $x+2y+3z=20$, $4x+5y+6z=47$.

By eliminating x, $y+2z=11$, and $y=11-2z$.

It is obvious that z must not be greater than 5.

When $z=$	1,	2,	3,	4,	5.
then $y=11-2z=$	9,	7,	5,	3,	1.
and $x=20-2y+3z=$	-1,	0,	1,	2,	3.

Hence the equations admit of only three solutions in positive integers

2. $6x+7y+4z=122$, $11x+8y-6z=145$.

Eliminate z, $40x+37y=656$,

from which $y=8-40p$, $x=37p+9$.

Here p must not be greater nor less than 0,

and $x=9$, $y=8$, $z=3$, or the equations admit of only one solution in positive integers.

3 and 4 require no remarks.

XXXIII.

1. Let x be the number, then $\frac{x-1}{5}$, $\frac{x-1}{4}$, $\frac{x}{3}$ are integers.

If $\frac{x}{3}=p$, then $x=3p$; substitute this in $\frac{x-1}{5}$, then $\frac{3p-1}{5}$ is an integer, also $\frac{6p-2}{5}$ is an integer, but $\frac{5p}{5}$ is an integer, $\therefore \frac{p-2}{5}$ is an integer. Let $\frac{p-2}{5}=q$, then $p=5q+2$. Hence $x=3(5q+2)$. Substitute this value of x in $\frac{x-1}{4}$, then $\frac{15q+6-1}{4}$ or $\frac{15q+5}{4}$ is an integer, $\therefore \frac{3q+1}{4}$ is an integer, but $\frac{4q}{4}$ is an integer, $\therefore \frac{q-1}{4}$ is an integer. Let $\frac{q-1}{4}=r$, $\therefore q=4r+1$, $\therefore x=15q+6=15(4r+1)+6=60r+21$, and in order that x may be a positive integer, r may be 0, 1, 2, &c. And the least value of $x=21$. The number of solutions is unlimited.

2. The number is 175.

3. The least number is 104, and the number of solutions is unlimited.

4. If x denote the number required, then $\frac{x-5}{6}$, $\frac{x-4}{5}$, $\frac{x-3}{4}$, $\frac{x-2}{3}$, $\frac{x-1}{2}$ are integers; the least value of x may be obtained by the same process as above in the first example, and will be found to be 59.

6. Let x, y be the two numbers, then $x+y=196$,

also $\frac{x-5}{7}$ and $\frac{y-1}{7}$ are integers. Let $\frac{x-5}{7}=p$, and $\frac{y-1}{5}=q$.

then $x=7p+5$ and $y=5q+1$, $\therefore x+y=7p+5+5q+1=196$,

and $7p+5q=190$, which admits of 5 solutions in positive integers.

$x=$ 40, 75, 110, 145, 180.

$y=$ 156, 121, 86, 51, 16.

8. Let $a=mp$, $b=mq$, then the equation $ax\pm by=\pm c$ becomes $mpx\pm mqy=\pm c$, and $px\pm qy=\frac{\pm c}{m}$, but m is not a divisor of c, $\therefore \frac{c}{m}$ is an irreducible fraction.

XXXIV.

1. Let $5x$, $8y$ be the numbers, then $5x+8y=78$.

2. One part is 406, and the other 594.

3. If x be the digit in the place of tens, y in that of units, then $10x+y=6x+38$, and there are nine solutions.

5. If x, y denote the numerators of the fractions, then

$$\frac{x}{5}+\frac{y}{9}=\frac{32}{45}, \text{ and } \frac{x}{5}-\frac{y}{9}=\frac{32}{45}.$$

7. If x, y be the multiples, $355x-452y=m$ any number assignable at pleasure.

XXXV.

1. If there be x weights of 9lb. each, and y of 14lb. each, then $9x+14y=2240$.

2. If x denote the number of half-guineas, and y the half-crowns,

then $\frac{21x}{2}+\frac{5y}{2}=400$, and $x=5p$, $y=160-21p$, from which it appears p must not exceed 7.

3. In 1999 different ways.

4. The least number required is 2 one-pound notes and 4 seven-shilling pieces.

5. Here 25 francs are equal to 20 shillings, ∴ 1 shilling is equal to $\frac{25}{20}$ or $\frac{5}{4}$ of one franc, and ∴ 28 shillings is equivalent to 35 francs.

Let x denote the sovereigns, y the Napoleons, then $25x-20y=35$, and $x=4p+3$, $y=5p+2$, p must not be less than 0, then $x=3$, $y=2$, are the least values of x and y. The number of solutions is unlimited.

6. It will be found that £13 6s. is the double of £6 13s.

7. Let x half-crowns and y shillings be required to pay the debt, then $\frac{5x}{2}+y=19\frac{1}{2}$, or $5x+2y=39$, whence $x=5p+2$, p not less than 1, and $y=7-2p$, p not greater than 3.

8. Here £100 stock in 3 per cents. = £64 sterling, ∴ £1 stock $=\frac{64}{100}=£\frac{16}{25}$ sterling, and £100 stock in 4 per cents. = £84 sterling, ∴ £1 stock $=\frac{84}{100}=£\frac{21}{25}$ sterling. Let x denote the stock in the 3 per cents., and y that in the 4 per cents., then $\frac{16x}{25}+\frac{21y}{25}=3480$, from which $x=5448-21p$, p not greater than 259, $y=16p-8$, p not less than 1. Hence there are 259 solutions, the least is when $p=1$.

9. The interest of £1 in 4 per cents. is $£\frac{4}{99}=\frac{1}{24}$, the interest of £1 in the 5 per cents is $£\frac{5}{105}=\frac{1}{21}$. Let x pounds be invested in the 4 per cents., and y in the 5 per cents. Then $\frac{x}{24}+\frac{y}{21}=4\frac{1}{2}$, and $7x+8y=756$, which gives 13 different solutions.

XXXVI.

1. The prime factors of 30 are 2, 3, 5. Let x, y, z be the respective numerators, then $\frac{x}{2}+\frac{y}{3}+\frac{z}{5}=\frac{53}{30}$, and $15x+10y+6z=53$.

2. Let x, y, z be the respective numerators of the fractions whose denominators are 6, 9, 18. Then $\frac{x}{6}+\frac{y}{9}+\frac{z}{18}=\frac{8}{3}$ and $\frac{x}{6}-\frac{y}{9}=\frac{y}{9}-\frac{z}{18}$. Then $3x+2y+z=48$, and $3x-4y+z=0$, ∴ $y=8$, and $z=32-3x$. In order that z be positive, x must not be greater than 10, nor less than 1.

6. Suppose x months of 28 days, y of 30, and z of 31, then $x+y+z=12$, and $28x+30y+31z=365$, if the year be not leap year.

7. Let v, x, y, z, denote the value of a ruby, sapphire, pearl, and diamond respectively. The result is $z=96y=6x=4v$, and the problem is indeterminate. If the value of any one be assigned, the values of each of the rest can be determined.

8. Let v, x, y, z denote respectively the price of a horse, camel, mule, and ox.

$$5v+2x+8y+7z=3v+7z+2y+z=6v+4x+y+2z=8v+x+3y+z.$$

Suppose the property of each denoted by p, the values of v, x, y, z can be found in terms of p, which being unassigned, the problem is indeterminate. The least values of v, x, y, z are 85, 76, 31, 4 respectively.

ELEMENTARY ALGEBRA,
WITH BRIEF NOTICES OF ITS HISTORY.

SECTION VIII.

QUADRATIC EQUATIONS.

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QUADRATIC EQUATIONS.

Art. 1. A quadratic equation, or an equation of the second degree, is one which involves the second power of the unknown quantity, with or without the first power of that quantity. Every equation of the second degree involving only one unknown quantity can be reduced to one of the forms, $ax^2+bx+c=0$, $ax^2+bx=0$, or $ax^2+c=0$, in which a, b, c may be positive or negative.

If the equation be reducible to the form $ax^2-c=0$, then $ax^2=c$, and $x^2=\frac{c}{a}$, and the value of the unknown quantity is found by extracting the square root of the terms on each side of the equation.

Taking the square root of $x^2=\frac{c}{a}$, $\therefore x=\pm\left\{\frac{c}{a}\right\}^{\frac{1}{2}}$, the values of x being $+\left\{\frac{c}{a}\right\}^{\frac{1}{2}}$ and $-\left\{\frac{c}{a}\right\}^{\frac{1}{2}}$, since the square of the same positive and negative quantity always produces the same positive quantity.*

If the equation, when reduced, assume the form $x^2+bx=0$, or $x(x+b)=0$, the values of x are $x=0$ and $x=-b$.

If the reduced equation involve both the first and second powers of the unknown quantity, as $ax^2+bx+c=0$; when each term is divided by a, it becomes $x^2+\frac{bx}{a}+\frac{c}{a}=0$, and putting p and q for $\frac{b}{a}$ and $\frac{c}{a}$ respectively, the equation becomes $x^2+px+q=0$, or $x^2+px=-q$.

The expression $x^2\pm px$ may constitute the first two terms of the square of a binomial quantity $x\pm a$, whose square is $x^2\pm 2ax+a^2$. And it is obvious that four times the product of the first and third terms of this expression is equal to the square of the second.

Hence if m be such a number as when added to $x^2\pm px$, the expression $x^2\pm px+m$ becomes a complete square; then $4mx^2=p^2x^2$, and $m=\frac{p^2}{4}=\left\{\frac{p}{2}\right\}^2$; or, the quantity to be added to $x^2\pm px$ to make it a

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* As the square root of every quantity is both positive and negative, the sign ± equally marks both the left and right sides of an equation, when the square root is either extracted or expressed.

When the square root of $x^2=\frac{c}{a}$ or $x^2=b$ (putting b for $\frac{c}{a}$ to simplify the expression), the equation assumes the form $\pm x=\pm b^{\frac{1}{2}}$, which gives four equalities:

$$+x=+b^{\frac{1}{2}},\ +x=-b^{\frac{1}{2}};\ -x=+b^{\frac{1}{2}},\ -x=-b^{\frac{1}{2}}.$$

Of these the first and fourth are obviously the same, as also the second and third. Hence $x=\pm b^{\frac{1}{2}}$ includes the four expressions, and it will be sufficient to mark only one side of the equation with the double sign ±.

complete square, is the square of half the coefficient of the first power of the unknown quantity, and $x^2 \pm px + \frac{p^2}{4}$ is a complete square.

By means of this property a complete quadratic equation can be reduced to a simple equation, and the values of the unknown quantity determined.

2. *To solve the quadratic equation $x^2+px+q=0$, or to find the values of x which satisfy the equation.*

Taking the form $x^2+px+q=0$, and transposing q to the other side of the equation, $x^2+px=-q$, then adding the square of half the coefficient of x to each side of the equation, in order to make the left side a complete square, it becomes $x^2+px+\frac{p^2}{4}=\frac{p^2}{4}-q=\frac{p^2-4q}{4}$,

extracting the square root of each side,

$$x+\frac{p}{2}=\frac{\pm(p^2-4q)^{\frac{1}{2}}}{2}, \text{ and } x=\frac{-p\pm(p^2-4q)^{\frac{1}{2}}}{2},$$

$$\therefore\ x=\frac{-p+(p^2-4q)^{\frac{1}{2}}}{2}, \text{ and } x=\frac{-p-(p^2-4q)^{\frac{1}{2}}}{2}$$

are the two values of the unknown quantity of the equation

$$x^2+px+q=0.$$

In the same manner, if $x^2+px-q=0$ be the form of the equation, then $x=\frac{-p\pm(p^2+4q)^{\frac{1}{2}}}{2}$.[*]

If p^2+4q be a complete square, then $(p^2+4q)^{\frac{1}{2}}$ is an integer, and the two values of x are positive or negative integers or fractions.

If p^2+4q be not a complete square, then $(p^2+4q)^{\frac{1}{2}}$ is a surd number, and the values of x can only be approximately determined by extracting the square root of p^2+4q to any number of decimal places required.

When the quadratic equation is of the form

$$x^2+px+q=0,\ x=\frac{-p\pm(p^2-4q)}{2}.$$

If $p^2-4q=0$, then $p^2=4q$, and the two roots of the quadratic equation are equal.

If p^2 be greater than $4q$, and if p^2-4q be a complete square, then the two roots of the equation are rational; if p^2-4q be not a complete square the two roots are irrational.

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[*] Solve the quadratic equation $x^2-12x=13$, and verify the correctness of the values of x. Here $x^2-12x=13$. Complete the square by adding to each side of the equation the square of half the coefficient of x, then $x^2-12x+36=13+36=49$. Extracting the square root of each side of the equation, $x-6=\pm 7$, $x=6\pm 7$, or $x=13$ and -1.

Verification: when $x=13$, $x^2-12x=(13)^2-12\times 13=169-156=13$.

When $x=-1$, $(-1)^2-12\times -1=1+12=13$.

And both values of x satisfy the proposed equation.

But if p^2 be less than $4q$, p^2-4q is a negative quantity, the square root of which is impossible, and both the roots of the equation are impossible.

3. *The quadratic equation* $ax^2+bx+c=0$ *may be solved without reducing the coefficient of* x^2 *to unity.*

Transposing c, the equation is $ax^2+bx=-c$, then multiplying each side by four times the coefficient of x^2, the equation becomes

$$4a^2x^2+4abx=-4ac, \text{ or } (2ax)^2+2b(2ax)=-4ac.$$

Next adding b^2, the square of the coefficient of x, to each side, then $(2ax)^2+2b(2ax)+b^2=b^2-4ac$, the left side of which is a complete square.

Extracting the square root of each side, $2ax+b=\pm(b^2-4ac)^{\frac{1}{2}}$,

$$\text{and } \therefore\ x=\frac{-b\pm(b^2-4ac)^{\frac{1}{2}}}{2a}.$$

The equation $ax^2+bx+c=0$ may also be solved by multiplying both sides of the equation by the coefficient of x^2, then $a^2x^2+abx=-ac$ or $(ax)^2+b(ax)=-ac$, in which the unknown quantity x is changed into ax, a multiple of that quantity.

Next add to each side the square of half the coefficient of ax, then $(ax)^2+b(ax)+\frac{b^2}{4}=\frac{b^2-4ac}{4}$, and extracting the square root,

$$\text{then } ax+\frac{b}{2}=\frac{\pm(b^2-4ac)^{\frac{1}{2}}}{2}, \text{ and } \therefore\ x=\frac{-b\pm(b^2-4ac)^{\frac{1}{2}}}{2a}.^*$$

4. It may be remarked, that when the second power of the unknown quantity is affected with a coefficient, this method of solution will be found the most convenient in practice.

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* Solve the equation $13x^2+7x=360$. Multiply each side by 4 times the coefficient of x^2, then $676x^2+364x=18720$. Add to each side the square of 7, and $676x^2+364x+49=18720+49=18769$. Extracting the square root,

$$26x+7=\pm137, \text{ and } 26x=-7\pm137=130 \text{ and } -144,$$

$$\therefore\ x=5 \text{ and } \frac{-72}{13}.$$

Verification:

When $x=5$, $13x^2+7x=13\times5^2+7\times5=325+35=360$,

$$x=\frac{-72}{13},\ 13x^2+7x=13\times\left\{\frac{-72}{13}\right\}^2+7\times\frac{-72}{13}=\frac{5184}{13}-\frac{504}{13}=\frac{468}{13}=360.$$

Here both values of x satisfy the equation.

Solve the equation $7x^2-4x=3$, and verify the correctness of the values of x.

Multiply each side of the equation by 7, then $(7x)^2-4(7x)=21$.

Next complete the square, $(7x)^2-4(7x)+4=21+4=25$.

Extract the square root, $7x-2=\pm5$, and $7x=2\pm5=7$ and -3,

$$\therefore\ x=1 \text{ and } \frac{-3}{7}.$$

Verification:

When $x=1$, then $7x^2-4x=7\times1^2-4\times1=7-4=3$,

$$x=\frac{-3}{7},\ 7\times\left\{\frac{-3}{7}\right\}^2-4\left\{\frac{-3}{7}\right\}=\frac{9}{7}+\frac{12}{7}=\frac{21}{7}=3.$$

Both values of x satisfy the proposed equation.

In the solution of equations of the second degree, certain anomalies arise, besides those which occur in the solution of equations of the first degree. It is possible that either one or both of the values of the unknown quantity obtained do not satisfy the proposed equation. This anomaly arises when a term in the proposed equation containing the unknown quantity is affected by the square root.

Every quantity has two equal square roots, one positive and the other negative, and if an expression of the form $+(mx+n)^{\frac{1}{2}}$ occur in an equation, the square of this quantity is $mx+n$, the same as the square of $-(mx+n)^{\frac{1}{2}}$.

Any proposed equation involving this expression, when reduced may be $px+q=+(mx+n)^{\frac{1}{2}}$ or $px+q=-(mx+n)^{\frac{1}{2}}$, and when each side of these equations is squared, they assume the same form

$$(px+q)^2=mx+n \text{ in both cases.}$$

Here an ambiguity is introduced in the course of the solution, and in the equations so reduced the distinction does not appear which existed in the proposed equations.

Hence it follows that a new condition is introduced, which will be manifest in the result, so that one or both values of the unknown quantity obtained may not satisfy the proposed equation. The roots so introduced are called roots of solution, as distinguished from the proper roots which always satisfy the equation.*

5. *A quadratic equation cannot have more than two values of the unknown quantity.*

Let a, b be the values of x in the equation $x^2+px+q=0$, and, if possible, let c be a third value of x.

---

* Solve the equation $2x+(7x+11)^{\frac{1}{2}}=9$.

Here $(7x+11)^{\frac{1}{2}}=9-2x$ by transposing $2x$, next $7x+11=81-36x+4x^2$, by squaring each side of the equation, and $4x^2-43x=-70$ by transposition.

Multiply each side by 4, and $(4x)^2-43(4x)=-280$.

Complete the square on the left side by adding to each side the square of half the coefficient of the first power of the unknown quantity,

$$\therefore\ (4x)^2-43(4x)+\frac{1849}{4}=\frac{1849}{4}-280=\frac{729}{4},$$

$$\text{extracting the square root, and } 4x-\frac{43}{2}=\frac{\pm 27}{2},\ 4x=\frac{43\pm 27}{2}=35 \text{ and } 8,$$

$$\therefore\ x=\frac{35}{4} \text{ and } 2.$$

If these two values of x satisfy the proposed equation, they are the proper roots of the equation.

Let $x=\frac{35}{4}$, then $2x+(7x+11)^{\frac{1}{2}}=\frac{35}{2}+\left\{\frac{245}{4}+11\right\}^{\frac{1}{2}}=\frac{35}{2}+\left\{\frac{289}{4}\right\}^{\frac{1}{2}}=\frac{35}{2}+\frac{17}{2}=26$, which ought to be 9, and $x=\frac{35}{4}$ does not satisfy the equation, and is a root of solution.

Let $x=2$, then $2x+(7x+11)^{\frac{1}{2}}=4+(14+11)^{\frac{1}{2}}=4+(25)^{\frac{1}{2}}=4+5=9$.

Hence $x=2$ does satisfy the proposed equation, and is a true root of it.

Then $a^2+pa+q=0$, $b^2+pb+q=0$, $c^2+pc+b=0$,
and $(a^2-b^2)+p(a-b)=0$, $(b^2-c^2)+p(b-c)=0$,
$\therefore$ $a+b+p=0$, and $b+c+p=0$,
whence $a-c=0$, or $a=c$.

Therefore a quadratic equation cannot have more than two values of the unknown quantity.

6. *To find the relation between the roots and the coefficients of a quadratic equation.*

Let a, b be the roots of the equation $x^2+px+q=0$, then both a and b, when substituted respectively for x, satisfies the equation,

$$\therefore\ a^2+pa+q=0, \text{ and } b^2+pb+q=0.$$

Then $(a^2-b^2)+p(a-b)=0$, by subtraction, and dividing by $a-b$,—

$$\therefore\ (a+b)+p=0, \text{ and } p=-(a+b).$$

Next substituting this value of p in either of the preceding equations, $a^2+pa+q=0$, or $b^2+pb+q=0$, and $a^2-(a+b)a+q=0$, or $q=ab$.

These results, $p=-(a+b)$ and $q=ab$ indicate, that the coefficient of the second term of the equation $x^2+px+q=0$, is equal to the sum of the roots with contrary signs, and the third term q is the product of the roots with contrary signs.

This property may be otherwise shewn by means of an identity.

For a, b are the roots of the equation $x^2+px+q=0$,

then $x=a$ and $x=b$; $x-a=0$, and $x-b=0$,
$\therefore$ $(x-a)(x-b)=0$, or $x^2-(a+b)x+ab=0$,

which is an equation identical to $x^2+px+q=0$.

Hence $p=-(a+b)$ and $q=ab$.

It also appears that when the roots of a quadratic equation are given the equation can be formed.

7. If one of the three roots of a cubic equation can be discovered by inspection, or be found by any process, the other two roots can always be determined by a quadratic equation.

The solution of all equations of the fourth degree of the form $x^4+qx^2+r=0$, and other equations that can be reduced to that form, can be effected by means of quadratic equations.

And in general, any equation in which the unknown quantity appears in two terms having the index of one, double the index of the other, can be reduced to the form of a quadratic equation, and the complete solution effected by a further operation of involution or evolution.

If the given equation when reduced be of the form $x^{2n}-px^n+q=0$, in which n is any number integral or fractional. If $x^n=y$, then by substitution, the equation becomes $y^2-py+q=0$, from which is determined $y=\frac{1}{2}\{p\pm(p^2-4q)^{\frac{1}{2}}\}$, and restoring x^n the value of y, $x^n=\frac{1}{2}\{p\pm(p^2-4q)^{\frac{1}{2}}\}$, and extracting the nth root of this equation,

$$x=\{\tfrac{1}{2}[p\pm(p^2-4q)^{\frac{1}{2}}]\}^{\frac{1}{n}}.$$

This method is also applicable for the solution of equations which may be reduced to the form $(x^{2n}+px^n+q)^{2n}-p'(x^{2n}+px^n+q)^n+q'=0$. For let $x^{2n}+px^n+q=y$, then by substitution the equation is reduced to $y^{2n}-p'y^n+q'=0$.

In quadratic equations involving two unknown quantities, when the equations are homogeneous, their solution may be facilitated by assuming one of the unknown quantities to be some unknown multiple of the other.

8. *To solve the homogeneous equations of the second degree,*

$$a_1x^2+b_1xy+c_1y^2=d_1,\quad a_2x^2+b_2xy+c_2y^2=d_2,$$

involving two unknown quantities.

Let $y=vx$, v being some unknown quantity, substituting this value of y in the given equations, they become $x^2(a_1+b_1v+c_1v^2)=d_1$, $x^2(a_2+b_2v+c_2v^2)=d_2$. Dividing the former by the latter,

$$\frac{a_1+b_1v+c_1v^2}{a_2+b_2v+c_2v^2}=\frac{d_1}{d_2},$$

whence $(c_1d_2-c_2d_1)v^2+(b_1d_2-b_2d_1)v=a_2d_1-a_1d_2$, a quadratic equation, from which v can be found; and by substituting this value of v in either of the given equations, the value of x is determined; also from $y=vx$, the value of y is also known.

In a similar manner, if there be given three homogeneous equations involving three unknown quantities, x, y, z, the solution in general may be simplified by assuming $y=ux$ and $z=vx$, where u and v are unknown quantities.

If these assumptions for y and z be substituted in the proposed equations, x may be eliminated, and the two resulting equations will involve u and v only, from which u and v can be found.

When, however, the equations are not homogeneous, the solution must be effected by considering the relation of the unknown quantities in the equations, and noting what operations are possible. Then by eliminating such unknown quantities in order as are least involved in the equations, the proposed equations may be reduced to an equation involving only one unknown quantity. And when this quantity is found, its value can be substituted in any of the given or derived equations, and the remaining unknown quantities determined.

Sometimes the solution of equations involving two unknown quantities may be rendered more easy by assuming $x+y$ and $x-y$ instead of x and y for the unknown quantities.

Also when an equation involves the sum, the product, the sum of the squares, cubes, &c., of the unknown quantities, the solution may become less operose by assuming $x+y=p$ and $xy=q$, and finding x^2+y^2, x^3+y^3, &c., in terms of p, q, and making these substitutions in the proposed equations.

With respect to indeterminate equations of the second degree, no general rules can be prescribed for their solution. A few of the more simple indeterminate equations involving one, two, and three unknown quantities have been added to this section.

9. In the solution of problems involving quadratic equations, since every quadratic equation, in general, admits of two values of the unknown quantity, which may be both positive or both negative, or one positive and the other negative; it is necessary, when the values are possible, to consider their meanings.

When both the values of the unknown quantity are positive, it will be found that these positive values give two solutions of the problem. If both be negative, neither gives a direct solution of the problem. When the values of the unknown quantity are, one positive and the other negative, it will be found that the positive value fulfils the direct conditions of the problem; and that the negative value will answer the conditions of the problem where one or more of the conditions are made the contrary of what they were in the proposed problem. This, with other anomalies, arises from the fact that the algebraical expression of an equation is more general than the verbal enunciation of the problem, and it may include other conditions than those expressed in the proposed problem.*

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* By selling an article for a pounds, as much per cent. is lost as the article cost; what was the prime cost of the article? Solve the question when the article was sold for £24, £25, £26 respectively.

Let x pounds denote the prime cost of the article, then $x - a$ is the loss on the sale of it. But x is the loss on £100,

$\therefore \frac{x}{100}$ is the loss on £1, and $\frac{x^2}{100}$ is the loss on x pounds, the prime cost,

$\therefore \frac{x^2}{100} = x - a$, and $x^2 - 100x = -100a$, from which $x = 50 \pm 10(25 - a)^{\frac{1}{2}}$.

If $a = £24$, then $x = 50 \pm 10$, or $x = £60$ and £40, two positive values of x, which indicate two solutions of the problem. If $a = £25$, then $x = 50$.

If $a = £26$, then $x = 50 \pm 10\sqrt{-1}$, and the problem is impossible.

A person buys a certain number of oxen for £80, and finds that if he had bought four more for the same money, they would have cost one pound a-piece less; how many did he buy?

Let x denote the number of oxen bought for £80, then $\frac{80}{x}$ is the price of one ox.

If 4 more, or $x + 4$ were bought for the same money, then $\frac{80}{x+4}$ would be the price of each.

And $\frac{80}{x+4} = \frac{80}{x} - 1$, by the conditions of the problem, $\therefore x^2 + 4x = 320$, $x = 16, -20$.

The positive value $x = 16$ gives the answer to the question. The negative value $x = -20$, gives the answer to the question of a contrary nature.

To explain the negative value of x. In the equation for x write $-x$,

$$\text{then } \frac{80}{-x+4} = \frac{80}{-x} - 1, \text{ or } \frac{80}{x-4} = \frac{80}{x} + 1,$$

which, when translated into words, expresses the following question—

A person sold a number of oxen for £80, and if he had sold 4 fewer for the same money, he would have received a pound a-piece more for each. How many did he sell?

EXERCISES.

EQUATIONS OF THE SECOND DEGREE.

I.

1. Upon what principle is the degree of an equation estimated when it involves the unknown quantity, affected by the index (1) of any power, and (2) of any root? Give examples.

2. What is a quadratic equation? State and prove the process for the solution of quadratic equations, both of those which do not and those which do require the completion of the square.

3. What is the distinction between *proper* roots and roots of *solution?* In which class of equations do the latter occur? Give examples of such roots.

4. In solving the quadratic equation $x^2-4x+2=0$, take every step in order, and describe the process leading to the result in plain and intelligible language.

5. Establish a proposition by means of which, if the roots of a quadratic equation be given, the equation itself may be formed.

Ex. Form the quadratic equations whose roots shall be respectively 3, 4; —3, —4; —3, 4; 3, —4; and conversely, find the roots of the equations so formed.

6. Under what circumstances will a quadratic equation have only one root?

7. What is the meaning of a negative quantity when it occurs as the answer to a question?

8. If $a^{\frac{1}{2}}+b^{\frac{1}{2}}$ be a root of the quadratic equation $x^2+px+q=0$, then shall $a^{\frac{1}{2}}-b^{\frac{1}{2}}$, $b^{\frac{1}{2}}-a^{\frac{1}{2}}$, $-a^{\frac{1}{2}}-b^{\frac{1}{2}}$ be also roots of the same equation.

9. What quadratic equation has its roots $m^{\frac{1}{2}}\div\{m^{\frac{1}{2}}\pm(m-n)^{\frac{1}{2}}\}$?

10. The trinomial ax^2+bx+c becomes 42, when $x=4$; 22 when $x=3$; and 8 when $x=2$: what is the value of a, b, c?

11. Is $2\{(x-a)(x-b)+(a-x)(a-b)+(b-x)(b-a)\}=$
$(x-a)^2+(x-b)^2+(a-b)^2$ an identity or an equation?

12. Resolve $x^2+73x+780$ into two factors of the first degree.

13. Separate the expression $mn(x^2+1)+(m^2+n^2)x$ into its primary factors.

14. Determine the value of the expression $\frac{x+a}{x-a}+\frac{x+b}{x-b}$, when x is a root of the equation $x^2+(a+b)x=3ab$.

II.

1. $x^2+4x=5$. 2. $x^2-4x=5$. 3. $x^2+x=20$.
4. $2x^2-24=22x$. 5. $3x^2-53x+34=0$. 6. $5x-3=-2x^2$.
7. $2x^2+4=7x$. 8. $5x^2-20x+14=0$. 9. $13x-7x^2+1=0$.

10. $7x^2-3x=160.$ 11. $3x^2-21x-450=6000.$ 12. $12x^2+x=1.$
13. $110x^2-21x+1=0.$ 14. $786x^2+73x+1=0.$
15. $625x^2-35x=86.$ 16. $12x^2-25x+12=0.$ 17. $x^2=5x+6000.$
18. $9x-10x^2+4=0.$ 19. $\cdot 3x^2+2\cdot 1x+1=0.$
20. $x^2-\cdot 318x=\cdot 000319.$ 21. $\cdot 0075x^2+\cdot 75x=5.$
22. $25x+2x^2=\cdot 0025.$ 23. $(x-4)^2+2(x-4)=8.$
24. $(9+5\sqrt{3})x^2+(15+7\sqrt{3})x+6=0.$

III.

1. $x+\frac{1}{x}=3.$ 2. $2x=4+\frac{6}{x}.$ 3. $x+\frac{1}{16x}=\frac{1}{2}.$
4. $3x+\frac{3}{x}=10.$ 5. $x-\frac{1}{x-3}=3.$ 6. $\frac{x}{3}=\frac{2}{x-1}.$
7. $\frac{2x}{x-1}=\frac{x-1}{x}.$ 8. $\frac{3(x^2+9)}{4}=\frac{x-3}{8}.$ 9. $x^2=\frac{x}{7}-\frac{4}{847}.$
10. $\frac{3x^2}{4}+\frac{2x}{7}=\frac{37}{112}.$ 11. $\frac{x+2}{x-2}+x=2x-1.$
12. $(x+5)^2+4x^2=160.$ 13. $1-\frac{2-x}{x}=\frac{x+2}{2}-1.$
14. $x+\frac{24}{x-1}=3x+4.$ 15. $\frac{x^2}{3}-\frac{x}{6}+5=6.$
16. $\frac{x+22}{3}-\frac{4}{x}=\frac{9x-6}{2}.$ 17. $\frac{x-1}{x-3}+2x=12.$
18. $\frac{1}{x-2}-\frac{2}{x+2}=\frac{3}{5}.$ 19. $\frac{x+1}{x}+\frac{x}{x+1}=\frac{25}{12}.$
20. $\frac{3(x^3-1)}{x-1}=\frac{19}{9}+x(7-x).$ 21. $\frac{x}{5}+\frac{5}{x}=\frac{26}{5}.$

IV.

1. $x^4+12x^2=13.$ 2. $3x+2x^{\frac{1}{2}}-1=0.$ 3. $6x^{\frac{4}{3}}-5x^{\frac{2}{3}}=-592.$
4. $4x^4-176x^2=-25.$ 5. $3x^4-7x^2=43076.$
6. $256x^4-16x^2-2=0.$ 7. $x^6-7x^3=8.$ 8. $x^6+6a^3x^3=16a^6.$
9. $x^{10}+31x^5=32.$ 10. $x+x^{\frac{1}{2}}+2=0.$ 11. $x^4-a^2x^2+b^4=0.$
12. $(2x)^{\frac{1}{2}}-7x=-52.$ 13. $x^4+x^{-4}=6.$ 14. $x^{-1}+ax^{-2}=2a^2.$
15. $x^{\frac{1}{3}}+x^{-\frac{1}{3}}=a-2.$ 16. $x^3-6x^{\frac{3}{2}}-16=0.$ 17. $3x^3+42x^{\frac{3}{2}}=3321.$
18. $(x-5)^3-3(x-5)^{\frac{3}{2}}=40.$ 19. $x^{\frac{3}{2}}-25x^{\frac{3}{4}}=54.$ 20. $x^{\frac{4}{3}}+7x^{\frac{2}{3}}=44.$
21. $2x^4-88x^2+13=0.$ 22. $7x^{\frac{1}{3}}-20x^{\frac{1}{6}}=3.$ 23. $x^{2n}+4x^n=5.$
24. $3x^{\frac{4n}{3}}+4x^{\frac{2n}{3}}=4.$ 25. $3x^{\frac{5}{3}}+x^{\frac{5}{6}}=3104.$ 26. $x^{\frac{6}{5}}+x^{\frac{3}{5}}=756.$
27. $x^{\frac{5}{2}}+x^{\frac{5}{4}}=1056.$

V.

1. $(x^2+4)^2-5(x^2+4)=44.$ 2. $(x^2+5)^2=7+x^2.$
3. $(x^2+7x+5)^2-3x^2-21x=19.$ 4. $x^2-5x-19=(x^2-5x+1)^{\frac{1}{2}}.$
5. $9x-4x^2+(4x^2-9x+11)^{\frac{1}{2}}=5.$ 6. $6(x^2-2x+6)^{\frac{1}{2}}=21+2x-x^2.$

7. $6x(x-1)-3\{3(x-2)(x+1)-2(x-5)\}^{\frac{1}{2}}=4(x+3)$.

8. $3x^2+8x-(28x^2-48x)^{\frac{1}{2}}=12-4x^2$.

9. $x^2+3=2\{(x^2-2x+2)^{\frac{1}{2}}+x\}$. 10. $7x-x^2+(2x^2-14x+20)^{\frac{1}{2}}=12$.

11. $\frac{x}{x+4}+\frac{4}{(x+4)^{\frac{1}{2}}}=\frac{21}{x}$. 12. $x+4+\left\{\frac{x+4}{x-4}\right\}^{\frac{1}{2}}=\frac{12}{x-4}$.

13. $\frac{10(8x-15)}{x^2-4x+5}-\frac{12(11x-15)}{x^2-6x+6}=13x^2$.

14. $\frac{2}{(x+2)^{\frac{1}{2}}}+\frac{(x+2)^{\frac{1}{2}}}{2}=\frac{17}{4(x+2)^{\frac{1}{2}}}$.

15. $\frac{x}{x^2+x+5}+\frac{5x}{(x^2+x+5)^{\frac{1}{2}}}=\frac{116}{25x}$.

VI.

The following equations, chiefly of the third degree, may be solved by means of equations of the second degree.

1. $x^3-12x=16$.
2. $(x+1)(x+2)(x+3)=x^3$.
3. $x^3+x^2-12=0$.
4. $x^3-2x^2-4x+8=0$.
5. $2x^3-x^2=1$.
6. $(x^2+1)(x+2)=2$.
7. $x^3-6x-9=0$.
8. $x+7x^{\frac{1}{3}}=22$.
9. $x^3-3x=2$.
10. $x^3+7x=49(x+1)$.
11. $x^3+x^2+4=0$.
12. $x^3-6x^2+10x-8=0$.
13. $x^3-x^2+3x-27=0$.
14. $4x^{\frac{2}{3}}-5x^{\frac{1}{3}}-44=0$.
15. $(x-1)^3=(x+1)^2+2$.
16. $x^3=8+4(x+3)(x-2)$.
17. $(2x^2-4x+1)(x^2-3x+4)=(x^2-5x+2)(2x^2-2x+3)$.
18. $x-3x^{\frac{1}{3}}-2=0$.
19. $x+1=2x^{\frac{1}{3}}$.
20. $x^{\frac{2}{3}}+(a-1)x^{\frac{1}{3}}=a$.
21. $(x+1)x^{\frac{1}{2}}=2$.
22. $x^2-2x^{\frac{3}{2}}+10x-9x^{\frac{1}{2}}=90$.
23. $x-3=2x^{-\frac{1}{2}}$.
24. $4(x^3+2x+5)x-4(5x^3+x^{\frac{1}{2}}-1)=5x^2$.

VII.

The following equations, chiefly of the fourth degree, may be solved by means of equations of the second degree.

1. $x^4-4x+3=0$.
2. $x^4-6=5x(x^2-x-1)$.
3. $x^4+4x^2=27$.
4. $x^4-2ax^3=2abx+b^2$.
5. $x^4+x^2+10x=0$.
6. $x^4+4x^3-8x-165=0$.
7. $x^4+4a^3x=a^4$.
8. $(x^2+1)^3=(x^2-1)^2(x-1)^2$.
9. $x^4+(1-x)^4=a$.
10. $1+x^4=a(1-x)^4$.
11. $a(1+x^4)=2x(1-x^4)^{\frac{1}{2}}$
12. $x^2(x^2+16x)+30x(2x-1)=45+2x$.
13. $x^4-2x^3+x=12$.
14. $x^4+4b^2(1+a^2x^4)^{\frac{1}{2}}=c^2$.
15. $(a+x)^4-(a^2+x^2)^2=4a^2x^2$.
16. $x^2(x^2+18)=4(x-12)$.
17. $(4x-2)^4-(3x-1)^4=(2x+4)^4-(x+3)^4$.
18. $(x^2-1)^4+(x-2)^4=2(x^2-1)^2(x^2-2)^2+4x^2-12x+9$.
19. $16x(x+1)(x+2)(x+3)=9$.
20. $(x+1)(x+2)(x+3)(x+4)=48$.
21. $3(x^6-1)+8x^2(x^2-1)=0$.
22. $x^4-8x^3+10x^2+24x+5=0$.

23. $(1+x^5)+(1+x)^5=2(1+x+x^2)^4$. 24. $x^4-2x^3-3x^2+12x+36=0$.

25. $(x^4-2x^2+1)+x(x^3-2x+1)=0$. 26. $x^4+4x^3-19x^2+4x+1=0$.

27. $x^4-12x^3+44x^2-48x=9009$. 28. $x^4+\frac{1}{3}x^3-39x=81$.

29. $\dfrac{x^6}{a^2b^4}=\dfrac{a^2+x^2}{b^2+x^2}$. 30. $\dfrac{x^2(x^4-a^4)}{x^6-a^6}=\dfrac{850}{931}$.

VIII.

1. $(3x-2)(x-4)=0$. 2. $(x+7)(x-10)=40600$.

3. $(6x-2)^2+(8x-3)^2=(7x-1)^2+3(2x-1)^2$.

4. $(x-9)(x-7)(x-5)(x-1)=(x-2)(x-4)(x-6)(x-10)$.

5. $(x-a)(x-b)(x-c)+abc=0$. 6. $x^2+(a-b)x=ab$.

7. $(a-x)(x-d)=(c-x)(x-b)$. 8. $(x+a)^2=5ax-(x-a)^2$.

9. $(x+a)(x+b)+(x-a)(x-b)=a^2+b^2$.

10. $abx^2-2x(a+b)(ab)^{\frac{1}{2}}=(a-b)^2$.

11. $(3a^2+b^2)(x^2-x+1)=(3b^2+a^2)(x^2+x+1)$.

12. $(x+2a)(x-a)^2=(x+2b)(x-b)^2$.

13. $3x^4-(x-a)^4-(x-b)^4-(x-c)^4+a^4+b^4+c^4=12abcx$.

14. $(x+b+c)(x+c+a)(x+a+b)=(x+a)(x+2b)(x+3c)$.

IX.

1. $\dfrac{12+2x}{x+3}+\dfrac{4x-3}{1+2x}=\dfrac{4x-1}{x-1}$. 2. $\dfrac{x(x-1)}{(x+1)(x+2)}=\dfrac{(x+3)(x+4)}{(x+5)(x+6)}$.

3. $\dfrac{2}{x^2-7x+3}-\dfrac{2}{x^2+7x+2}=4$. 4. $\dfrac{25x^2-16}{10x+8}=\dfrac{3x^2-12}{2x-4}$.

5. $\dfrac{x^3-10x^2+1}{x^2-6x+9}=x-3$. 6. $\dfrac{4x+3}{9}=\dfrac{8x-19}{18}-\dfrac{7x-29}{5x-12}$.

7. $\dfrac{1}{1-x}+\dfrac{1}{1+x}+\dfrac{1}{1+2x}=\dfrac{3}{(1+x)(1-x)(1+2x)}$.

8. $\dfrac{1}{x-1}+\dfrac{1}{x-2}+\dfrac{1}{x-3}=\dfrac{3x^2}{(x-1)(x-2)(x-3)}$.

9. $\dfrac{1}{x-1}-\dfrac{4}{x-2}+\dfrac{4}{x-3}-\dfrac{1}{x-4}=\dfrac{1}{30}$.

10. $\dfrac{x-1}{(x-2)(x-3)}+\dfrac{x-2}{(x-3)(x-1)}+\dfrac{x-3}{(x-1)(x-2)}=\dfrac{7}{6}$.

11. $\dfrac{12}{x(x-2)}-\dfrac{6}{(x-2)(x-1)}-\dfrac{21}{(x+2)(x+3)}+\dfrac{126}{(x-3)(x+3)}=18$.

12. $\dfrac{40}{x^2+2x-48}-\dfrac{20}{x^2+9x+8}-\dfrac{8}{x^2+10x}+\dfrac{12}{x^2+5x+50}+1=0$.

13. $\dfrac{(x+1)(x+2)}{6}+\dfrac{(x+2)(x+3)}{4}=2+\dfrac{(x+3)(x+4)}{10}$.

14. $\dfrac{x+2}{3}\cdot\dfrac{x+3}{4}\cdot\left\{\dfrac{x+1}{2}-\dfrac{2+4}{5}\right\}=\dfrac{x(x-1)(x+5)}{2.4.5}$.

15. $\dfrac{2x^2-x-1}{x-2}+\dfrac{2x^2-3x-8}{x-3}=\dfrac{8x^2-9}{2x-3}$.

16. $\frac{12x^2-28x+100}{x-3}+\frac{x^2+19x+4}{x-5}=\frac{13x^2-14x+100}{x-2}+\frac{20x+4}{x-6}.$

17. $\frac{(x-4)^2}{x-5}+\frac{(x-8)^2}{x-9}=\frac{(x-7)^2}{x-8}+\frac{(x-5)^2}{x-6}.$

X.

1. $\frac{1+a}{1-ax}=\frac{b-x}{1+bx}.$ 2. $\frac{a}{x-a}-\frac{b}{x-b}=\frac{a-b}{x-c}.$

3. $\frac{a+b}{x+b}-\frac{a-c}{x-c}=\frac{2(a+b+c)}{x+b+c}.$ 4. $\frac{a^2(x-b)}{a-b}+\frac{b^2(x-a)}{b-a}=x^2.$

5. $\frac{a^2+ax+x^2}{a^2-ax+x^2}=\frac{a^2}{x^2}.$ 6. $\frac{x^2+c^2}{x+a+b}=\frac{x^2+a^2}{x+c+d}.$

7. $\frac{x+ab}{x^2-a^2b^2}-\frac{x-cd}{x^2-c^2d^2}=\frac{ab-cd}{x^2-abcd}.$ 8. $\frac{x^3+a^3}{x+a}+\frac{x^3-a^3}{x-a}=4a^2.$

9. $\frac{x+a}{x-a}+\frac{x+b}{x-b}+\frac{x+c}{x-c}=3.$ 10. $\frac{a(b-c)}{a+x}+\frac{b(c-a)}{b+x}+\frac{c(a-b)}{c+x}=0.$

11. $\frac{1}{c+(b-a)x}-\frac{1}{a+(b-c)x}=1.$ 12. $\frac{1}{x+a}+\frac{1}{x+b}+\frac{1}{x+c}=0.$

13. $\frac{x-a}{x-b}+\frac{x-a-b}{x-b-c}=\frac{x+a}{x+b}+\frac{x+a+b}{x+b+c}.$

14. $\frac{a}{x+b+c}+\frac{b}{x+a+c}=\frac{a-c}{x+b}+\frac{b+c}{x+a}.$ 15. $\frac{1}{x+a+b}=\frac{1}{x}+\frac{1}{a}+\frac{1}{b}.$

16. $\frac{1}{a+x}+\frac{1}{b+x}=\frac{1}{a-c+x}+\frac{1}{b-c+x}.$

17. $\frac{a-bx}{a+bx}+\frac{ax-b}{ax+b}=\frac{2(a-b)}{a+b}.$ 18. $\frac{(x+a)(x+mb)}{(x-ma)(x-b)}=\frac{(mx+a)(x+b)}{(x-a)(x-mb)}.$

19. $\frac{1}{x+a+b}+\frac{1}{x-a+b}+\frac{1}{x+a-b}+\frac{1}{x-a-b}=0.$

20. $\frac{x-b}{x-a}+\frac{x-a}{x-b}=\frac{x+b-2a}{x+a-2b}+\frac{x+a-2b}{x+b-2a}.$

21. $x^2+\frac{b^2c^2}{4}\cdot\frac{1}{x^2}-\frac{b^2+c^2+a^2}{2}=\frac{(a+b-c)(a+c-b)}{2}.$

22. $\frac{a}{x}+\frac{x}{bc}+\frac{b}{x}+\frac{x}{ca}+\frac{c}{x}+\frac{x}{ab}=\frac{2(a+b+c)}{\surd(abc)}.$

23. $\frac{1}{(a+b)(x-c)}+\frac{1}{(a+c)(x-b)}+\frac{1}{(b+c)(x-a)}=0.$

24. $\frac{(x+b)(x+c)}{(x-b)(x-c)}+\frac{(x+c)(x+a)}{(x-c)(x-a)}+\frac{(x+a)(x+b)}{(x-a)(x-b)}=3.$

25. $\frac{(x-b)(x-c)a^2}{(a-b)(a-c)}+\frac{(x-c)(x-a)b^2}{(b-c)(b-a)}=x^2.$

26. $\frac{(x-b)(x-c)a^3}{(a-b)(a-c)}+\frac{(x-c)(x-a)b^3}{(b-c)(b-a)}+\frac{(x-a)(x-b)c^3}{(c-a)(c-b)}=x^3.$

27. $\frac{(x+a)(x+b)}{(x-a)(x-b)}+\frac{(x-a)(x-b)}{(x+a)(x+b)}=\frac{(x+c)(x+d)}{(x-c)(x-d)}+\frac{(x-c)(x-d)}{(x+c)(x+d)}.$

XI.

1. $(3x-2)^{\frac12}(2x+3)^{\frac12}=12.$ 2. $(x^2-3x+7)^{\frac12}+(x^2-4x+6)^{\frac12}=14.$
3. $(1+7x)^{\frac12}-(7-2x)^{\frac12}=(3x-18)^{\frac12}.$
4. $(a+x)^{\frac12}+(b+x)^{\frac12}=2(a+b+x)^{\frac12}.$
5. $(x^2-8x+15)^{\frac12}+(x^2+2x-15)^{\frac12}=(4x^2-12x+18)^{\frac12}.$
6. $(a+x)^{\frac13}+(b+x)^{\frac13}+(c+x)^{\frac13}=0.$
7. $(a+b)(a^2+b^2+x^2)^{\frac12}-(a-b)(a^2+b^2-x^2)^{\frac12}=a^2+b^2.$
8. $x^2+x-4=x^{\frac12}(x+x^{\frac12}+2).$
9. $(a+x)^{\frac23}-(a-x)^{\frac23}=(a^2-x^2)^{\frac13}.$ 10. $x^2-2x+4=2(x^3-1)^{\frac12}.$
11. $\{4x^6+x^2(x^2+3a^2)^2\}^{\frac12}=(a^2+x^2)(4a^2+x^2)^{\frac12}.$ 12. $x(x+1)=48+14x^{\frac12}.$
13. $(x+a)^{\frac12}+(x+b)^{\frac12}+(x+c)^{\frac12}=(x+a+b+c)^{\frac12}.$
14. $(1+x+x^2)^{\frac12}-(1-x+x^2)^{\frac12}=mx.$
15. $mx=\{(1-x)^{\frac12}-1\}\cdot\{(1-x)^{\frac12}+1\}.$ 16. $(x^4-1)^{\frac12}+(x^2-1)^{\frac12}=x^2.$
17. $\{(1+a)^2+(1-a)^2x\}^{\frac12}+\{(1-a)^2+(1+a)^2x\}^{\frac12}=2a.$
18. $(x^2-x^{-2})^{\frac12}+(1-x^2)^{\frac12}=x^2.$ 19. $(2x^2+1)^{\frac12}=\{x^2+(2x^4-x^2-3)^{\frac12}\}^{\frac12}.$
20. $(x^{-\frac12}+1)^{-\frac12}+(x^{-\frac12}-1)^{-\frac12}=x^{\frac12}.$
21. $x^{\frac12}+\dfrac{9}{x^{\frac12}+2}+\dfrac{8}{x}=0.$ 22. $9(112-8x)^{\frac12}=19+(3x-7)^{\frac12}.$

XII.

1. $\dfrac{x+x^{\frac12}}{x-x^{\frac12}}=\dfrac{3x^{\frac12}+6}{2x^{\frac12}}.$ 2. $\dfrac{x+x^{\frac12}}{x-x^{\frac12}}=\dfrac{x^3-x}{4}.$ 3. $\left\{\dfrac{x+2a}{x+2b}\right\}^{\frac12}=\dfrac{x+a}{x+b}.$
4. $\dfrac{(1+x)^{\frac12}}{1+(1+x)^{\frac12}}=\dfrac{(1-x)^{\frac12}}{1-(1-x)^{\frac12}}.$ 5. $\dfrac{2x-a}{(x^2-ax+a^2)^{\frac12}}=\dfrac{2x-b}{(x^2-bx+b^2)^{\frac12}}.$
6. $\dfrac{x+(x^2-16)^{\frac12}}{x-(x^2-16)^{\frac12}}=(x-13)^2.$ 7. $\dfrac{x+(x^2-a^2)^{\frac12}}{x-(x^2-a^2)^{\frac12}}=\dfrac{x}{a}.$
8. $\dfrac{(a^2+x^2)^{\frac12}-(a^2-x^2)^{\frac12}}{(a^2+x^2)^{\frac12}+(a^2-x^2)^{\frac12}}=x^2.$ 9. $\dfrac{a+x}{(b^2+x^2)^{\frac12}-x}=c.$
10. $\dfrac{a+x}{(a-x)^{\frac12}}+\dfrac{a-x}{(a+x)^{\frac12}}=2a^{\frac12}.$ 11. $\left\{\dfrac{x}{x-1}\right\}^2-\left\{\dfrac{x}{x+1}\right\}^2=4(x^2-1).$
12. $\left\{\dfrac{a+x}{a-x}\right\}^2-1=\dfrac{cx}{ab}.$ 13. $\left\{\dfrac{2x+a+c}{2x+b+c}\right\}^2=\dfrac{x+a}{x+b}.$
14. $\left\{\dfrac{x-a}{x-b}\right\}^3=\dfrac{x-2a+b}{x+a-2b}.$

XIII.

1. $(40-x)^2=5x^2.$ 2. $(x^4+a^4)^2-(x^4-a^4)^2=2a^8.$
3. $(x^3-2x^2-2x+3)(x^3-4x^2+4x-3)=$
$(x^3+2x^2-2x-3)(x^3+4x^2+4x+3).$
4. $(x^2+4)x^2=1+4x^2.$ 5. $(x+1)^4-(x-1)^4=14x.$
6. $(x+9a)(x+8a)=72a^2.$ 7. $(a+x)^{\frac13}+(a-x)^{\frac13}=b^{\frac13}.$
8. $(50+x)^{\frac13}+(50-x)^{\frac13}=2.$ 9. $(a^4+x^4)^{\frac14}=(b^2+x^2)^{\frac12}.$
10. $\{(b+c)^2+x^2\}^{\frac12}+\{(b-c)^2+x^2\}^{\frac12}=2a.$
11. $(a^2+1)(x^3+1)^2=2(x^3+1).$ 12. $(x^2+1)^3=(x^2-1)^2(x-1)^2.$

13. $\dfrac{x^{\frac12}+8}{x^{\frac12}+3}=\dfrac{x^{\frac12}+48}{x^{\frac12}+23}$. 14. $\dfrac{a-(a^2-x^2)^{\frac12}}{a+(a^2-x^2)^{\frac12}}=b$.

15. $\dfrac{(a+x)+(a^2-x^2)^{\frac12}}{(a+x)-(a^2-x^2)^{\frac12}}=\dfrac{b}{x}$. 16. $\dfrac{ax+1+(a^2x^2-1)^{\frac12}}{ax+1-(a^2x^2-1)^{\frac12}}=\dfrac{b^2x}{2}$.

17. $\dfrac{a+x+(2ax+x^2)^{\frac12}}{a+x-(2ax+x^2)^{\frac12}}=\dfrac{m^2}{n^2}$. 18. $\left\{\dfrac{x^3}{a}\right\}^{\frac12}+\left\{\dfrac{a^3}{x}\right\}^{\frac12}=x+a$.

19. $x+(a^2+x^2)^{\frac12}=\dfrac{2a^2}{(a^2+x^2)^{\frac12}}$. 20. $\left\{\dfrac{x-a}{x+a}\right\}^{\frac12}+\left\{\dfrac{x+a}{x-a}\right\}^{\frac12}=3^{\frac12}$.

21. $a-b=(x+a)\left\{\dfrac{x^2+b^2}{x^2+a^2}\right\}^{\frac12}-(x+a)\left\{\dfrac{x^2+a^2}{x^2+b^2}\right\}^{\frac12}$.

XIV.

1. Determine the relation between a, b, and c, in order that the roots of the equation $ax^2+bx+c=0$ may be, (1) real and unequal; (2) real and equal; or (3) imaginary. What is the relation between p and q if the equation were reduced to the form $x^2+px+q=0$.

2. State the criterion, in quadratic equations which have possible roots, when both are positive or both negative, or when one is positive and the other negative.

3. Solve the equations

$$\frac{(2x+3)}{(x+5)(x+1)}+\frac{x+2}{x^2-1}=\frac{x-7}{(x+5)(x-1)}$$

$$\text{and } \frac{x-7}{(x+5)(x-1)}=\frac{(2x+3)}{(x+5)(x+1)}-\frac{x+2}{x^2-1},$$

and explain how the latter fails to give any definite value of the unknown quantity.

4. Determine the real values of x which will render x^2+2x+1, and $x^4-x^3-\frac34x^2+\frac12x+\frac14$ numerically equal.

5. Divide $x^5-2x^4-4x^3+13x^2-11x-7$ by x^3-3x+7, and find what must be the numerical value of x in order that the quotient may vanish.

6. If $ax^3+bx^2+c=0$, and $cx^2+bx+a=0$, find the equation which involves only the constants, a, b, c.

7. Eliminate x between the equations

$$x^3+x^{-3}+3(x+x^{-1})=m,\quad x^3-x^{-3}-3(x-x^{-1})=m.$$

8. Find the value of β in terms of μ when $x=-1$ from the equation $(\mu+2)x^2+2(\mu^2-1)(\beta-1)x-2(\mu+1)(\mu-1)^2\beta+\mu(\mu-1)^2\beta^2+\mu^3=0$.

9. Find x from the equation $(225-6x)^{1{\cdot}25}-{\cdot}25x^2(225-6x)^{{\cdot}25}=0$.

10. Solve the equation $\dfrac{a}{x+b}-\dfrac{b}{x+a}=\dfrac{c}{x+d}-\dfrac{d}{x+c}$, when $a+d=b+c$.

11. Solve $\dfrac{ax-bc}{x+b}+\dfrac{bx-ca}{x+c}+\dfrac{cx-ab}{x+a}=0$, when $a+b+c=0$.

12. Shew by means of a quadratic equation that the four roots of the equation $(x+a)(x+2a)(x+3a)(x+4a)=b^4$ are exhibited by $x=\frac12\{-5a\pm[5a^2\pm4(b^4+a^4)^{\frac12}]^{\frac12}\}$.

13. State the reason why one of the roots of a quadratic equation does not always satisfy the conditions of the problem. Obtain the values 18 and 3 for x from the equation $x+(5x+10)^{\frac{1}{2}}=8$, and shew which value satisfies the equation, and to what equation the other value belongs.

14. Explain the reason why both the values of x obtained from the equation $\{2-\sqrt{4-x^2}\}^{-1}-\{2+\sqrt{4-x^2}\}^{-1}=\sqrt{2}$ do not satisfy it on substitution?

15. Shew that of these two equations, $(x-4)^{\frac{1}{2}}=1+(x-1)^{\frac{1}{2}}$, and $(x-4)^{\frac{1}{2}}=(x-1)^{\frac{1}{2}}-1$; one has, and the other has not, a root.

16. Prove that in general an equation whose coefficients are whole numbers has no fractional root. Shew why it does not hold true in the equation $x^2-x+1=0$.

17. Is it true, and if so, in what sense, that an equation containing radicals has in general any solution? Examine the equation

$$\sqrt{2x+7}+\sqrt{3x-18}=\sqrt{7x+1}.$$

18. For what values of x is $\sqrt{14-(3x-2)(x-1)}$ possible?

19. Prove that a quadratic equation can have only two roots, and if any quadratic equation be satisfied by more than two values of the unknown quantity, the equation is an identity. How is explained the paradox of there appearing to be three roots of the quadratic equation $x^2-x=2$; namely, 2, 1, and

$$\{\tfrac{1}{2}(-1+\sqrt{-3})\}^{\frac{1}{3}}+\{\tfrac{1}{2}(-1-\sqrt{-3})\}^{\frac{1}{3}}?$$

20. Shew that the roots of the quadratic equation

$$(3a-x)^{-1}+(3b-x)^{-1}+(3c-x)^{-1}=0$$

are real, and depend on the fact that $a^2+b^2+c^2$ is greater than $ab+ac+bc$.

XV.

1. If x be real, prove that $x^2-8x+22$ can never be less than 6.

2. Shew that $x^2-6x+10$ can never be negative for any value of x.

3. Shew that the values of the expression $\dfrac{x^2-4x+4}{x-1}$ cannot lie between 0 and -4.

4. Within what limits is the value of $\dfrac{x^2-3x-3}{2x^2+2x+1}$ an integer or rational fraction?

5. Shew that the values of $\dfrac{n^2-n+1}{n^2+n+1}$ lie between 3 and $\frac{1}{3}$.

6. Shew that $(x-a)(b-x)$ can never exceed $\frac{1}{4}(a-b)^2$.

7. Shew that the least value of $\dfrac{(x+a)(x+b)}{x}$ is $(a^{\frac{1}{2}}+b^{\frac{1}{2}})^2$; and the greatest value of $\dfrac{(x+a)(x-b)}{x^2}$ is $\dfrac{(a+b)^2}{4ab}$.

8. What relation exists between the constants involved in the equations $ax^2+bx+c=0$, and $x^2+b'x+c'=0$?

9. If x be real, prove that $\frac{x^2+5x+4}{x^2+5x}$ can have no real value between $\frac{9}{25}$ and 1.

10. The greatest value which the expression $\frac{a(x+a)}{x^2+a^2}$ admits for any value of x is $\frac{\sqrt{2}+1}{2}$.

11. If x be real, $\frac{9}{x-1}-\frac{4}{x-6}$ can never lie between 5 and $\frac{1}{5}$.

12. Shew that by giving an appropriate real value to x, $\frac{4x^2+36x+9}{12x^2+8x+1}$ can be made to assume any real value.

13. The expression $\frac{(x-a)(x-b)}{(x-c)(x-d)}$ admits of all possible values provided that one, and *only one*, of the quantities a or b lies between c and d; and otherwise will have two limits between which it cannot lie.

14. The expression $\frac{ax^2+2bx+c}{a'x^2+2b'x+c'}$ will be capable of all values for real values of x, provided that $(ac'-a'c)^2-(a'b-ab')(b'c-bc')$ is positive.

Shew that this includes the conditions b^2 greater than ac, and b'^2 greater than $a'c'$.

15. If a be greater than b, and c be positive, prove that the greatest value which the expression $(x-a)(x-b)(x-a-c)(x-b+c)$ can have for values of x between a and b, is $\frac{(a-b)^2.(a-b+2c)^2}{16}$.

XVI.

1. If α, β, be the roots of the equation $ax^2+bx+c=0$, and that $\alpha^2+\beta^2=1$, shew that $2ac=b^2-a^2$.

2. If one third of the sum of the squares of the roots of the equation $ax^2+bx+c=0$ be equal to their product, then $b^2=5ac$.

3. If α, β, be the roots of the equation $ax^2+bx+c=0$, then the equation whose roots are $\alpha^2+\beta^2$ and $\alpha^{-2}+\beta^{-2}$ will be

$$a^2c^2x^2-(b^2-2ac)(a^2+c^2)x+(b^2-2ac)^2=0.$$

4. If α, β, be the roots of the equation $x^2-px+q=0$, find the equation whose roots shall be respectively, (1) $\alpha+\beta$, $\alpha-\beta$. (2) $\alpha\pm\beta\sqrt{-1}$. (3) $\alpha^2\pm\alpha\beta+\beta^2$. (4) $(2\alpha-\beta)^{-1}$, $(\alpha-2\beta)^{-1}$. (5) $\alpha^3\beta^{-2}$, $\alpha^{-2}\beta^3$.

5. If α, β, be the roots of $ax^2+(a+b)x+b=0$, then shall

$$\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{a}{b}+\frac{b}{a}, \text{ and } \left\{\frac{\alpha}{\beta}\right\}^{\frac{1}{2}}+\left\{\frac{\beta}{\alpha}\right\}^{\frac{1}{2}}+\left\{\frac{b}{a+b}\right\}^{\frac{1}{2}}=0.$$

6. If the equation $(a^2+b^2)x^2-2(ac+bd)x+(c^2+d^2)=0$ have real roots, shew that these roots are equal to the sum and difference of $ac+bd$ and $ad-bc$.

7. If a quadratic equation be satisfied by more than two different values of x, it will be satisfied by all values of x; and shew that if α, β, γ, each satisfy the equation $x^2+px+q=0$, then must

$$(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)=0.$$

8. If α, β, be the roots of $x^2+px+q=0$, and α', β' the roots of $x^2-p'x+q'=0$, then $x^2-pp'x+p^2q'+p'^2q-4qq'=0$ is the equation whose roots are $\alpha'\beta+\alpha\beta'$ and $\alpha\beta+\alpha'\beta'$.

9. Shew that the equations $c^2x^2+(2ac-b^2)x+a^2=0$, and $a^4x^2-2a^2(b^2-4ac)x+b^2(b^2-4ac)=0$, have respectively their roots, the former the squares of the reciprocals, the latter the squares of the sum and difference of the roots of the equation $ax^2+bx+c=0$.

10. If $x^2-3x=a$, and $x^2-4x=5$, have one root in common, shew that $a^2-14a+40=0$.

11. If the roots of the equation $x^2+px+q=0$ be respectively any equimultiples of the roots of the equation $x^2+rx+s=0$; prove that $sp^2=qr^2$.

12. If α, β be the roots of the equation $ax^2+bx+c=0$; find in terms of the coefficients the sum of the second, third, and fourth powers of the roots of the equation; also the sum of the reciprocals of the same powers of the roots.

13. If α, β be roots of the equation $x^2+px+q=0$, and α, γ roots of $x^2+rx+s=0$, then $\beta+\gamma$ is a root of the equation

$$x^2+(p+r)x+2(q+s)=0.$$

14. State the relations which subsist between the coefficients of the quadratic equation $ax^2+2bx+c=0$, when the roots are possible or imaginary, equal or unequal; and if the roots be imaginary, shew that the result of the substitution of any value of x in the expression $ax^2+2bx+c$ is always positive, and find the least value.

15. If the difference between the roots of the two equations $x^2+(p-a)x+b^2=0$, and $x^2+(p-b)x+a^2=0$, be equal, then shall $2p=5(a+b)$.

16. If $x^2+p_1x+q_1=0$, and $x^2+p_2x+q_2=0$, have a common root, then shall $(q_1-q_2)^2=(p_1-p_2)(p_2q_1-p_1q_2)$, and the equation which contains the two roots not common is

$$(q_1-q_2)^2x^2-(p_1-p_2)(q_1^2-q_2^2)x+(p_1-p_2)^2q_1q_2=0,$$

also these roots are $\dfrac{(p_1-p_2)q_1}{q_2-q_1}$ and $\dfrac{(p_2-p_1)q_2}{q_1-q_2}$. Find also the root common to the two equations.

17. If α, β, be the roots of $x^2-px+q=0$, and α', β' the roots of $x^2-qx+p=0$, shew that $\dfrac{1}{\alpha\alpha'}+\dfrac{1}{\alpha\beta'}+\dfrac{1}{\alpha'\beta}+\dfrac{1}{\beta\beta'}=1$.

18. Determine the relation which subsists between the coefficients

of the three equations $x^2-p_1x+q_1=0$, $x^2-p_2x+q_2=0$, $x^2-p_3x+q_3=0$, when they have a common root; and prove that

$$p_1^2+p_2^2+p_3^2+4(q_1+q_2+q_3)=2(p_1p_2+p_1p_3+p_2p_3).$$

XVII.

1. What number is that, the treble of which added to the double of its square root, is equal to unity?

2. Find that number whose square added to its cube is nine times the next higher number.

3. Divide the number 20 into two such parts that their product may be 64.

4. A certain number consists of two digits. The left-hand digit is double of the right-hand digit; and if the digits be reversed, the product of the number thus formed and the original number is 2268. Find the number.

5. Find three consecutive numbers whose product is three times the middle number.

6. Divide 42 into four parts which shall consist of four consecutive numbers.

7. The sum of £1,024 is lent at a certain rate of interest, and at the end of two years the sum repaid for capital and compound interest is £1,156; what was the rate of interest?

8. *A*, *B*, *C* execute a work in a certain time; *A* alone could have done it in 6 hours more; *B* in 15 hours more; *C* in twice the time. How long did the work occupy them?

XVIII.

1. Divide a given line into two parts such that the sum of their squares shall be double the square of another line; and determine the limit of the possibility of the problem.

2. A piece of ground whose length exceeds the breadth by 6 yards has an area of 91 square yards; what are its dimensions?

3. The length of a rectangular field exceeds the breadth by one yard, and the area contains 10,100 square yards; find the length and breadth.

4. A rectangular court, of which the sides are a and b yards respectively, is surrounded by a path of uniform width equal in area to one mth part the inner rectangle; find the width of the path.

5. A garden in the form of a rectangle is twice as long as it is broad, and contains three acres of land, and is divided by two paths into four equal parts; find the area of each of the beds if the paths are five feet wide.

6. The three edges of a rectangular parallelopiped are in the ratio of 2, 3, 4, and if these edges be respectively increased by 1, 2, 3 inches,

the solidity will be augmented by 153 cubic inches; determine the edges.

7. An officer in forming his regiment into a solid square found that he had 76 too many; upon augmenting the side of the square by 2 he had 88 too few; of how many did the regiment consist?

8. A company of men is formed into a hollow square 8 deep, and afterwards into a hollow square 4 deep; and it is found that the number of men in the front rank of the latter exceeds the number in the front rank of the former square by 19; find the number of men in the company.

9. A body of troops is drawn up in column, the depth being 10 men more than the front rank of the column; but when drawn up in line 5 deep the front is 780 more than it was; find the number of men. Can any meaning be given to the negative result?

10. A body of men drawn up in a hollow square 4 deep, having 404 drafted from them, it is found necessary to form the remainder in a hollow equilateral wedge, which they are just sufficient to do three deep; the side of the wedge being 10 more than the square root of the side of the square; required the number of men.

11. Find the edge of that cube whose surface and content are expressed by the same number.

XIX.

Shew that a quadratic equation cannot be satisfied by more than two values of x; and that a pair of simultaneous quadratic equations in x and y can be satisfied by not more than four pairs of values of x and y.

Solve the following equations, verify the results, and express in words at length the first six of them:—

1. $x+y=8$, $xy=15$.
2. $x-y=3$, $xy=18$.
3. $x^2+y^2=113$, $x-y=1$.
4. $x^2+y^2=244$, $x+y=22$.
5. $x^2+y^2=74$, $xy=35$.
6. $x^2-y^2=144$, $x+y=18$.
7. $x^2-y^2=175$, $x-y=5$.
8. $x^2+xy=140$, $y^2+xy=56$.
9. $x^2-xy=9$, $xy-y^2=4$.
10. $x^2+y^2=8xy$, $x+y=5$.

XX.

Eliminate x or y between the respective equations:—

1. $x^2y-7x+2=0$ and $(y-1)x^2-3x-2=0$.
2. $x^3y-3x+1=0$ and $x^2(y-1)+x-2=0$.
3. $x^3+3x^2y+3y^2x+2y^3=0$ and $x^2+xy+y^2=0$.
4. $x^3-3yx^2+(3y^2-y+1)x-y^3+y^2-2y=0$ and $x^2-2xy-y=0$.
5. $x^3-3(y-1)x^2+(3y^2-6y-1)x-y^3+3y^2+y-3=0$,
   and $x^2+2(y+2)x+y^2+4y+3=0$.
6. $(x-y)(x^2-y^2)=5$ and $(x+y)(x^2+y^2)=65$.

7. $x^4-(2y^2+1)x^2+y^4+y^2-20=0$ and $x^2-2xy+y^2-4y=0$.
8. $x^2+y^2+(2y^2+1)(x+1)=0$ and $x^2-y^2-xy-1=0$.
9. $(y-2)x^2-2x+5y-2=0$ and $x^2y-5x+4y=0$.

XXI.

Eliminate x and y between the respective equations:—

1. $x^2+ab=mx,\ y^2+ac=ny,\ cx^2+by^2=pxy$.
2. $ax+by=c(x^2+y^2)^{\frac{1}{2}}$ and $a_1x+b_1y=c_1(x^2+y^2)^{\frac{1}{2}}$.
3. $y^2(bx-ay)=x^2(ax-by)=x^2y^2,\ x^2+y^2=c^2$.
4. $ax+by=c^2,\ xy=c^2-ab,\ \frac{x^3}{a^3}+\frac{y^3}{b^3}=\frac{1}{8}$.
5. $ax^2+bxy+cy^2=0$ and $a'x^2+b'xy+c'y^2=0$.
6. $a(x^2+y^2)+bxy+c(x+y)+d=0$
   and $a'(x^2+y^2)+b'xy+c'(x+y)+d'=0$.
7. $x^2+xy+y^2=a^2,\ x^4+x^2y^2+y^4=b^4,\ x^8+x^4y^4+y^8=c^8$.
8. $x+\frac{ab}{x}=2m,\ y+\frac{ac}{y}=2n,\ \frac{cx}{y}+\frac{by}{x}=p$.
9. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ \frac{x^2}{b^2}-\frac{y^2}{a^2}=0,\frac{x^4}{a^4}-\frac{y^4}{b^4}=\frac{m}{n}$.
10. $a^2by=mb^2x,\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ bx+my=p$.

XXII.

1. $x^2+xy=66,\ x^2-y^2=11$.   2. $x^2+5y^2=6xy,\ x^2-5y^2=4xy$.
3. $x^2+y=(x-y)^2,\ 7x=9y$.
4. $y^2(a-x)=ax^2-x,\ x^2(b-y)=by^2-y$.
5. $xy=a(x+y),\ x^2y^2=b^2(x^2+y^2)$.
6. $(x-y)^2=a^2(x+y),\ (x+y)^2=b^2(x-y)$.
7. $x^2+x-(y^2+y)-6=0,\ x^3+y^3-xy(x+y)-5=0$.
8. $x^4y^3-x^3y^4=216,\ x^2y-x^2y=6$.
9. $a(x^2+y^2)-b(x^2-y^2)=2a,\ (a^2-b^2)(x^2-y^2)=4ab$.
10. $x^3y^2+xy^4=156,\ 2x^2y^3-x^2y^3=144$.
11. $(x^2-y^2)(x-y)=3xy,\ (x^4-y^4)(x^2-y^2)=45x^2y^2$.
12. $(ax-by)^3+(bx-ay)^3=(a+b)^3,\ x-y=1$.
13. $x^2+y^2=c^2,\ n(x+y)=m(x-y)$.
14. $x^3+12xy^2=140,\ 3x^2y+4y^3=38$.   15. $\frac{x}{y}+\frac{y}{x}=\frac{5}{2},\ x^2-y^2=\frac{3}{2}$.
16. $x^2=ax+by,\ y^2=bx+ay$.   17. $xy=(6-x)^2,\ \frac{x}{y}=\frac{3-x}{12-y}$.
18. $x^3-y^3=19,\ \frac{x}{y}-\frac{y}{x}=\frac{5}{6}$.   19. $\frac{x^2}{a^2}+\frac{a^2}{x^2}+\frac{y^2}{b^2}+\frac{b^2}{y^2}=m,\ \frac{x}{a}+\frac{a}{x}+\frac{y}{b}+\frac{b}{y}=n$.
20. $x^3=(a^2-xy)y,\ y^3=(b^2-xy)x$.   21. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,\frac{x}{a}+\frac{y}{b}=2$.

22. $x^2y+y^2x=20,\ \frac{1}{x}+\frac{1}{y}=\frac{5}{4}$. 23. $x^2y+xy^2=ab,\ x^3+y^3=a^3-ab+b^3$.

24. $\frac{x^4}{y^2}+\frac{y^4}{x^2}=136\frac{1}{3}-2xy,\ x+y=10$. 25. $xy^2+y=52,\ x^2y^4+y^2=2320$.

26. $(x^3+y^3)-9(x+y)=-6,\ x^2y+xy^2=2$.

27. $ax^3+3bx^2y=4ab^3,\ by^3+3axy^2=4a^3b$.

XXIII.

1. $(x^2+a^2)=(y^2+b^2)=(x+y)^2+(a-b)^2$.
2. $2x^2-xy+y^2=2y,\ 2x^2+4xy=5y$.
3. $2x^2+3y=24,\ x^2+2x=3(y-1)(3y-1)$.
4. $y^2-x^2-90000=0,\ yx-300y-125x=0$.
5. $x+y+xy=19,\ x^2y+xy^2=84$.
6. $4(4x-5y)=5x-3,\ 5x^2-7xy=3y^2+1$.
7. $xy+xy^2=12,\ x+xy^3=18$.
8. $(a-x)^2+(b-y)^2=2(a+x)(b+y),\ 4xy=(a-b)^2$.
9. $x^2+2y^2+4x^2y=70,\ 3x^2+y=5$.
10. $x^2+y^2=7+xy,\ x^3+y^3=6xy-1$.
11. $(1+x^2)(1+y^2)=2(xy-1)^2,\ x+y=5$. 12. $x^3+y^3=19,\ xy=6$.
13. $1+xy=(1-xy)\sqrt{3},\ x-y=(1+xy)\sqrt{2}$.
14. $2ax+y^2=b^2,\ a^2x^2+y^4=c^2$.
15. $(x-y)(x-y-3)=18,\ xy=4$. 16. $ax-by=m,\ (ax)^3-(by)^3=nxy$.
17. $x+y=5,\ x^3+y^3=35$.
18. $(x^2+y^2)(ax+by)=2ab(a+b),\ x^2-y^2=xy-ab$.
19. $x^2y^2+6xy=16,\ x=y-1$. 20. $m(x^2+y^2)+nxy=a,\ x+y=b$.
21. $a^2b^2x^2-y^2=m,\ (a^2+b^2)y^2-2a^2b^2xy=n$.
22. $x^2y^2=5(x+y)+2,\ x+xy+y=34$.
23. $x^2+xy=8x-3,\ y^2+xy=8y+6$.
24. $(x+y)(x^3+y^3)=1225,\ x^2-xy+y^2=49$.
25. $2ab(a+b)x+y^2=abx^2+2aby,\ abx+(a+b)y=xy$.
26. $x+y=x^2,\ 3y-x=y^2$.

XXIV.

1. $\frac{2}{x}+\frac{3}{y}=8,\ 7xy=6$. 2. $\frac{x+y}{1+xy}=a,\ \frac{x-y}{1-xy}=b$.

3. $ax+by=c,\ \frac{b}{x}+\frac{a}{y}=d$. 4. $(xy-1)\left(x+\frac{1}{y}\right)=\frac{3}{2},\ y+\frac{1}{x^2y}=\frac{5}{2}$.

5. $\frac{x-1}{y-1}+\frac{y-1}{x+1}=\frac{5}{2},\ x+y=4$. 6. $\frac{x+y}{a+b}=\frac{x-y}{a-b},\ xy=ab$.

7. $\frac{1}{x}+\frac{1}{y}=\frac{x+y}{6}=\frac{5}{x+y+1}$. 8. $\frac{x}{y}-\frac{y}{x}=\frac{x+y}{x^2+y^2},\ \frac{x^2}{y^2}-\frac{y^2}{x^2}=\frac{x-y}{yx}$.

9. $\frac{x^4+y^4}{x^2y^2}+\frac{x^2+y^2}{xy}=\frac{17}{4},\ x-y=2$. 10. $ax+by=c,\ \frac{b}{x}+\frac{a}{y}=\frac{1}{c}$.

11. $\frac{bx+ay}{c^2-xy+ab}=\frac{x+y}{a+b},\ \frac{xy+ab}{c^2-bx+ay}=\frac{x+b}{y+a}$. 12. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ \frac{x}{a}-\frac{b}{y}=0$.

13. $\frac{x}{a}+\frac{b}{y}=p$, $\frac{y}{a}-\frac{b}{x}=q$.

14. $ax+by=\frac{a}{x}+\frac{b}{y}+m$, $a'x+b'y=\frac{a'}{x}+\frac{b'}{y}+m'$.

15. $\frac{1+x}{1-y}+\frac{1+y}{1-x}=\frac{9}{13}$, $\frac{1+x}{1+y}+\frac{1-y}{1-x}=\frac{4}{13}$.

XXV.

1. $x^2+xy+x=14$, $y^2+xy+y=28$.
2. $3x^2=2xy+12$, $x^4=6x^2y+12$.   3. $x^4+y^4=641$, $x^3y+xy^3=290$.
4. $x^4+6y^3=5x^2y$, $y^4+6x^3=5y^2x$.
5. $x^4+y^4+2xy(x^2+y^2)=6x^2y^2$, $x+y-xy=0$.
6. $x^2+xy+y^2=49$, $y^4+x^4=961-x^2y^2$.
7. $x^2+y^2-(x+y)=a$, $x^4+y^4+(x+y)-2(x^2+y^2)=b$.
8. $y^4-x^2y=69$, $3y^3+3x^2=31y$.
9. $x^4-x^2+y^4-y^2=84$, $x^2+x^2y^2+y^2=49$.
10. $(x+y)(x+y+1)=56$, $(x-y)(x-y-1)=12$.
11. $2(x^4+y^4)+7xy(x^2+y^2)=740$, $2(x^2+y^2)-xy=20$.
12. $x^4+y^4=641$, $xy(x^2+y^2)=290$.
13. $x^4y^4+1=313y^4-6x^2y^2$, $x^2y^2-1=5y^2$.
14. $x^2+xy+y^2=19$, $x^4+x^2y^2+y^4-4(x^2+\frac{5}{2}xy+y^2)=21$.
15. $x^4+y^4=1+2xy+3x^2y^2$, $x^3+y^3=2y^2x+2y^2+x+1$.
16. $x^2+y^2-x^2y^2+2xy=5$, $x^2y(y+2)+xy^2(x+2)=20$.

XXVI.

1. $(x^2+y^2)y=ax$, $(x^2-y^2)x=by$.   2. $(ax)^{\frac12}+(by)^{\frac12}=a+b=\frac12(x+y)$.
3. $x+y+(x+y)^{\frac12}=12$, $x^3+y^3=189$.
4. $x^3=31x^2-4y^2$, $y^3=31y^2-4x^2$.
5. $(x^2+12y)^{\frac12}+(y^2+12x)^{\frac12}=33$, $x+y=23$.
6. $y^2-64=8x^{\frac12}y$, $y-4=2y^{\frac12}$.
7. $4y^2+(4-x^2)^2=18$, $\{(1+x)^2+y^2\}^{\frac12}+\{(1-x)^2+y^2\}^{\frac12}=4$.
8. $x^{\frac12}+y^{\frac12}=3x$, $x^{\frac12}-y^{\frac12}=x$.
9. $x+y-\left\{\frac{x+y}{x-y}\right\}^{\frac12}=\frac{6}{x-y}$, $x^2+y^2=41$.
10. $\frac{y}{x}\cdot\frac{x^{\frac12}}{y^{\frac12}}+\frac{x^{\frac12}}{(2y)^{\frac12}}\cdot\frac{y^{\frac32}}{x^{\frac32}}=3$, $\frac{3x^2}{y}-\frac{x}{4y^{\frac12}}=11\frac12$.
11. $x^2+y^2=17$, $(x+y)x^{\frac12}y^{\frac12}=10$.   12. $\frac{x^{\frac32}}{y^{\frac12}}-\frac{y^{\frac32}}{x^{\frac12}}=34x^{\frac12}y^{\frac12}$, $x-y=12$.
13. $(ax)^{\frac23}+(by)^{\frac23}=(a^2-b^2)^{\frac23}$, $b^2x^2+a^2y^2=a^2b^2$.
14. $(x^3+3xy^2)^2-(3x^2y+y^3)^2=\frac45(x^6-y^6)$, $(x+y)^{\frac23}+(x-y)^{\frac23}=a^{\frac23}$.

XXVII.

1. $x+(xy)^{\frac12}+y=19$, $x^2+xy+y^2=133$.   2. $4x^{\frac12}+y^{\frac12}=x$, $3y^{\frac12}-x^{\frac12}=y$.
3. $x^{\frac12}+y^{\frac12}=6$, $x^{\frac32}+y^{\frac32}=126$.   4. $x^{\frac12}+y^{\frac12}=4$, $x^{\frac32}+y^{\frac32}=28$.

5. $x+x^{\frac{1}{2}}y^{\frac{1}{2}}=10,\ y+x^{\frac{1}{2}}y^{\frac{1}{2}}=15.$ 6. $x^2+x^{\frac{4}{3}}y^{\frac{2}{3}}=208,\ y^2+x^{\frac{2}{3}}y^{\frac{4}{3}}=1053.$

7. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=85,\ x^{\frac{1}{3}}+y^{\frac{1}{3}}=11.$ 8. $x^{\frac{1}{2}}+y^{\frac{1}{2}}=11,\ x^{\frac{1}{4}}+y^{\frac{1}{4}}=4.$

9. $(x+y)^{\frac{1}{2}}+(x-y)^{\frac{1}{2}}=a,\ (x+y)^{\frac{1}{4}}+(x-y)^{\frac{1}{4}}=b.$

10. $x^{\frac{2}{3}}y^{\frac{4}{3}}=4y^2,\ x^{\frac{1}{3}}+y^{\frac{1}{3}}=12.$

11. $\dfrac{x^{\frac{1}{2}}+(x-y)^{\frac{1}{2}}}{x^{\frac{1}{2}}-(x-y)^{\frac{1}{2}}}=4,\ 4x^{\frac{1}{2}}=y.$ 12. $\dfrac{x-y}{x^{\frac{1}{2}}-y^{\frac{1}{2}}}=8,\ x^{\frac{1}{2}}y^{\frac{1}{2}}=15.$

13. $\dfrac{x+(x^2-y^2)^{\frac{1}{2}}}{x-(x^2-y^2)^{\frac{1}{2}}}+\dfrac{x-(x^2-y^2)^{\frac{1}{2}}}{x+(x^2-y^2)^{\frac{1}{2}}}=\dfrac{82}{9},\ \dfrac{(x^2-y^2)^{\frac{1}{2}}+y}{(x^2-y^2)^{\frac{1}{2}}-y}-\dfrac{(x^2-y^2)^{\frac{1}{2}}-y}{(x^2-y^2)^{\frac{1}{2}}+y}=\dfrac{48}{7}.$

14. $\left\{\dfrac{a^2-x^2}{y^2-b^2}+\dfrac{y^2-b^2}{a^2-x^2}\right\}^{\frac{1}{2}}+\left\{\dfrac{a^2+x^2}{y^2+b^2}+\dfrac{y^2+b^2}{a^2+x^2}\right\}^{\frac{1}{2}}=4,\ xy=ab.$

15. $\dfrac{x^{\frac{1}{2}}(x^{\frac{3}{2}}-1)}{y^{\frac{1}{2}}}+\dfrac{y^{\frac{1}{2}}(2x^{\frac{3}{2}}-1)}{x^{\frac{1}{2}}}=\dfrac{4y^{\frac{1}{2}}(y^{\frac{1}{2}}+x^{\frac{1}{2}})}{x^{\frac{1}{2}}}+\dfrac{3y}{x^{\frac{1}{2}}}+2,$

$$\frac{x^{\frac{5}{2}}}{y^{\frac{1}{2}}}-\frac{2x^{\frac{3}{2}}}{y}-\frac{2x^{\frac{1}{2}}}{y^{\frac{1}{2}}}=\frac{133}{36y^{\frac{3}{2}}}-\frac{2}{x^{\frac{1}{2}}}-\frac{y^{\frac{3}{2}}}{x^{\frac{3}{2}}}.$$

XXVIII.

1. $x^2+xy=12,\ xy-2y^2=1.$
2. $x^2+6xy=235,\ 4xy-3y^2=-7.$
3. $5x^2+7xy-87=0,\ 5y^2+7xy-67=0.$
4. $3x^2-4xy=7,\ 3xy-4y^2=5.$
5. $x^2+3xy+y^2=20,\ 5x^2+4y^2=41.$
6. $x^2-3xy+2y^2=3=3x^2-8xy+4y^2.$

XXIX.

1. Shew that $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ may be put under the form

$\left\{\dfrac{x}{a}+\dfrac{\mu y}{b}\right\}^2+\left\{\dfrac{\mu x}{a}-\dfrac{y}{b}\right\}^2=\mu^2+1$, when μ is any numerical quantity.

2. Resolve $2y^2+2x^2-5xy-4ay-ax-6a^2$ into two simple factors.

3. If a, b, c be real quantities, shew that no real values of x and y can satisfy the equation $ay-bx=c\{(x-a)^2+(y-b)^2\}^{\frac{1}{2}}$ unless c^2 be less than a^2+b^2.

4. Solve the equations $2y^2-xy-2x^2-4ay+6ax-4a^2=0,$
and $y^2+xy-3x^2-2ay+4ax-5a^2=0.$

5. Prove that the two following equations coincide:—

$$(x-x')^2(y'^2-b^2)-2(x-x')(y-y')x'y'+(y-y')^2(x'^2-a^2)=0,$$

$$\left\{\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right\}.\left\{\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right\}=\left\{\frac{xx'}{a^2}+\frac{yy'}{b^2}-1\right\}^2.$$

6. Find x and y from the equation $(x+y\sqrt{-1})^2=a+b\sqrt{-1}.$

7. Find the real values of x and y which satisfy the equation $(3x+2y\sqrt{-1})(8x+3y\sqrt{-1})=15(5\sqrt{-1}-2).$

8. If the equations $x+y=a$, $x^4+y^4=b^4$ be solved by means of an intermediate quadratic equation from which xy is obtained, the

negative value in this last equation must always be taken in order to obtain the possible values of x and y.

9. If $y^2+axy+bx^2+cy+dx+e=0$ can be resolved into two rational factors of the first degree, then $(ac-2d)^2=(a-4b)(c^2-4e)$. Resolve $2x^2-21xy-11y^2-x+34y-3$ into two rational factors.

10. If $ax^2+by^2+2a'y+2b'x+2c'xy+c$ be the product of two lineal factors involving x and y; shew that $abc+2a'b'c'-aa'^2+bb'^2-cc'^2=0$.

XXX.

1. If $y^2-m(2x+m)=a^2$, $x^2-m(2y+m)=b^2$, $a+x=b+y$; shew that $m=0$ or $m=a+b$.

2. If $x=(a^2+s^2)^{\frac{1}{2}}$ and $ae^{\frac{y}{a}}=x+(x^2-a^2)^{\frac{1}{2}}$;

$$\text{Then shall } 2x=a(e^{\frac{y}{a}}+e^{-\frac{y}{a}}) \text{ and } 2s=a(e^{\frac{y}{a}}-e^{-\frac{y}{a}}).$$

3. Given $(x-\alpha)^2-(y-\beta)^2=c^2$, $\alpha^2+\beta^2=a^2$, $\alpha y=\beta x$; prove that

$$x^2+y^2=(a\pm c)^2.$$

4. If $(x-\alpha)^2+(y-\beta)^2=c^2$, $(y^2-x^2)(x-\alpha)=2xy(y-\beta)$; shew that $\alpha y-\beta x=\pm cx$.

5. Given $2y=m^2+b^2$, $2nx=n^2+a^2$, and $4mnxy=m^2n^2+a^2b^2$; shew that $a^2y^2+b^2x^2=a^2b^2$.

6. Solve $a^2y^2+b^2x^2=a^2b^2$, $my+nx=mn$; and shew that the two values of x and y coincide when $m^2b^2+n^2a^2=a^2b^2$.

7. If $a^2yy'+b^2xx'-a^2b^2=a^2y'^2+b^2x'^2-a^2b^2=b^2x'y-a^2y'x=0$; prove that $(x^2+y^2)^2=a^2x^2+b^2y^2$.

Also shew that the same result can be obtained from the equations,

$$a^2(y_1+y_2)x=b^2(x_1+x_2)y,\ b^2(x_1+x_2)x+a^2(y_1+y_2)y=a^2b^2,$$
$$b^2x_1^2+a^2y_1^2=a^2b^2,\ b^2x_1x_2+a^2y_1y_2=0,\ b^2x_2^2+a^2y_2^2=a^2b^2.$$

8. If $\{(a+x)^2+y^2\}^{\frac{1}{2}}\times\{(a-x)^2+y^2\}=a^2$,

$$\text{then shall } (x^2+y^2)^2=2a^2(x^2-y^2).$$

9. $\dfrac{a^2}{m^2}+\dfrac{b^2}{n^2}=1$, $\dfrac{x^2}{m^2}+\dfrac{y^2}{n^2}=1$, and $\dfrac{by}{ax}=\dfrac{n^2}{m^2}$; then $\dfrac{m}{n}=\dfrac{a}{y}$.

10. If $mx_1^2+ny_2^2=a^2$, $mx_2^2+ny_1^2=a^2$, $mx_1x_2+ny_1y_2=0$; then

$$m(x_1^2+x_2^2)=a^2=n(y_1^2+y_2^2).$$

11. If $\dfrac{x-\alpha}{x}=\dfrac{(a^4y^2+b^4x^2)}{a^4b^2}$, $\dfrac{y-\beta}{y}=\dfrac{(a^4y^2+b^4x^2)}{a^2b^4}$ and $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$;

$$\text{shew that } (b\beta)^{\frac{2}{3}}+(a\alpha)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}.$$

12. If $\dfrac{m^2a}{b}=\dfrac{b'}{m^2a'}=\dfrac{\alpha'a}{\alpha\alpha'}$, and $\left\{\dfrac{ma}{\alpha}\right\}^2+\left\{\dfrac{ma'}{\alpha'}\right\}^2=1$; shew that

$$ab'+a'b=\alpha\alpha'.$$

13. If $a^3(b^2-y^2)=xy$ and $x(b^2-y^2)^{\frac{1}{2}}=b^3y^2$, then $x=a^2b^3(a^2+b^2)^{-\frac{1}{2}}$.

14. If m, n be roots of $ax^2+b^3x+c^3=0$, and p, q roots of

$$cy^2+b^2y+a^3=0;\ \text{then shall } \frac{m+n}{p+q}=\left\{\frac{mn}{pq}\right\}^{\frac{1}{2}}.$$

15. If $axz+bxy+cy^2=ab^2$, $cx^2+bxy+ay^2=cb^2$, and $x+y=d$; then shall $\frac{a+c+b}{a+c-b}=\frac{b^2}{d^2}$.

16. If $(\beta-\alpha)(y-m\alpha)=(n\beta-m\alpha)(x-\alpha)$, $(\beta'-\alpha')(y^2-m\beta')=(m\beta'-n\alpha')(x-\beta')$; shew that $\left\{\frac{1}{\alpha\alpha'}-\frac{1}{\beta\beta'}\right\}x=\frac{\alpha+\alpha'}{\alpha\alpha'}-\frac{\beta+\beta'}{\beta\beta'}$.

17. If $x^2(y-p)=q$ and $y^2(x-q)=p$; shew that if x, y be rational, pq is not greater than $\frac{1}{4}$.

18. If $ax+by=x+y+xy=x^2+y^2-1=0$, shew that $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{(a-b)^2}$.

19. If $\{x+(x^2+4)^{\frac{1}{2}}\}^{\frac{2}{3}}+\{y+(y^2+4)^{\frac{1}{2}}\}^{\frac{2}{3}}=4$, then $(x^2y)^{\frac{1}{3}}+(xy^2)^{\frac{1}{3}}=1$.

XXXI.

1. Divide 41 into two parts, so that the difference of their square roots may be equal to unity.

2. The sum of the squares of the digits composing a number of two places of figures is 25, and the product of the digits is 12; find the number.

3. The sum of the squares of two numbers is 170, and the difference of their squares is 72. What are the numbers?

4. A number divided by the product of its digits gives the quotient 2; and if 27 be added to the number, the digits are reversed. What is the number?

5. A number consists of three digits, the first two being the same, and is equal to the product of the sum of the digits and of the number formed by the last two digits; if the product of the last two digits be 10, find the numbers.

6. Find two numbers such that their difference may be 4, and that twice their product may be equal to the cube of the less.

7. There is a number consisting of two digits, the number is equal to eight times the sum of its digits; and if it be increased by nine, it shall be equal to the square of the sum of its digits.

8. Find two numbers such that three times their product is equal to the sum of their squares, and their quotient equal to the difference of their squares.

9. Determine two numbers whose product shall be equal to the difference of the squares, and the sum of their squares to the difference of their cubes.

10. Find two numbers such that *twice* their sum and *three* times their product may each be equal to *twice* the difference of their squares.

11. A number consisting of two digits is such that when divided by the sum of the digits the result is the second digit; and if the

digits be reversed, the square of the number thus formed is four times the cube of the sum of the digits. Find the number.

12. Is it possible to determine two integral numbers whose sum, product, and sum of the squares are equal to one another?

13. Find two numbers whose product is equal to the difference of their squares, and the sum of the squares is equal to the difference of their cubes.

XXXII.

1. The perimeter of a rectangular field is 680 yards, and its diagonal is 260; find the sides and the area.

2. There is a rectangular enclosure such that its boundary fence is n times as long as the difference between its longer and its shorter side, and its area contains m times as many square yards as that difference contains lineal yards; find the length of the sides.

3. The two sides of a rectangle expressed in feet have the sum of their cubes equal to 109 times their sum, and the difference of their cubes equal to 229 times their difference; find the area and the diagonal.

4. A field in the form of a rectangular parallelogram has an area of p square yards, and the length exceeds the breadth by m feet; what are the dimensions?

5. A person travelling along a straight railway, with a uniform velocity v, observes at intervals of time a, b, c that he is in the plane of the faces of a square tower; find the area of a horizontal section of the tower, and the distance of the nearest corner of it from the rail.

6. The contents of a rectangular parallelopiped having two of its edges equal is a^3 cubic inches, its surface is b^2 square inches, and the sum of its edges is c lineal inches; find the lengths of the edges.

7. The difference between the edges of the first and second of three cubes is equal to that of the second and third, and the sum of the three edges is a inches, also the content of the three cubes is c^3 inches; find the lengths of the edges of the cubes.

8. There is a chest whose width is equal to its depth, and its solid contents in inches is exactly equal to the solid contents of the boards which compose it at one inch thick; also the external surface is $\frac{88}{57}$ of the internal surface; what are the dimensions of the chest?

9. The fore-wheel of a carriage makes six revolutions more than the hind-wheel in going 120 yards; but if the circumference of each wheel were increased by 3 feet, the fore-wheel would make only four revolutions more than the hind-wheel in the same space; what is the circumference of each wheel?

10. *A* and *B* gained by trading £100. Half of *A*'s stock was

less than *B*'s by £100, and *A*'s gain was $\frac{3}{20}$ of *B*'s stock; what did each put into the stock, and what are the respective shares of the gain?

11. A certain number of oranges were sold for a certain number of pence; but if 5 more were sold for the same money, they would have cost each one halfpenny less, and if 5 less had been sold for the same money they would have been sold each for one halfpenny more; what was the number of oranges and the price of each?

XXXIII.

From the following equations eliminate the unknown quantities x, y, z, and find in each case the respective resulting equation:—

1. $x+y+z=a,\ x^2+y^2+z^2=b^2,\ x^3+y^3+z^3-3xyz=c^3.$
2. $x+y+z=a,\ x^2+y^2+z^2=b^2,\ x^3+y^3+z^3=c^3,\ xyz=d^3.$
3. $x+y+z=a,\ x^2+y^2+z^2=b^2,\ \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=c,\ xyz=d.$
4. $x+y+z=0,\ \frac{x}{a}+\frac{a}{x}=\frac{b}{y}+\frac{y}{b}=\frac{c}{z}+\frac{z}{c}.$
5. $x^3+y^3+z^3=3xyz,\ 3a-x+z=3b-y+x=3c-z+y.$
6. $a^2x^3+b^2y^3+c^2z^3=m^2x+n^2y+p^2z,\ ax=by=cz,\ x^2+y^2+z^2=1.$
7. $xyz=a^3,\ (x+y)(y+z)(z+x)=c^3,\ \frac{x}{y}+\frac{y}{z}+\frac{z}{x}=\alpha,\ \frac{x}{z}+\frac{y}{x}+\frac{z}{y}=\beta.$
8. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\ \frac{x^2}{a^2+k^2}+\frac{y^2}{b^2+k^2}+\frac{z^2}{c^2+k^2}=1,\ \frac{x}{a^2x'}=\frac{y}{b^2y'}=\frac{z}{c^2z'}.$
9. $x^2=y^2+z^2-2ayz,\ y^2=x^2+z^2-2bxz,\ z^2=x^2+y^2-2cxy.$
10. $x^2(y+z)=a^3,\ y^2(z+x)=b^3,\ z^2(x+y)=c^3,\ (x+y)(y+z)(z+x)=abc.$
11. $(x+y)^2=4cxy,\ (x+z)^2=4bxz,\ (y+z)^2=4ayz.$
12. $x+z=y,\ (a+b)x+(b+c)z=(a+c)y,\ abx+bcz=1+acy.$
13. $2x+z=0,\ x^2z+2xy^2=b^2,\ x^2+y^2+2xz=a^2,\ x^2y^2=c^4.$
14. $x^2+a^2=y^2+b^2=z^2+c^2=(x+y+z)^2+(a+b+c)^2=m^2.$
15. $ax+yz=bc,\ by+xz=ac,\ cz+xy=ab,\ xyz=c^3.$
16. $bx+ay=z,\ az+cz=y,\ cy+bz=x.$
17. $\frac{x}{y}+\frac{y}{x}=c,\ \frac{x}{z}+\frac{z}{x}=b,\ \frac{y}{z}+\frac{z}{y}=a.$

XXXIV.

Solve the following equations:—

1. $xy=8,\ yz=25,\ xz=14.$ 2. $xy+xz=27,\ xy+yz=32,\ xz+yz=35.$
3. $xy^3=a^3,\ yz^3=b^3,\ zx^3=c^3.$ 4. $x^2yz=a^3,\ xy^2z=b^3,\ xyz^2=c^3.$
5. $x(1-y)=y(1-z)=z(1-x)=a.$
6. $x+y=axy,\ x+z=bxz,\ y+z=czy.$
7. $xy=a(x+y),\ xz=b(x+z),\ yz=c(y+z).$
8. $x(y+z)=a^2,\ z(x+y)=b^2,\ y(x+z)=c^2.$
9. $x(x+y)=10,\ y(y+z)=21,\ z(z+x)=24.$
10. $xyz=a^2(x+y)=b^2(y+z)=c^2(x+z).$

11. $xyz=a(y^2+z^2)=b(z^2+x^2)=c(x^2+y^2)$.
12. $a^2y+b^2x=b^2z+c^2y=c^2x+a^2z=xyz$.
13. $y^2+z^2=ayz$, $x^2+z^2=bxz$, $x^2-y^2=z(bx-ay)$.
14. $ayz+bcx=bxz+cay=cxy+abz=abc$.
15. $x^2+a(y+z)=y^2+a(z+x)=z^2+a(x+y)=b^2$.
16. $yz=a(y+z)+p$, $xz=a(z+x)+q$, $xy=a(x+y)+r$.
17. $x^2+y^2=mxy$, $x+z^2=nxz$, $y^2+z^2=pyz$.
18. $x(x-y)(y-z)=0$, $(y-1)(x+z-1)=0$, $(z+1)(x+z)=0$.
19. $(x+y)(x+z)=a^2$, $(x+y)(y+z)=b^2$, $(x+z)(y+z)=c^2$.
20. $a^2x^2(y+z)^2=(a^2+x^2)y^2z^2$, $b^2y^2(z+x)^2=(b^2+y^2)z^2x^2$,
$c^2z^2(x+y)^2=(c^2+z^2)x^2y^2$.

XXXV.

1. $xy+z=26$, $xz+y=29$, $x+y+z=15$.
2. $x+y-z=0$, $xy+yz+xz=11$, $z^2-x^2-y^2=4$.
3. $x+y+z=a$, $x^2+y^2+z^2=b^2$, $xz=c^2$.
4. $ax+by+cz=0$, $px+qy+rz=0$, $x^2+y^2+z^2=0$.
5. $x^2+y^2+z^2=110$, $x+y+z=18$, $x(y+z)=65$.
6. $x^2+y^2+z^2=14$, $x^2+y=3$, $y+z^2=11$.
7. $x+y+z=3c$, $xy+xz+yz=3c^2$, $xyz=c^3$.
8. $xyz=x+y+z$, $2x+2z=y$, $xy-x=z$.
9. $x+y+z=a$, $y^2+2xy=x^2=z^2+2xz+2yz$.
10. $x+2y+3z=20$, $x^2+4y^2+9z^2=136$, $2xy-3xz-6yz=36$.
11. $xy+xz+yz=11$, $2xy+3xz+5yz=43$, $3xy+5xz+7yz=63$.
12. $x^2-y^2=y^2-z^2$, $x+y+z=13$, $x^2+y^2+z^2=75$.
13. $a(y+z-x)=(x+y+z)^2-2by$, $b(z+x-y)=(x+y+z)^2-2cx$,
$c(x+y-z)=(x+y+z)^2-2ax$.
14. $yz+xz+xy=a^2-x^2=b^2-y^2=c^2-z^2$.
15. $xy+xz+z^2=103$, $yz+yx+x^2=64$, $zx+zy+y^2=89$.
16. $x+y+z=xyz$, $xy+xz+yz=0$, $(x^2+1)(y^2+1)(z^2+1)=2xyz$.
17. $x(x+y+z)-(y^2+yz+z^2)=a^2$, $y(x+y+z)-(z^2+xz+x^2)=b^2$,
$z(x+y+z)-(x^2+xy+y^2)=c^2$.
18. $x^2+xy+y^2=c^2$, $x^2+xz+z^2=b^2$, $y^2+yz+z^2=a^2$.
19. $x^3+y^3+z^3=57a^3$, $x^2+y^2+z^2=21a^2$, $x+y+z=3a$.
20. $zx+xy-yz=x^2-a^2$, $xy+yz-xz=y^2-b^2$, $yz+xz-xy=z^2-c^2$.
21. $(ay-bx)(bz-cy)+(bz-cy)(cx-az)+(cx-az)(ay-bx)=0$,
$x^2+y^2+z^2=\mu^2(a^2+b^2+c^2)$, $ax+by+cz=\mu(a^2+b^2+c^2)$.
22. $x+y+z=10$, $yz+xz+xy=13$, $(y+z)(z+x)(x+y)=294$.

XXXVI.

1. $x+y+z=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{7}{2}$, $xyz=1$.
2. $\frac{1}{x}-\frac{1}{y}=\frac{1}{z}$, $x^2+y^2+z^2=36$, $x+2y+3z=12$.

3. $yz=bc,\ \frac{x}{a}+\frac{y}{b}=p,\ \frac{x}{a}+\frac{z}{c}=q.$

4. $\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}$ and $(x-a)^2+(y-b)^2+(z-c)^2=r^2.$

5. $\frac{x}{x-a}+\frac{y}{y-b}=\frac{2z}{z-b},\ \ \frac{x-y}{y}=\frac{y-z}{z},\ y^2=bz.$

6. $\frac{y+z}{a-x}=\frac{z+x}{b-y}=\frac{x+y}{c-z},\ ax+by+cz=a^2+b^2+c^2.$

7. $bx+ay=\frac{1}{c}-\frac{1}{z},\ \ cz+by=\frac{1}{a}-\frac{1}{x},\ \ az+cx=\frac{1}{b}-\frac{1}{y}.$

8. $\frac{1}{x}+\frac{1}{y}=\frac{z}{c^2},\ \ \frac{1}{x}+\frac{1}{z}=\frac{y}{b^2},\ \ \frac{1}{y}+\frac{1}{z}=\frac{x}{a^2}.$

9. $x(x+y+z)=a^2,\ y(x+y+z)=b^2,\ z(x+y+z)=c^2.$

10. $(x+y+z)a^2=x,\ (x+y+z)b^2=y,\ (x+y+z)c^2=z.$

11. $x+y+z=1,\ x^2+y^2+z^2+6xy=0,\ \ \frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}=0.$

12. $x=\frac{ay}{(1-a)(1-y)},\ \ y=\frac{xz}{(1-x)(1-z)},\ z=\frac{by}{(1-b)(1-y)}.$

13. $\frac{ax+cy-bz}{a^2}=\frac{by+az-cx}{b^2}=\frac{cz+bx-ay}{c^2},\ x^2+y^2+z^2=bc+ca+ab.$

14. $\frac{ax-yz}{(ac-y^2)^{\frac{1}{2}}(ab-z^2)^{\frac{1}{2}}}=\frac{p}{a},\ \frac{by-zx}{(ab-z^2)^{\frac{1}{2}}(bc-x^2)^{\frac{1}{2}}}=\frac{q}{b},\ \frac{cz-xy}{(bc-x^2)^{\frac{1}{2}}(ca-y^2)^{\frac{1}{2}}}=\frac{r}{c}.$

15. $\frac{x(a^2-b^2)^{\frac{1}{2}}}{a^3}+\frac{z(b^2-c^2)^{\frac{1}{2}}}{c^3}=0,\ \ \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$

$$x^2+y^2+z^2-2ax\left\{\frac{a^2-b^2}{a^2-c^2}\right\}^{\frac{1}{2}}-2cz\left\{\frac{b^2-c^2}{a^2-c^2}\right\}^{\frac{1}{2}}=b^2.$$

XXXVII.

1. Find the number of rational solutions of the equation $(x^2-a^2)^2+(y^2-b^2)^2+(z^2-c^2)^2=0.$

2. If $ax^2+by^2=bz(y-x)$ and $bx^2+ay^2=ay(x-z)$ determine the equation which connects x, y, and z independently of a and b.

3. If $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, and $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=0$, then shall $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

4. If $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=0$, and $(b-c)^2x+(c-a)^2y+(a-b)^2z=0$; then shall $(b-c)x=(c-a)y=(a-b)z.$

5. If $bz+cy=a,\ cx+az=b,\ ay+bx=c,$ and $2s=a+b+c$; shew that $\frac{x+1}{x-1}\cdot\frac{y+1}{y-1}\cdot\frac{z+1}{z-1}=\frac{s^3}{(s-a)(s-b)(s-c)}.$

6. Eliminate a from the equations $\frac{x}{a^2+x^2}=\frac{2y}{a^2+y^2}=\frac{4z}{a^2+z^2}$, and prove that $x(y^2-z^2)+2y(z^2-x^2)+4z(x^2-y^2)=0.$

7. If any one of the following equations be true, prove that other two are also true, $x+y(1-z^2)^{\frac{1}{2}}+z(1-y^2)^{\frac{1}{2}}=0$,

$$y+z(1-x^2)^{\frac{1}{2}}+x(1-z^2)^{\frac{1}{2}}=0,\ z+x(1-y^2)^{\frac{1}{2}}+y(1-x^2)^{\frac{1}{2}}=0.$$

8. From $ax+by-z=(m^2a^2+n^2b^2+c^2)^{\frac{1}{2}}=n^2by^{-1}=m^2ax^{-1}$, find an equation which does not contain a and b.

9. Find the equation between x, y, and z, independent of a and b, from the equations $a^3+ax=y$, $b^3+bx=z$, $a^2+b^2=1$.

10. Given the equations $x^2=\dfrac{1-z}{1+z}$ and $y=\dfrac{2z}{1-z^2}$; shew that if z be eliminated, the following equation will hold between x and y,

$$\{y+(y^2+1)^{\frac{1}{2}}\}^{\frac{1}{2}}+\{y-(y^2+1)^{\frac{1}{2}}\}^{\frac{1}{2}}=x^{-1}-x.$$

11. Eliminate y and z from the equations $y+z=a$, $x^2+y^2-2mxy=b^2$, $x^2+z^2+2mxz=c^2$; and shew that

$$x^2=\frac{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}{4a^2(1-m^2)}.$$

12. The equation $\{a+(a^2-yz)^{\frac{1}{2}}\}\cdot\{b+(b^2-xz)^{\frac{1}{2}}\}\cdot\{c+(c^2-xy)^{\frac{1}{2}}\}=xyz$, is equivalent to the equation $a^2x+b^2y+c^2z=2abc+xyz$ when x, y, z are finite quantities.

13. If $z=\dfrac{(ay^2-a^2)^{\frac{1}{2}}}{y}$ and $y=\dfrac{(ax^2-a^2)^{\frac{1}{2}}}{x}$; then shall $\dfrac{(az^2-a^2)^{\frac{1}{2}}}{z}=x$.

14. In the system of equations $a(1+y^2z^2)+b(y^2+z^2)+cyz=0$, $a(1+z^2x^2)+b(z^2+x^2)+czx=0$, $a(1+x^2y^2)+b(x^2+y^2)+cxy=0$; prove that if $a^2-b^2+bc=0$, any one of the equations is deducible from the other two.

15. Find the value of a from the equations

$$3x+ay-z=0,\ ax-2y+3z=0,\ x+3ay-2z=0.$$

16. If $2a-3y=\dfrac{(x-z)^2}{y}$, and $2a-3z=\dfrac{(y-x)^2}{z}$; then $x+y+z=a$.

17. If $x(2a^2-3x^2)^{\frac{1}{2}}+y(2a^2-3y^2)^{\frac{1}{2}}+z(2a^2-3z^2)^{\frac{1}{2}}=0$, where a^2 stands for $x^2+y^2+z^2$; shew that $(x+y+z)(-x+y+z)(x-y+z)(x+y-z)=0$.

XXXVIII.

1. Find three numbers, the third of which exceeds the first by 5, such that their sum multiplied by the first is 48, and the sum multiplied by the second is 128.

2. Find a number greater than 100 and less than 1,000, in which the middle digit is double of the first, and if the first and last digits be interchanged, the number is less than what it was before by 99; also the sum of the squares of the digits is 21.

3. What three numbers are those which have the sum 15, the sum of their cubes 495, and the difference between the first and second equal to the difference between the second and third.

4. There are three numbers, the difference of whose differences is 5, their sum is 20, and their continued product 130; find the numbers.

22. $x^2y+y^2x=20,\ \frac{1}{x}+\frac{1}{y}=\frac{5}{4}.$ 23. $x^2y+xy^2=ab,\ x^3+y^3=a^2-ab+b^2.$
24. $\frac{x^4}{y^2}+\frac{y^4}{x^2}=136\frac{1}{3}-2xy,\ x+y=10.$ 25. $xy^2+y=52,\ x^2y^4+y^2=2320.$
26. $(x^3+y^3)-9(x+y)=-6,\ x^2y+xy^2=2.$
27. $ax^3+3bx^2y=4ab^3,\ by^3+3axy^2=4a^3b.$

XXIII.

1. $(x^2+a^2)=(y^2+b^2)=(x+y)^2+(a-b)^2.$
2. $2x^2-xy+y^2=2y,\ 2x^2+4xy=5y.$
3. $2x^2+3y=24,\ x^2+2x=3(y-1)(3y-1).$
4. $y^2-x^2-90000=0,\ yx-300y-125x=0.$
5. $x+y+xy=19,\ x^2y+xy^2=84.$
6. $4(4x-5y)=5x-3,\ 5x^2-7xy=3y^2+1.$
7. $xy+xy^2=12,\ x+xy^3=18.$
8. $(a-x)^2+(b-y)^2=2(a+x)(b+y),\ 4xy=(a-b)^2.$
9. $x^2+2y^2+4x^2y=70,\ 3x^2+y=5.$
10. $x^2+y^2=7+xy,\ x^3+y^3=6xy-1.$
11. $(1+x^2)(1+y^2)=2(xy-1)^2,\ x+y=5.$ 12. $x^3+y^3=19,\ xy=6.$
13. $1+xy=(1-xy)\sqrt{3},\ x-y=(1+xy)\sqrt{2}.$
14. $2ax+y^2=b^2,\ a^2x^2+y^4=c^2.$
15. $(x-y)(x-y-3)=18,\ xy=4.$ 16. $ax-by=m,\ (ax)^3-(by)^3=nxy.$
17. $x+y=5,\ x^3+y^3=35.$
18. $(x^2+y^2)(ax+by)=2ab(a+b),\ x^2-y^2=xy-ab.$
19. $x^2y^2+6xy=16,\ x=y-1.$ 20. $m(x^2+y^2)+nxy=a,\ x+y=b.$
21. $a^2b^2x^2-y^2=m,\ (a^2+b^2)y^2-2a^2b^2xy=n.$
22. $x^2y^2=5(x+y)+2,\ x+xy+y=34.$
23. $x^2+xy=8x-3,\ y^2+xy=8y+6.$
24. $(x+y)(x^3+y^3)=1225,\ x^2-xy+y^2=49.$
25. $2ab(a+b)x+y^2=abx^2+2aby,\ abx+(a+b)y=xy.$
26. $x+y=x^2,\ 3y-x=y^2.$

XXIV.

1. $\frac{2}{x}+\frac{3}{y}=8,\ 7xy=6.$ 2. $\frac{x+y}{1+xy}=a,\ \frac{x-y}{1-xy}=b.$
3. $ax+by=c,\ \frac{b}{x}+\frac{a}{y}=d.$ 4. $(xy-1)\left(x+\frac{1}{y}\right)=\frac{3}{2},\ y+\frac{1}{x^2y}=\frac{5}{2}.$
5. $\frac{x-1}{y-1}+\frac{y-1}{x+1}=\frac{5}{2},\ x+y=4.$ 6. $\frac{x+y}{a+b}=\frac{x-y}{a-b},\ xy=ab.$
7. $\frac{1}{x}+\frac{1}{y}=\frac{x+y}{6}=\frac{5}{x+y+1}.$ 8. $\frac{x}{y}-\frac{y}{x}=\frac{x+y}{x^2+y^2},\ \frac{x^2}{y^2}-\frac{y^2}{x^2}=\frac{x-y}{yx}.$
9. $\frac{x^4+y^4}{x^2y^2}+\frac{x^2+y^2}{xy}=\frac{17}{4},\ x-y=2.$ 10. $ax+by=c,\ \frac{b}{x}+\frac{a}{y}=\frac{1}{c}.$
11. $\frac{bx+ay}{c^2-xy+ab}=\frac{x+y}{a+b},\ \frac{xy+ab}{c^2-bx+ay}=\frac{x+b}{y+a}.$ 12. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ \frac{x}{a}-\frac{b}{y}=0.$

4. Find x, y, z from the equations—

(1) $x^a y^b z^c = m$, $x^d y^e z^f = n$, $x^g y^h z^k = p$.

(2) $x^m y^n = a$, $y^m z^n = b$, $z^m x^n = c$.

5. If $x^y = y^x$, $x^m = y^n$, find the values of x and y.

6. Given $x^{\frac{p+q}{p-q}} = x(2 - x^{\frac{q}{p-q}})$, shew that $(-2)^{\frac{p-q}{q}}$ is one of the values of x.

7. If $x^{\frac{p+q}{2pq}} = \frac{1}{2} \cdot \frac{a^2-b^2}{a^2+b^2}(x^{\frac{1}{p}} + x^{\frac{1}{q}})$; shew that $x = \left\{\frac{a+b}{a-b}\right\}^{\frac{2pq}{p-q}}$.

XL.

1. What values of x render the following expressions square numbers?

(1) $x \pm a$. (2) $x^2 \pm 1$. (3) $2x^2 \pm a$. (4) $a + bx + cx^2$. (5) $6x^2 + 13x + 6$.

2. Solve the following equations:—

(1) $5x + 7y - 2y = xy$. (2) $3y - xy + 4x + 2 = 0$. (3) $x^2 - y^2 = 60$.
(4) $2xy + x + y - 195 = 0$. (5) $x^2 + 2y^2 = 59$. (6) $x^2 - 7y^2 = 1$.
(7) $2x(x+y) + y^2 - 6(y-3) = 0$. (8) $x + y + xy = 79$.
(9) $x^2 + xy - 4x - 3y = 27$. (10) $x^2 - y^2 - 100y = 1{,}000$.

3. Find two square numbers, (1) whose sum, (2) whose difference shall be square numbers.

4. Find what pairs of rational values of x and y will satisfy the equation $(x-a)(y-b) \pm (x-c)(y-d) = 0$.

5. Find the limits for the real values of x and y which satisfy the equation $x^2 + 4xy + 4y^2 + 4x + 8y - 21 = 0$.

6. Find three numbers in arithmetical progression that the sum of every two of them may be square numbers.

7. Find a series of positive integral values of x, y, z which satisfy the equations $x^2 + y^2 = (y+1)^2$ and $x^2 + y^2 + z^2 = (z+1)^2$, and shew that if $x^2 + y^2$ be a complete square, y is not greater than $\frac{1}{2}(x^2+1)$ nor less than $(2x+1)^{\frac{1}{2}}$.

8. To determine three square numbers which shall be in arithmetical progression.

9. The number of men in a regiment is such that they could be formed into a solid square; and they could also be formed into 5 hollow squares, each 5 deep, and each square having 5 men more in the front rank than the former square; find the whole number of men.

RESULTS, HINTS, ETC., FOR THE EXERCISES ON EQUATIONS OF THE SECOND DEGREE.

I.

1. See Section VII., p. 2.

2. See Arts. 1, 2, pp. 1—3.

3. See Art. 4, p. 4.

5. See Art. 6, p. 5.

The equations are $x^2 - 7x + 12 = 0$, $x^2 + 7x + 12 = 0$, $x^2 - x - 12 = 0$, $x^2 + x - 12 = 0$.

6. When the two roots of a quadratic equation are equal, the equation may be said to have one root.

7. The nature of the question will determine its meaning to be of a contrary nature to what it would have been if it had been positive.

8. The square root of every algebraic quantity is both positive and negative.

9. $x^2 - \frac{2mx}{n} + \frac{m}{n} = 0$.

10. $a = 3$, $b = -1$, $c = -2$.

11. Reduce and arrange the terms on each side, and compare them.

12. The factors are $x + 13$ and $x + 60$.

13. By solving the equation $mn\,(x^2 + 1) + (m^2 + n^2)x = 0$, $x = -\frac{m}{n}$, and $x = \frac{-n}{m}$,

$\therefore\ x + \frac{m}{n} = 0$, and $x + \frac{n}{m} = 0$, $\therefore\ x + \frac{m}{n}$ and $x + \frac{n}{m}$ are the factors required.

14. As the equation $x^2 + (a + b)x = 3ab$ admits of two values, each may be substituted for x, and two values of the expression may be obtained.

II.

1. $x^2 + 4x = 5$, complete the square, $x^2 + 4x + 4 = 9$. Extract the square root of each side, $x + 2 = \pm 3$, $\therefore\ x = -2 \pm 3$, that is $x = 1$ and -5.

2. $x = 5$ and -1. 3. $x = 3$ and -4. 4. $x = 12$ and -1.

5. $x = 17$ and $\frac{2}{3}$. 6. $x = 1$ and -6. 7. $x = \frac{1}{2}(7 \pm \sqrt{17})$.

8. $x = \frac{1}{3}(10 \pm \sqrt{30})$. 9. $x = \frac{1}{14}(13 \pm \sqrt{197})$. 10. $x = 5$ and $\frac{-32}{7}$.

11. $x = 50$ and -43. 12. $x = \frac{1}{2}$ and $-\frac{1}{3}$. 13. $x = \frac{1}{10}$ and $\frac{1}{11}$.

14. $x = \frac{-73 \pm \sqrt{2115}}{1572}$. 15. $x = \frac{4}{5}$ and $\frac{-86}{125}$. 16. $x = 37$ and -12.

17. $x = \frac{1}{2}(5 \pm \sqrt{24025})$. 18. $x = \frac{1}{20}(7 \pm \sqrt{241})$. 19. $x = \frac{1}{6}(-21 \pm \sqrt{321})$.

20. $x = \cdot 319$ and $-\cdot 001$. 21. $x = \frac{-\cdot 075 \pm \sqrt{\cdot 0007}}{\cdot 0015}$.

22. $x = \frac{1}{2}(-12\cdot 25 \pm \sqrt{156\cdot 255})$. 23. $x = 6$ and 0.

24. $x = \frac{-(15 + 7\sqrt{3}) \pm \sqrt{(156 + 90\sqrt{3})}}{2(9 + 5\sqrt{3})}$.

III.

1. $x = \frac{1}{2}(3 \pm \sqrt{5})$. 2. $x = 3$ and -1.

3. Clearing from fractions the equation $x + \frac{1}{16x} = \frac{1}{2}$ becomes

$16x^2 + 1 = 8x$, or $(4x)^2 - 2(4x) + 1 = 0$; $\therefore\ (4x - 1)^2 = 0$, $x = \frac{1}{4}$ and $\frac{1}{4}$.

4. $3x + \frac{3}{x} = 10$, then $3x^2 - 10x = -3$. Multiply each side by 3,

$\therefore (3x)^2-10(3x)=-9$, completing the square, $(3x)^2-10(3x)+25=25-9=16$;
$\therefore 3x-5=\pm 4$; $\therefore 3x=9$ and 1, and $x=3$ and $\frac{1}{3}$.

5. $x=4$ and 2. 6. $x=3$ and -2. 7. $x=-1\pm\sqrt{2}$.

8. Clearing the equation from fractions $6x^2-x=-57$. Multiply by 6,
$\therefore (6x)^2-(6x)=-342$, from which $x=\frac{1}{12}(1\pm\sqrt{-1367})$.

9. $x^2=\frac{x}{7}-\frac{4}{847}$, clearing from fractions $847x^2=121x-4$,
and $(77x)^2-11(77x)=-28$, from which $x=\frac{1}{11}$ and $\frac{4}{77}$.

10. $x=\frac{1}{14}(-4\pm\sqrt{793})$. 11. The equation reduced is
$x^2-4x=0$, or $x(x-4)$; $\therefore x=0$ and $x=4$.

12. $x=-1\pm 2\sqrt{7}$. 13. $x=\cdot 2$ and 2. 14. $x=\frac{1}{2}(-1\pm\sqrt{57})$.

15. $x=2$ and $-\frac{2}{3}$. 16. $x=2$ and $1\frac{2}{3}$. 17. $x=\frac{1}{4}(17\pm\sqrt{109})$.

18. $x=\frac{1}{8}(-5\pm\sqrt{649})$. 19. $x=3$ and -4. 20. $x=\frac{2}{3}$ and $\frac{1}{3}$. 21. $x=25$ and 1.

IV.

These examples may be reduced to the form of a quadratic equation by assuming y for the unknown power or root in the second term of the equation, and restoring in the solution the value of y.

1. $x^4+12x^2=13$. Let $x^2=y$, then $y^2+12y=13$, and completing the square, $y^2+12y+36=13+12=25$; $\therefore y+6=\pm 5$, and $y=-6\pm 5=-1$, and -11, restoring the value of y, $x^2=-1$, and $x^2=-11$; $\therefore x=\pm(-1)^{\frac{1}{2}}$ and $\pm(-11)^{\frac{1}{2}}$.

2. $3x+x^{\frac{1}{2}}=1$. Let $x^{\frac{1}{2}}=y$, then $3y^2+2y=1$, and from this equation $y=\frac{1}{3}$ and -1, restoring the value of y; $\therefore x^{\frac{1}{2}}=\frac{1}{3}$ and -1, and $x=\frac{1}{9}$ and $+1$.

3. $6x^{\frac{4}{3}}-5x^{\frac{8}{3}}=-592$, changing signs of the terms of the equation $5x^{\frac{8}{3}}-6x^{\frac{4}{3}}=592$. Let $y=x^{\frac{4}{3}}$, then $5y^2-6y=592$, and $y=\frac{1}{5}(3\pm\sqrt{2969})$, restoring the value of y,
$\therefore x^{\frac{4}{3}}=\frac{1}{5}(3\pm\sqrt{2649})$, and $x=\{\frac{1}{5}3\pm\sqrt{2649})\}^{\frac{3}{4}}$.

4. $x=\pm\{\frac{1}{2}(44\pm\sqrt{1911})\}^{\frac{1}{2}}$. 5. $x=\pm 11$ and $\pm\left\{\frac{-356}{3}\right\}^{\frac{1}{2}}$.

6. $x=\frac{\pm\sqrt{2}}{4}$ and $\frac{\pm\sqrt{-1}}{4}$. 7. $x=2$ and -1. 8. $x=a2^{\frac{1}{2}}a$ and $-2a$.

9. $x=1$ and -2. 10. $x=\frac{1}{2}(-3\mp\sqrt{-7})$. 11. $x=\pm\{\frac{1}{2}(a^2\pm\sqrt{(a^4-4b^4)})\}^{\frac{1}{2}}$.

12. $x=8$ and $\frac{338}{49}$. 13. $x=\pm(3\pm 2\sqrt{2})^{\frac{1}{2}}$.

14. $x=\frac{1\pm\sqrt{(1+8a^3)}}{4a^2}$. 15. $x=\frac{1}{4}\{(a-2)\pm\sqrt{(a^2-4a)}\}^2$.

16. $x=4$ and $-2^{\frac{3}{2}}$. 17. $x=(41)^{\frac{3}{2}}$ and $+9$. 18. $x=9$ and $5+(25)^{\frac{1}{2}}$.

19. $x=81$ and $(-2)^{\frac{4}{3}}$. 20. $x=\pm 8$ and $\pm(-11)^{\frac{3}{2}}$.

21. $x=\pm\{\frac{1}{2}(44\pm\sqrt{1910}\}^{\frac{1}{2}}$. 22. $x=3^6$ and $\left\{\frac{-1}{7}\right\}^6$.

23. $x=1$ and $(-5)^{\frac{1}{2}}$. 24. $x=\pm(\frac{2}{3})^{\frac{3}{2n}}$ and $\pm(-2)^{\frac{3}{2n}}$.

25. $x=2^6$ and $\left\{\frac{-97}{3}\right\}^{\frac{3}{2}}$. 26. $x=3^3$ and $(-28)^{\frac{1}{2}}$.

27. $x=16$ and $(-33)^{\frac{3}{2}}$.

V.

1. $(x^2+4)^2-5(x^2+4)=44$. Let $x^2+4=y$, then the equation becomes $y^2-5y=44$, and $y=\frac{1}{2}(5\pm\sqrt{201})$, restoring value of y, and $x^2+4=\frac{1}{2}(5\pm\sqrt{201})$,
$\therefore x^2=-4+\frac{1}{2}(5\pm\sqrt{201})$, and $x=\pm\{-4+\frac{1}{2}(5\pm\sqrt{201})\}^{\frac{1}{2}}$.

2. $(x^2+5)^2=7+x^2$, $(x^2+5)^2-(x^2+5)=2$. Let $x^2+5=y$, then $y^2-y=2$, y may be found as above.

3. $(x^2+7x+5)^2-3x^2-21x=19$, subtracting 15 from each side of the equation, $(x^2+7x+5)^2-3(x^2+7x+5)=4$. Let $x^2+7x+5=y$, then $y^2-3y=4$, and $y=4$ and -1, and restoring the values of y, there arise two equations $x^2+7x+5=4$ and $x^2+7x+5=-1$, from which the values of x can be determined.

4. $x^2-5x-19=(x^2-5x+1)^{\frac{1}{2}}$ is reducible to the equation

$$(x^2-5x+1)-(x^2-5x+1)^{\frac{1}{2}}=20.$$ Let $(x^2-5x+1)^{\frac{1}{2}}=y$, then $y^2-y=20$.

5. $9x-4x^2+(4x^2-9x+11)^{\frac{1}{2}}=5$, may be reduced to

$$(4x^2-9x+11)-(4x^2-9x+11)^{\frac{1}{2}}=6.$$

6. The same remark applies to this equation.

7. $6x(x-1)-3\left\{3(x-2)(x+1)2x\right\}^{\frac{1}{2}}=4(x+3)$. By transposing $4(x+3)$ and reducing the terms the equation becomes $2(3x^2-5x-6)-3(3x^2-5x-6)^{\frac{1}{2}}=0$.

8. The equation $3x^2+8x-(28x^2+32x)^{\frac{1}{2}}=12-4x^2$, may be reduced to

$$(7x^2+8x)-2(7x^2+8x)^{\frac{1}{2}}=12.$$

9. $x^2+3=2\left\{(x^2-2x+2)^{\frac{1}{2}}+x\right\}$ is reducible to $(x^2-2x+2)-2(x^2-2x+2)^{\frac{1}{2}}=-1$.

10. $7x-x^2+(2x^2-14x+20)^{\frac{1}{2}}=10$. Multiply each side of the equation by 2, and

$$(2x^2-14x+20)-2(2x^2-14x+20)^{\frac{1}{2}}=0.$$

11. $\dfrac{x}{x+4}+\dfrac{4}{(x+4)^{\frac{1}{2}}}=\dfrac{21}{x}$. Multiply each side by x, and $\dfrac{x^2}{x+4}+\dfrac{4x}{(x+4)^{\frac{1}{2}}}=21$.

$$\text{Let } y=\frac{x}{(x+4)^{\frac{1}{2}}}, \text{ then } y^2+4y=21.$$

12. $x+4+\left\{\dfrac{x+4}{x-4}\right\}^{\frac{1}{2}}=\dfrac{12}{x-4}$. Multiply by $x-4$, and $(x^2-16)+(x^2-16)^{\frac{1}{2}}=12$.

13. $\dfrac{10(8x-15)}{x^2-4x+5}-\dfrac{12(11x-15)}{x^2-6x+6}=13x^2$. By reducing the left side of the equation it becomes $\dfrac{-52x^3+78x^2}{(x^2-4x+5)(x^2-6x+6)}=13x^2$, or $\dfrac{-4x+6}{(x^2-4x+5)(x^2-6x+6)}=1$,

$$\text{and } x^4-10x^3+35x^2-54x+30=6-4x;$$
$$\therefore\ x^4-10x^3+35x^2-50x+24=0,$$
$$\text{and } x^4-10x^3+25x^2+10x^2-50x=-24,$$
$$\text{or } (x^2-5x)^2+10(x^2-5x)=-24.$$
$$\text{Let } x^2-5x=y, \text{ then } y^2+10y=-24.$$

14. $\dfrac{2}{(x+2)^{\frac{3}{2}}}+\dfrac{(x+2)^{\frac{1}{2}}}{2}=\dfrac{17}{4(x+2)^{\frac{1}{2}}}$. Multiply each side of the equation by $4(x+2)^{\frac{3}{2}}$, and $8+2(x+2)^2=17(x+2)$.

15. $\dfrac{x}{x^2+x+5}+\dfrac{5x}{(x^2+x+5)^{\frac{1}{2}}}=\dfrac{116}{25x}$. Multiply each side by $25x$,

then $\dfrac{25x^2}{x^2+x+5}+\dfrac{25\times5x}{(x^2+x+5)^{\frac{1}{2}}}=116$. Let $\dfrac{5x}{(x^2+x+5)^{\frac{1}{2}}}=y$, then $y^2+25y=116$.

VI.

1. $x^3-12x=16$, $x^3-4x=8x+16$, or $x(x^2-4)=8(x+2)$.
2. The equation reduced is $6x^2+11x+6=0$.
3. $x^3-8=-(x^2-4)$.
4. $x^3-2x^2-4x+8=0$, $x^2(x-2)-4(x-2)=0$.
5. $x^3-x^2=-(x^2-1)$.
6. $(x^2+1)(x+2)=2$, $x(x^2+2x+1)=0$.
7. $x^3-6x-9=0$, by adding $-3x$ to each side, and transposing -9,

$$x^3-9x=-3x+9, \text{ or } x(x^2-9)=-3(x-3).$$

8. $x+7x^{\frac{1}{2}}=22$, $x-8=-7(x^{\frac{1}{2}}-2)$. 9. $x^3-x=2(x+1)$.
10. $x^3+7x=49(x+1)$, $x^3-49x=-7x+49$, or $x(x^2-49)=-7(x-7)$.
11. Add 4 $\therefore$ $x^3+8=-x^2+4=-(x^2-4)$.

12. $x^3-6x^2+10x+8=0$, may be arranged $x(x^2-6x+8)+2(x-4)=0$,
or $x(x-2)(x-4)+2(x-4)=0$, or $(x-4)(x^2-2x+2)=0$.

13. $x^3-x^2+3x-27=0$, or $(x^3-27)-x(x-3)=0$.

14. $x=64$ and $-\frac{1331}{64}$.

15. $(x-1)^3=(x+1)^2+2$, or $x^3-3x^2+3x-1=x^2+2x+1+2$,
$\therefore x^3-4x^2+2-4=0$, and $x(x^2+1)-4(x^2+1)=0$.

16. $x^3=8+4(x+3)(x-2)$, $\therefore x^3-8=4(x+3)(x-2)$.

17. The equation when reduced is $x^3-2x^2+1=0$, $x^2(x-1)=x^2-1$.

18. $x=3x^{\frac{1}{2}}+2$. Add 1 to each side, $x+1=3(x^{\frac{1}{2}}+1)$. 19. $x^{\frac{3}{2}}-1=-a(x^{\frac{1}{2}}-1)$.

20. $x^{\frac{1}{2}}(x-1)=-a(x^{\frac{1}{2}}-1)$. 21. $x^{\frac{2}{3}}-1=-(x^{\frac{1}{3}}-1)$.

22. The equation may be put into the form $(x-x^{\frac{1}{2}})^2+9(x-x^{\frac{1}{2}})=90$.

23. $x^{\frac{2}{3}}-3x^{\frac{1}{3}}=2$. Add 1, $\therefore x^{\frac{2}{3}}+1=3(x^{\frac{1}{3}}+1)$.

24. $\frac{4(x^3+2x+5)x}{4x^2}-\frac{4(5x^3+x^2-1)}{4x^2}=\frac{5x^2}{4x^2}$, by dividing each side by $4x^2$.

$$\therefore \frac{x^3+5}{x}+2-\frac{5x^3-1}{x^2}-1=\frac{5}{4} \;\therefore\; x^2+\frac{5}{x}-5x+\frac{1}{x^2}=\frac{1}{4},$$

$$\text{whence } \left(x-\frac{1}{x}\right)^2-5\left(x-\frac{1}{x}\right)=\frac{1}{4}-2.$$

VII.

1. $x^4-4x+3=0$, $(x^4-1)-4(x-1)=0$, $x-1$ is a factor,
then $(x^3+x^2+x+1)-4=0$, or $(x^3-1)+(x^2-1)+(x-1)=0$.

2. $x^4-6=5x(x^2-x-1)$, $x^4-5x^3+5x^2+5x-6=0$, $(x^4+5x^2-6)-5x(x^2-1)$,
$(x^2-1)(x^2+6)-5x(x^2-1)=0$.

4. $x^4-2ax^3=2abx+b^2$, add to each side a^2x^2,
$\therefore x^4-2ax+a^2x^2=a^2x^2-2abx+b^2$.

5. $x^4+x^2+10x=0$, $x(x^3+x+10)=0$, $\therefore x=0$,
and $x^3+x+10=0$, $(x^3+8)+(x+2)=0$.

6. $x^4+4x^3-8x-165=0$, add and subtract $4x^2$,
$(x^4+4x^3+4x^2)-4(x^2+2x)=165$, or $(x^2+2x)^2-4(x^2+2x)=165$.

7. $x^4+4a^3x=a^4$, $x^4=-4a^3x+a^4$, add $2a^2x^2+a^4$ to each side,
$\therefore x^4+2a^2x^2+a^4=2a^2x^2-4a^3x+2a^4=2a^2(x^2-2ax+a^2)$.

8. $(x^2+1)^3=(x^2-1)^2(x-1)^2=0$, the equation reduced becomes $(x^4+x^2+1)x=0$.

9. $x^4+(1-x)^4=a$, or $x^4+1-4x+6x^2-4x^3+x^4=a$,
$\therefore x^4-2x^3+3x^2-2x=\frac{1}{2}(a-1)$, and $(x^4-2x^3+x^2)+2(x^2-x)=\frac{1}{2}(a-1)$.

10. $1+x^4=a(1-x)^4$. The equation reduced becomes

$$x^2+\frac{1}{x^2}-\frac{4a}{a-1}\left\{x+\frac{1}{x}\right\}=\frac{6a}{a-1}=0.$$

11. $a(1+x^4)=2x(1-x^4)^{\frac{1}{2}}$, the equation reduced becomes

$$x^4+\frac{1}{x^4}+\frac{4}{a^2}\left\{x^2-\frac{1}{x^2}\right\}+2=0.$$

12. $x^2(x^2+16x)+30x(2x-1)=45+2x$, the equation reduced becomes
$(x^2+8x)^2-4(x^2+8)=45$.

13. $x^4-2x^3+x=12$, $(x^4-2x^3+x^2)-(x^2-x)=12$.

14. $x^4+4b^2(1+a^2x^4)^{\frac{1}{2}}=c^2$. Multiply each side by a^2,
then $a^2x^4+4a^2b^2(1+a^2x^4)^{\frac{1}{2}}=a^2c^2$, add 1 to each side
and $(1+a^2x^4)+4a^2b^2(1+a^2x^4)^{\frac{1}{2}}=1+a^2c^2$.

15. $(a+x)^4-(a^2+x^2)^2=4a^2x^2$, the equation reduced is $4ax(a^2+x^2)=0$.

16. $x^2(x^2-18)=-4(x+12)$, or $x^4-18x^2=-4x-48$, add $4x^2+49$ to each side, and $x^4-14x^2+49=4x^2-4x+1$.

17. $(4x-2)^4-(3x-1)^4=(2x+4)^4-(x+3)^4$, by transposing
$(4x-2)^4-(2x+4)^4=(3x-1)^4-(x+3)^4$, which when reduced becomes
$20(x^2+1)32x=10(x^2+1)12x$.

18. By transposition $(x^2-1)^4-2(x^2-1)^2(x-2)^2+(x-2)^4=4x^2-12x+9$, extracting the square root, $(x^2-1)^2-(x-2)^2=\pm(2x-3)$, taking the positive value, $(x^2-1)^2-(x-2)^2=2x-3=2x-4+1=2(x-2)+1$.

19. $16x(x+1)(x+2)(x+3)=9$. Divide by 16, $\therefore\ x(x+1)(x+2)(x+3)=\frac{9}{16}$.

Since $x(x+1)(x+2)(x+3)$ is the product of four consecutive natural numbers, if 1 be added to this product it becomes a complete square.

$\therefore\ x(x+1)(x+2)(x+3)+1=1+\frac{9}{16}=\frac{25}{16}$, and extracting the square root,

$\therefore\ x^2+3x+1=\pm\frac{5}{6}$, from which equation the four values of x may be found,

20. $(x+1)(x+2)(x+3)(x+4)=48$, and $(x+1)(x+2)(x+3)(x+4)+1=49$, extracting the square root $x^2+5x+5=\pm 7$, from which the values of x can be determined.

21. $3(x^6-1)+8x^2(x^2-1)=0$, $(x^2-1)\{3(x^4+x^2+1)+8x^2\}=0$,

$$(x^2-1)(3x^4+11x^2+3)=0\ ;\ \therefore\ x^2-1=0, \text{ and } 3x^4+11x^2+3=0.$$

22. $x^4-8x^3+10x^2+24x+5=0$, $x^4-8x^3+16x^2-6x^2+24x+5=0$,

$$\text{or } (x^2-4x)^2-6(x^2-4x)+5=0.$$

23. $(1+x^8)+(1+x)^8=2(1+x+x^2)^4$. The equation when reduced becomes $8x^2(x^4+3x^3+4x^2+3x+1)=0$; $\therefore 8x^2=0$ and $x^4+3x^3+4x^2+3x+1=0$, divide by x^2,

$$\therefore\ \left(x^2+2+\frac{1}{x^2}\right)+3\left(x+\frac{1}{x}\right)+2=0.$$

24. $x^4-2x^3-3x^2+(2x+36)=0$, $x^4-2x^3+x^2-4x^2+12x+36=0$,

$$\text{or } (x^2-x)^2-4(x^2-x)+36=0.$$

25. $(x^4-2x^2+1)+x(x^2-2x+1)=0$, $(x+1)^2(x-1)^2+x(x-1)^2=0$,

$$\text{or } (x-1)^2\{(x+1)^2+x\}=0,\ \therefore\ (x-1)^2=0, \text{ and } (x+1)^2+x=0.$$

26. See Example 23.

27. $x^4-12x^3+44x^2-48x=9009$, $x^4-12x^3+36x^2+8x^2-48x=9009$,

$$\text{or } (x^2-6x)^2+8(x^2-6x)=9009.$$

28. $x^4+\frac{13}{3}x^3-39x=81$, $3x^4+13x^3-117x=243$, $3(x^4-81)+13x(x^2-9)=0$.

29. $\frac{x^6}{a^2b^4}=\frac{a^2+x^2}{b^2+x^2}$, $b^2x^6+x^8=a^4b^4+a^2b^4x^2$,

$$(x^8-a^4b^4)=-b^2x^6+a^2b^4x^2=-b^2x^2(x^4-a^2b^2),$$

$$\therefore\ (x^4-a^2b^2)\{x^4+a^2b^2+b^2x^2\}=0.$$

30. $\frac{x^2(x^4-a^4)}{x^6-a^6}=\frac{850}{931}$ may be expressed thus—

$$\frac{x^2(x^2+a^2)(x^2-a^2)}{(x^2-a^2)(x^4+a^2x^2+a^4)}=\frac{850}{931}.$$

VIII.

The equations under this number offer no difficulties in reducing them to the required form for solution.

IX.

1. $\frac{12+2x}{x+3}+\frac{4x-3}{2x+1}=\frac{4x-1}{x-1}$, $2+\frac{6}{x+3}+2-\frac{5}{2x+1}=4+\frac{3}{x-1}$;

or $\frac{6}{x+3}-\frac{5}{2x+1}=\frac{3}{x-1}$, which, when cleared of fractions and reduced, becomes $5x^2+37x=0$.

2. The equation cleared of fractions is

$$x^4+10x^3+19x^2-30x=x^4+10x^3+35x^2+50x+24.$$

3. The equation reduced is $2x^4-83x^2+13=0$.

5. The reduced equation is $x^2+27x-28=0$.

7. The reduced equation is $x^2-4x+3=0$.

9. By adding the fractions on the left side of the equation,
$\frac{x^2-5x-14}{(x-1)(x-2)(x-3)(x-4)}=\frac{1}{30}$, and $30(x^2-5x-14)=(x-1)(x-2)(x-3)(x-4)$. But since the product of four consecutive numbers increased by unity is always a complete square, add 1 to each side, and

$$30(x^2-5x-14)+1=(x-1)(x-2)(x-3)(x-4)+1=(x^2-5x+5)^2.$$

10. The equation reduced becomes $\frac{x^2-4}{(x-1)(x-2)(x-3)}=\frac{7}{6}$.

12. By combining the first two fractions on the left side of the equation, and then the other two, the equation becomes

$$\frac{20(x+8)^2}{(x^2+2x-48)(x^2+9x+8)}+\frac{4(x-10)^2}{(x^2+10x)(x^2+5x-50)}+1=0,$$

$$\text{or}\quad\frac{20(x+8)^2}{(x+8)(x-6)(x+8)(x+1)}+\frac{4(x+10)^2}{x(x+10)(x-5)(x+10)}+1=0.$$

15. By reducing each term of the equation to the form of a mixed quantity, the equation is

$$2x+3+\frac{5}{x-2}+2x+3+\frac{1}{x-3}=4x+6+\frac{9}{2x-3},\ \text{and}\ \frac{5}{x-2}+\frac{1}{x-3}=\frac{9}{2x-3}.$$

17. On reducing each side of this equation as above in example 15, it will be found that the conditions are incompatible or inconsistent.

X.

1. $\frac{1+a}{1-ax}=\frac{b-x}{1+bx}$, the equation reduced is $ax^2-(1+b+2ab)x-(1+a-b)=0$.

3. $\frac{a+b}{x+b}-\frac{a-c}{x-c}=\frac{2(a+b+c)}{x+b+c}$, this equation when reduced becomes

$$(2a+b+c)x^2-\{2ab+3(b+c)^2\}x+2(b^2c+bc^2+ac^2+2abc)=0.$$

5. $\frac{a^2+ax+x^2}{a^2-ax+x^2}=\frac{a^2}{x^2}$, when reduced becomes $(x^2+a^2)(x^2-ax-a^2)=0$.

6. The reduced equation is $(a+b-c-d)x^2+(a^2-c^2)x+a^2(a+b)-c^2(c+d)=0$.

7. $\frac{x+ab}{x^2-a^2b^2}-\frac{x-cd}{x^2-c^2d^2}=\frac{ab-cd}{x^2-abcd}$, is equivalent to

$$\frac{1}{x+ab}\div\frac{1}{x+cd}=\frac{ab-cd}{x^2-abcd},\ \text{which when reduced becomes}$$

$$2cdx^2-(ab-cd)^2x-2abc^2d^2=0.$$

12. The reduced equation is $3x^2-2(a+b+c)x+bc+ac+ab=0$.

16. Transposing and $\frac{1}{a+x}-\frac{1}{a-c+x}=\frac{1}{b-c+x}-\frac{1}{b+x}$;

$\therefore\ \frac{-c}{(a+x)(a-c+x)}=\frac{c}{(b-c+x)(b+x)}$, and $-(a+x)(a-c+x)=(b-c+x)(b+x)$.

17. The reduced equation is $abx^2-2abx+ab=0$.

18. $x=\pm(ab)^{\frac{1}{2}}$.

19. $\left\{\frac{1}{x+(a+b)}+\frac{1}{x-(a+b)}\right\}+\left\{\frac{1}{x-(a-b)}+\frac{1}{x+(a-b)}\right\}=0,$

$$\text{or}\quad\frac{2x}{x^2-(a+b)^2}+\frac{2x}{x^2-(a-b)^2}=0.$$

20. Reduce each term to a mixed number, and 2 may be removed from each side, and then each side is divisible by $a-b$, and the equation becomes

$$\frac{1}{x-a}\div\frac{1}{x-b}=\frac{3}{x+a-2b}-\frac{3}{x+b-2a};$$

whence $9(x-a)(x-b)=(x+b-2a)(x+a-2b)$ and $4(x-a)(x-b)=-(a-b)^2$, which gives two equal roots, each $\frac{1}{2}(a+b)$.

21. $x=\pm\frac{1}{2}\{2bc(1\pm\sqrt{2})\}^{\frac{1}{2}}$.

23. By clearing the equation from fractions it becomes

$$\{a^2+b^2+c^2+3(ab+ac+bc)\}x^2-3(a+b)(b+c)(c+a)x$$
$$=-\{a^2b^2+b^2c^2+c^2a^2+3(a+b+c)abc\}.$$

24. By aggregating the terms on the left side of the equation it becomes

$$\frac{3x^3+(a+b+c)x^2-(ab+ac+bc)x-3abc}{(x-a)(x-b)(x-c)}=3.$$

XI.

1. $(3x-2)^{\frac{1}{2}}(2x+3)^{\frac{1}{2}}=12$. By squaring each side, $(3x-2)(2x+3)=144$, and the reduced equation is $6x^2+5x=150$.

2. $(x^2-3x+7)^{\frac{1}{2}}+(x^2-4x+6)^{\frac{1}{2}}=14$. This equation reduced becomes

$$783x^2-2746x=-33321.$$

3. The reduced equation is $5x^2-7x-54=0$.

5. By squaring this equation, and reducing, it becomes $(x^2-8x+15)^{\frac{1}{2}}(x^2+2x-15)^{\frac{1}{2}}=x^2-3x+9$. Squaring again, and reducing, the resulting equation is $43x^2-204x+306=0$.

6. The reduced equation is $3x^2+2(a+b+c)x=(a+b+c)^2-4ab$.

8. Here $x^2+x-4=x^{\frac{1}{2}}(x+x^{\frac{1}{2}}+2)$,

$$\therefore \frac{x^2+x-4}{x^{\frac{1}{2}}}=x^2+x^{\frac{1}{2}}+2, \text{ or } \frac{x^2-4}{x^{\frac{1}{2}}}+x^{\frac{1}{2}}=x+x^{\frac{1}{2}}+2, \text{ or } \frac{x^2-4}{x^{\frac{1}{2}}}=x+2.$$

9. Since $(a-b)^3=a^3-b^3-3ab(a-b)$. Employ this form in raising each side of the equation to the third power, and it becomes $x^2+ax-a^2=0$. Otherwise, divide each side by $(a^2-x^2)^{\frac{1}{3}}$, then $\left\{\frac{a+x}{a-x}\right\}^{\frac{1}{3}}-\left\{\frac{a-x}{a+x}\right\}^{\frac{1}{3}}=1$, and raise each side to the third power.

10. The equation reduced is $(x^2-4x)^2-4(x^2-4x)+20=0$.

12. Add $x+1$ to each side, and $x^2+2x+1=x+14x^{\frac{1}{2}}+49$.

16. $(x^4-1)^{\frac{1}{2}}+(x^2-1)^{\frac{1}{2}}=x^2$, $(x^4-1)^{\frac{1}{2}}=x^2-(x^2-1)^{\frac{1}{2}}$, squaring each side

$$x^4-1=x^4-2x^2(x^2-1)^{\frac{1}{2}}+x^2-1, \ \therefore 2x(x^2-1)^{\frac{1}{2}}=x^4-x^2+1,$$

and squaring this equation and reducing, it becomes

$$\left(x^2-\frac{1}{x^2}\right)^2-2\left(x^2-\frac{1}{x^2}\right)+1=0, \text{ whence } x=\pm\sqrt{\tfrac{1}{2}(1\pm\sqrt{5})}.$$

19. The equation reduced is $x^4-3x^2=11$.

20. $(x^{-\frac{3}{2}}+1)^{-\frac{1}{2}}+(x^{-\frac{3}{2}}-1)^{-\frac{1}{2}}=x^{\frac{3}{4}}$, by changing the negative to positive integers,

the equation becomes $\left\{\frac{x^{\frac{3}{2}}}{1+x^{\frac{3}{2}}}\right\}^{\frac{1}{2}}+\left\{\frac{x^{\frac{3}{2}}}{1-x^{\frac{3}{2}}}\right\}^{\frac{1}{2}}=x^{\frac{3}{4}}$

$\therefore \frac{1}{(1+x^{\frac{3}{2}})^{\frac{1}{2}}}+\frac{1}{(1-x^{\frac{3}{2}})^{\frac{1}{2}}}=1$, squaring each side $\frac{1}{1+x^{\frac{3}{2}}}+\frac{1}{1-x^{\frac{3}{2}}}+\frac{2}{(1-x^3)^{\frac{1}{2}}}=1$,

or $\frac{2}{1-x^3}+\frac{2}{(1-x^3)^{\frac{1}{2}}}=1$, and $\frac{1}{1-x^3}+\frac{1}{(1-x^3)^{\frac{1}{2}}}=\frac{1}{2}$.

21. Clearing the equation from fractions, $x^2+2x^{\frac{3}{2}}+9x+8x^{\frac{1}{2}}+16=0$,

$$\text{or } (x+x^{\frac{1}{2}})^2+8(x+x^{\frac{1}{2}})+16=0.$$

XII.

1. $\frac{x+x^{\frac{1}{2}}}{x-x^{\frac{1}{2}}}=\frac{3x^{\frac{1}{2}}+6}{2x^{\frac{1}{2}}}$, the equation being reduced becomes $x+x^{\frac{1}{2}}=6$.

2. $\frac{x+x^{\frac{1}{2}}}{x-x^{\frac{1}{2}}}=\frac{x^2-x}{4}$, $\frac{x^2-x}{4}-\frac{x+x^{\frac{1}{2}}}{x-x^{\frac{1}{2}}}=0$, and

$(x+x^{\frac{1}{2}})\left\{\frac{x-x^{\frac{1}{2}}}{4}-\frac{1}{x-x^{\frac{1}{2}}}\right\}=0$, $\therefore$ $x+x^{\frac{1}{2}}=0$, and $\frac{x-x^{\frac{1}{2}}}{4}-\frac{1}{x-x^{\frac{1}{2}}}=0$.

4. Rationalise the denominators on each side of the equation, and the equation reduced gives $x=\frac{\pm\sqrt{3}}{2}$, and two other values each zero.

5. $x=\pm(ab)^{\frac{1}{2}}$.

6. Rationalise the denominators of the left side, then extract the square root of each side, and the equation reduced is $8x^2-312x+2703=0$.

7. $\frac{x+(x^2-a^2)^{\frac{1}{2}}}{x-(x^2+a^2)^{\frac{1}{2}}}=\frac{x}{a}$, the equation reduced is $(x-a)(4ax^2+3a^2x+a^3)=0$.

8. The reduced equation is $x^4(x^4-2a^2+1)=0$.

9. The equation reduced is $(1+2c)x^2+2a(1+c)x+a^2-b^2c^2=0$.

10. $\frac{a+x}{(a-x)^{\frac{1}{2}}}+\frac{a-x}{(a+x)^{\frac{1}{2}}}=2a^{\frac{1}{2}}$, $(a+x)^{\frac{3}{2}}+(a-x)^{\frac{3}{2}}=2a^{\frac{1}{2}}(a^2-x^2)^{\frac{1}{2}}$.

Squaring each side $(a+x)^3+(a-x)^3+2(a^2-x^2)^{\frac{3}{2}}=4a(a^2-x^2)$

and $5ax^2-a^3=(a^2-x^2)^{\frac{3}{2}}$, $\therefore$ $25a^2x^4-10a^4x^2+a^6=(a^2-x^2)^3$,

whence $x^2(x^4+22a^2x^2-18a^4)=0$.

11. The equation reduced is $x=x^2-1$.

12. The reduced equation is $4ab=c(a-x)^2$.

13. The given equation may be reduced to the form

$$1+\frac{a-b}{x+b}=\left\{1+\frac{a-b}{2x+b+c}\right\}^2=1+\frac{2(a-b)}{2x+b+c}+\frac{(a-b)^2}{(2x+b+c)^2}.$$

14. Subtracting unity from each side, the equation becomes

$$\frac{(x-a)^3-(x-b)^3}{(x-b)^3} \text{ or } \frac{-3x^2(a-b)+3x(a^2-b^2)-(a^3-b^3)}{(x-b)^3}=\frac{-3(a-b)}{x+a-2b}.$$

Hence $-3x^2+3x(a+b)-(a^2+ab+b^2)=\frac{-3(x-b)^3}{(x-b)+(a-b)}$

$$=-3\left\{(x-b)^2+(a-b)(x-b)+(a-b)^2+\frac{3(a-b)^3}{x+a-2b}\right\}.$$

$$=-3x^2+6xb-3b^2-3ax+3bx+3(a-b)b-3(a-b)^2-\frac{3(a-b)^3}{x+a-2b},$$

$\therefore$ $6(a-b)x=-2(a-b)(a-4b)-\frac{3(a-b)^3}{x+a-2b}$, and dividing by $a-b$,

$$6x+2(a-4b)=-\frac{3(a-b)^2}{x+a-2b}.$$

XIII.

The exercises under this number are pure quadratic equations.

1. $(40-x)^2=5x^2$, $\therefore$ $\pm\sqrt{5}x=40-x.(\pm\sqrt{5}+1)x=40$, and

$$x=\frac{40}{\pm\sqrt{5}+1}=\frac{40(\pm\sqrt{5}-1)}{(\pm\sqrt{5}+1)(\pm\sqrt{5}-1)}=\frac{40(\pm\sqrt{5}-1)}{5-1}=10(\pm\sqrt{5}-1).$$

2. $x=\frac{\pm a^2}{\sqrt{2}}$. 3. $x=\pm 3^{\frac{1}{2}}$. 4. $x=\pm 1$. 5. $x=\frac{\pm\sqrt{3}}{2}$.

8. $x=0$, $x=-17a$. 9. $x=\pm\left\{\frac{27a^2b^2-(b-2a)^3}{27b}\right\}^{\frac{1}{2}}$.

10. $x=\pm\left\{\frac{(a^2-bc)^2-a^2(b-c)}{a^2}\right\}^{\frac{1}{2}}$.

11. $(x^3+1)(x^{\frac{1}{2}}+1)^2=2(x^{\frac{1}{2}}+1)$, $\therefore$ $x^3(x^{\frac{1}{2}}+1)^{\frac{1}{2}}=2(x^{\frac{1}{2}}+1)-(x^{\frac{1}{2}}-1)^2=(x^{\frac{1}{2}}+1)^2$.

12. The equation reduced is $x^4+2x^3-4x^2+2x+1=0$.

13. $x=4$. 14. $x=\frac{\pm 2a\sqrt{b}}{1+b}$.

15. $\frac{a+x+(a^2-x^2)^{\frac{1}{2}}}{a+x-(a^2-x^2)^{\frac{1}{2}}}=\frac{b^2}{x}$, dividing the numerator and denominator of the left side by $(a+x)^{\frac{1}{2}}$, then $\frac{(a+x)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}}{(a+x)^{\frac{1}{2}}-(a-x)^{\frac{1}{2}}}=\frac{b}{x}$.

16. A similar remark applies to this equation.

17. Rationalise the denominator and $\frac{\{a+x+(2ax+x^2)^{\frac{1}{2}}\}^2}{a^2}=\frac{m^2}{x^2}$.

17. $\left\{\frac{x^3}{a}\right\}^{\frac{1}{2}}+\left\{\frac{a^3}{x}\right\}^{\frac{1}{2}}=x+a$, $\therefore x\left\{\frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}}}-1\right\}=a\left\{1-\frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}\right\}$,

$\therefore \frac{x}{a}=\frac{1-\frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}}{\frac{x^{\frac{1}{2}}}{a^{\frac{1}{2}}}-1}=\frac{x^{\frac{1}{2}}-a^{\frac{1}{2}}}{x^{\frac{1}{2}}-a^{\frac{1}{2}}}\cdot\frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}=\frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}$, $\therefore \frac{x^2}{a^2}=\frac{a}{x}$, and $x^3=a^3$, $\therefore x=a$. Otherwise, square each side of the given equation, and $\frac{x^3}{x}+\frac{a^3}{x}+2ax=x^2+2ax+a^2$.

19. $x=\frac{\pm a}{\sqrt{3}}$. 20. $x=\frac{\pm a\sqrt{-3}}{3}$.

21. Here $a-b=(x+a)\cdot\left\{\frac{x^2+b^2}{x^2+a^2}\right\}^{\frac{1}{2}}-(x+a)\cdot\left\{\frac{x^2+a^2}{x^2+b^2}\right\}^{\frac{1}{2}}$,

By squaring each side of the equation

$$a^2-2ab+b^2=x^2+b^2+\frac{2ax(x^2+b^2)}{x^2+a^2}+x^2+a^2+\frac{2bx(x^2+a^2)}{x^2+b^2}-2x^2-2(a+b)x-2ab,$$

$$\therefore a+b=a\left\{1-\frac{a^2-b^2}{x^2+a^2}\right\}+b\left\{1+\frac{a^2-b^2}{x^2+b^2}\right\},$$

$$\text{and } 0=-\frac{a(a^2-b^2)}{x^2+a^2}+\frac{b(a^2-b^2)}{x^2+b^2}, \therefore \frac{a}{x^2+a^2}=\frac{b}{x^2+b^2},$$

$$\text{and } ax^2+ab^2=bx^2+a^2b, \therefore (a-b)x^2=ab(a-b),\ x^2=ab,\ x=\pm(ab)^{\frac{1}{2}}.$$

XIV.

1. See Art. 6, p. 5.
2. See Section IV., p. 19.
3. The first equation gives two values of x, the second fails—explain why.
4. Here $x^4-x^3-\frac{3}{4}x^2+\frac{1}{2}x+\frac{1}{4}=x^2+2x+1$,
   and $\frac{1}{4}\{(2x^2-x)^2-2(2x^2-x)+1\}=(x+1)^2$, $\therefore \frac{1}{2}(2x^2-x-1)=\pm(x+1)$.
5. The quotient is x^2-2x-1, if $x^2-2x-1=0$, then $x=1\pm\sqrt{2}$.

6 Multiply the first equation by c and the second by ax, and subtract the latter from the former, and $(bc-ab)x^2-a^2x+c^2=0$. Eliminate x^2 between this and the second equation, and $x=\frac{abc-a^2b-c^2}{a^2c+b^2c-ab^2}$: If this value of x be substituted in the second equation, an equation will be found which contains only a, b, c.

7. Extracting the cube roots of each side of the equations,

$$x+x^{-1}=m^{\frac{1}{3}},\ x-x^{-1}=n^{\frac{1}{3}},\ \therefore 2x=m^{\frac{1}{3}}+n^{\frac{1}{3}},\ 2x^{-1}=m^{\frac{1}{3}}-n^{\frac{1}{3}},$$

$$\therefore 4=(m^{\frac{1}{3}}+x^{\frac{1}{3}})(m^{\frac{1}{3}}-n^{\frac{1}{3}})=m^{\frac{2}{3}}-n^{\frac{2}{3}}.$$

9. The equation is reducible to $225-6x=\cdot 25x^2$.

10. Here $\frac{a}{x+b}-\frac{b}{x+a}=\frac{c}{x+d}-\frac{d}{x+c}$, next $\frac{(a-b)x+(a^2-b^2)}{(x+a)(x+b)}=\frac{(c-d)x+(c^2-d^2)}{(x+d)(x+c)}$, but $a+d=b+c$, $\therefore$ also $a-b=c-d$.

11. The equation reduced becomes

$$(a+b+c)x^3+(a^2+b^2+c^2)x^2-3abcx-abc(a+b+c)=0.$$

But since $a+b+c=0$ is a condition, $(a^2+b^2+c^2)x^2-3abcx=0$, which gives

$$x=0, \text{ and } x=\frac{3abc}{a^2+b^2+c^2}.$$

14. The values of x are $\pm\sqrt{2}$ and $\pm 2\sqrt{-1}$.

18. See Art. 4, p. 4.

19. See p. 4.

20. The equation reduced is $x^2-2(a+b+c)x+3(ab+ac+bc)=0$.

XV.

1. Let $a^2-8x+22=m$, then $x^2-8x+16=m-6$, and $x-4=\pm\sqrt{(m-6)}$. In order that $x-4$ may be possible $\sqrt{(m-6)}$ must be rational, m must not be less than 6, and any number greater than 6, not excluding 6, satisfies the condition.

3. Let $\dfrac{x^2-4x+4}{x-1}=m$, then $x-\frac{1}{2}(m+4)=\pm\frac{1}{2}(m^2+4m)^{\frac{1}{2}}$.

4. If $\dfrac{x^2-3x-3}{2x^2+2x+1}=p$, then p lies between $\dfrac{3}{2}$ and $\dfrac{-7}{2}$.

5. Let $\dfrac{x^2-x+1}{x^2+x+1}=p$, then $x=\dfrac{-(p+1)\pm\{(3p-1)(3-p)\}^{\frac{1}{2}}}{2(p-1)}$.

Here $(3p-1)(3-p)$ must not be negative, it is equal to zero when $p=3$ and $p=\frac{1}{3}$, $\therefore$ the value of the fraction lies within these limits.

6. Let $(x-a)(b-x)=m$, then $x+\frac{1}{2}(a+b)=\pm\{m-\frac{1}{4}(a-b)^2\}^{\frac{1}{2}}$.

15. The given expression may be put into the form

$$(x-a)^2(x-b)^2-(a-b-c)c(x-a)(x-b).$$

XVI.

3. Here $\alpha+\beta=-\dfrac{b}{a}$, $\alpha\beta=\dfrac{c}{a}$: then $\alpha^2+\beta^2=\dfrac{b^2-2ac}{a^2}$, and $\alpha^{-2}+\beta^{-2}=\dfrac{b^2-2ac}{c^2}$.

The equation required is therefore $x^2-\left\{\dfrac{b^2-2ac}{a^2}+\dfrac{b^2-2ac}{c^2}\right\}x+\dfrac{(b^2-2ac)^2}{a^2c^2}=0$.

4. (1) $x^2+\{p-(p^2-4q)^{\frac{1}{2}}\}x-p(p^2-4q)^{\frac{1}{2}}=0$.
   (2) $x^2-\{p\pm(p^2-4q)^{\frac{1}{2}}\}x+p^2-2q=0$.
   (3) $\{x-(\alpha^2+\beta^2+\alpha\beta)\}.\{x-(\alpha^2+\beta^2-\alpha\beta)\}=\{x-(p^2-q)\}.\{x-(p^2-3q)\}$
   $=x^2-2(p^2-2q)x+(p^4-4p^2q+3q^2)=0$.
   (4) $(2p^2+q)x^2+3px+1=0$.
   (5) $q^2x^2-(p^5-5p^3q+3pq^2)x+q^2=0$.

5. Here when the equation is reduced to $x^2+\dfrac{(a+b)x}{a}+\dfrac{b}{a}=0$,

$$\alpha+\beta=-\frac{a+b}{a},\ \alpha\beta=\frac{b}{a}.$$

$$\text{Then } \frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\left\{\frac{(a+b)^2}{a^2}-\frac{2b}{a}\right\}\frac{a}{b}=\frac{a^2+b^2}{a^2}\times\frac{a}{b}=\frac{a}{b}+\frac{b}{a}.$$

6. The equation whose roots are $(ac+bd)+(ad-bc)$ and $(ac+bd)-(ad-bc)$, is $x^2-2(ac+bd)x+(ac+bd)^2-(ad-bc)^2=0$. And this equation is identical to

$$x^2-\frac{2(ac+bd)x}{a^2+b^2}+\frac{c^2+d^2}{a^2+b^2}=0$$

and has the same roots. Find the relation between the coefficients that the roots may be real.

9. Let α, β denote the roots of the proposed equation.

First transform $ax^2+bx+c=0$ into an equation whose roots shall be α^2, β^2, and this last equation into one whose roots shall be the reciprocals of α^2, β^2.

Next transform $ax^2+bx+c=0$ into an equation whose roots shall be $\alpha+\beta$, $\alpha-\beta$, and this last equation into one whose roots shall be $(\alpha+\beta)^2$, $(\alpha-\beta)^2$.

A quadratic equation, $x^2-px+q=0$, whose roots are α, β, may be transformed into another whose roots are the squares of the roots of the original equation, thus:

Here $x^2+q=px$, $\therefore\ x^4+2qx^2+q^2=p^2x^2$, and $x^4-(p^2-2q)x^2+q^2=0$.

Let $x^2=y$, then $y^2-(p^2-2q)y+q^2=0$, is the equation whose roots are α^2, β^2.

14. The roots of the equation $ax^2+2bx+c=0$ are real, equal, or imaginary, according as $ax^2+2bx+c$ is less than, equal to, or greater than a complete square.

15. If α, β be the roots of the first, and α', β' be the roots of the second equation, then $\alpha-\beta=\alpha'-\beta'$. Also $(\alpha-\beta)^2=(p-a)^2-4b^2$, and

$$(\alpha'-\beta')^2=(p-b)^2-4a^2. \quad \text{Whence } 2p=5(a+b).$$

16. Let α, β be the roots of the first, α, β' the roots of the second equation. From the first equation $\alpha+\beta=p_1$, $\alpha\beta=q_1$ from the second; $\alpha+\beta'=p_2$, $\alpha\beta'=q_2$.

$$\text{Then } \frac{p_1-p_2}{q_2-q_1}=\frac{\beta-\beta'}{\alpha(\beta-\beta')}=\frac{1}{\alpha},\ \therefore\ \beta=\frac{(p_1-p_2)q_1}{q_2-q_1},\ \text{and } \beta'=\frac{(p_1-p_2)q_2}{q_2-q_1}.$$

also $\alpha=\dfrac{q_2-q_1}{p_1-p_2}$ is the root common to the two equations.

17. The equation $x^2-px+q=0$, gives $\alpha+\beta=-p$, $\alpha\beta=q$. The equation $x^2-qx+p=0$ gives $\alpha'+\beta'=-q$, $\alpha'\beta'=p$. Then $\dfrac{(\alpha+\beta)(\alpha'+\beta')}{\alpha\alpha'\beta\beta'}=1$.

XVII.

1. ⅕ answers the question. Explain why the second value of x does not.

2. If x be the number $x^3+x^2=9(x+1)$ and $(x+1)(x^2-9)=0$, $\therefore\ x+1=0$, and $x^2-9=0$, $\therefore\ x=-1$, $x=\pm3$.

3. 16 and 4. 4. The number is 63. 5. 1, 2, and 3.

6. 9, 10, 11, and 12. 7. 6¼ per cent. per annum.

8. Let A, B, C execute the work w in x units of time, then they do $\dfrac{w}{x}$ of it in an unit of time. Similarly A executes $\dfrac{w}{x+6}$, B in $\dfrac{w}{x+15}$, C in $\dfrac{w}{2x}$.

$$\text{Then } \frac{w}{x}=\frac{w}{x+6}+\frac{w}{x+15}+\frac{w}{2x}.$$

XVIII.

1. Let a denote the given line, x one part, then $a-x$ will denote the other part. If b denote the other line, then $x^2+(a-x)^2=2b^2$.

2. Let x be the breadth, then $x+6$ is the length, then $x(x+6)=91$.

3. Let x be the breadth, $x+1$ is the length, and $x(x+1)=10100$.

4. Let x be the width of the path, then $a-2x$, $b-2x$ are the sides of the inner rectangle,

$$ab=\text{area of the path and inner rectangle,}$$
$$(a-2x)(b-2x)=\text{area of inner rectangle,}$$
$$\therefore\ ab-(a-2x)(b-2x)=\text{area of the path}=\frac{(a-2x)(b-2x)}{m}.$$

6. Let $2x$, $3x$, $4x$ denote the three edges,

$2x+1$, $3x+2$, $4x+3$, the edges when increased.

Then $(2x+1)(3x+2)(4x+3)-2x\times3x\times4x=153$.

7. 1676 men. 8. 961 men. 9. 4200.

10. See note, Art. 3, p. 20, note on triangular numbers, Section X.

XIX.

1. Here $x^2+2xy+y^2=64$, squaring the first equation,

$4xy=60$, 4 times the second.

$x^2-2xy+y^2=4$, subtracting,

$x-y=\pm 2$, extracting the square root,

but $x+y=8$

$\therefore 2x=10$ and 6, $\therefore x=5$ and 3,

$2y=6$ and 10, $y=3$ and 5.

If x, y be supposed to denote two numbers, the two equations may be expressed in words:—If the sum of two numbers be 8, and their product 15, what are the numbers?

2. Here $x^2-2xy+y^2=9$

$4xy=72$

$x^2+2xy+y^2=81$

$x+y=\pm 9$

$x-y=3$

$\therefore 2x=12$ and -6, $\therefore x=6$ and -3,

$2y=6$ and -12, $y=3$ and -6.

3. Here $2x^2+2y^2=226$

$x^2-2xy+y^2=1$

$x^2+2xy+y^2=225$

$x+y=\pm 15$

$x-y=1$

$\therefore 2x=16$ and -14, $\therefore x=8$ and -7,

$2y=14$ and -16, $y=7$ and -8.

5. $x^2+y^2=74$

$2xy=70$

$x^2+2xy+y^2=144$, $\therefore x+y=\pm 12$

$x^2-2xy+y^2=4$, $x-y=\pm 2$

$\therefore 2x=+14,\ -10,\ +10,\ -14$,

$2y=+10,\ -14,\ +14,\ -10$,

and $x=+7,\ -5,\ +5,\ -7$,

$y=+5,\ -7,\ +7,\ -5$.

6. Divide the first by the second equation,

$x-y=8$

$x+y=18$

$2x=26$, $\therefore x=13$,

$2y=10$, $y=5$.

8. $x^2+xy=140$

$y^2+xy=56$

$x^2+2xy+y^2=196$

$x+y=\pm 14$.

Then $x(x+y)=\pm 14x=140$, $\therefore x=\pm 10$

$y(x+y)=\pm 14y=56$, $\therefore y=\pm 4$.

10. $x^2 - 8xy + y^2 = 0$, $x^2 + 2xy + y^2 = 25$, $\therefore$ $10xy = 25$. See example 5.

NOTE.—Not only $x^2 \pm y^2 = b^2$, $x \pm y = a$: but also $x \pm y = a$, $x^3 \pm y^3 = b^3$; $x \pm y = a$, $x^4 \pm y^4 = b^4$; $x \pm y = a$, $x^5 \pm y^5 = b^5$, can be solved by means of quadratic equations.

XX.

1. $x^2 - 4x = 0$. 2. $x^3 - x^2 - 3x + 1 = 0$.

3. By dividing the first by the second equation, $x + 2y = 0$.

4. The given equations may be put into the forms $(x-y)^3 - (y-1)x + y^2 - 2y = 0$, and $(x-y)^2 = y^2 + y$. From which x may be eliminated, and $(y+1)^3 - y = 0$.

5. The second equation may be put into the form $(x+y)^2 + 4(x+y) = -3$, which gives $x = -(y+3)$ and $-(y+1)$, and each of these values of x substituted in the first equation will give two different equations involving y only.

The first equation can be put into the form

$$(x-y)^2 + 3x^2 - (6y+1)x + 3y^2 + y - 3 = 0.$$

6. Divide the second equation by the first, and from the resulting equation $x = \frac{3y}{2}$ and $\frac{2y}{3}$, each of these values of x can be substituted, and different equations may be found involving y only. If $x = \frac{3y}{2}$ be substituted in the second equation, $13y^3 - 104 = 0$.

7. The equations may be put into the forms $(x^2 - y^2)^2 - (x^2 - y^2) = 20$, and $(x-y)^2 = 4y$.

8. From the second equation $x^2 - xy = y^2 + 1$, add y^2 to each side, then $x^2 - xy + y^2 = 2y^2 + 1$. Substitute this value $2y^2 + 1$ in the first equation, and $(x^3 + y^3) + (x^2 - xy + y^2)(x+1) = 0$,

or $(x^2 - xy + y^2)(x + y + x + 1) = 0$, $\therefore$ $x^2 - xy + y^2 = 0$, and $y + 2x + 1 = 0$.

Hence $x = -\frac{y+1}{2}$. Substitute this value of x in $x^2 - xy + y^2 = 0$, and reduce the equation, the result is $7y^2 + 4y + 1 = 0$.

9. $2x^4 - 8x^3 - 15x^2 + 8x + 8 = 0$.

XXI.

1. Find x from the first, and y from the second equation, and substitute the values of x and y in the third, and reduce the result to its simplest form.

2. Divide the first by the second equation, $\frac{ax + by}{a_1x + b_1y} = \frac{c}{c_1}$, and $y = \frac{(a_1c - ac_1)x}{bc_1 - cb_1}$. substitute this value of y in the first equation, and

$$ba_1 - ab_1 = \{(bc_1 - cb_1)^2 + (ca_1 - ac_1)^2\}^{\frac{1}{2}}.$$

3. Here $bx - ay = x^2$, and $ax - by = y^2$,

$\therefore$ $(a-b)(x+y) = y^2 - x^2$, and $(a+b)(x-y) = x^2 + y^2$, also $x^2 + y^2 = 8$.

4. From the first and second equations may be found

$$x = \frac{c^2 - ab}{a} \text{ and } b;\ y = a \text{ and } \frac{c^2 - ab}{b}.$$

$$\text{If } x = \frac{c^2 - ab}{a} \text{ and } y = a, \text{ then } \frac{x^3}{a^3} + \frac{y^3}{b^3} = \frac{(c^2 - ab)^3}{a^6} + \frac{a^3}{b^3} = \frac{1}{8}.$$

$$\text{If } x = b \text{ and } y = \frac{c^2 - ab}{b}, \text{ then } \frac{x^3}{a^3} + \frac{y^3}{b^3} = \frac{b^3}{a^3} + \frac{(c^2 - ab)^3}{b^6} = \frac{1}{8}.$$

5. By eliminating x^2 from the two equations may be found $y = \frac{(a'b - ab')x}{ac' - a'c}$, then substitute this value of y in either of the given equations, and an equation is found involving only the constants.

6. Eliminate $x^3 + y^3$ from the two equations, and from the resulting equation

find the value of x in terms of y, or y in terms of x, and then substitute in either of the given equations.

7. Divide the second equation by the first, and $x^2 - xy + y^2 = \frac{b^4}{a^2}$,

$$\text{but } x^2 + xy + y^2 = a^2, \therefore xy = \frac{a^4 - b^4}{2a^2}.$$

Next divide the third by the second, and $x^4 - x^2y^2 + y^4 = \frac{c^8}{b^4}$.

$$\text{But } x^4 + x^2y^2 + y^4 = b^4, \therefore x^2y^2 = \frac{b^8 - c^8}{2b^4}.$$

$$\text{Hence } \frac{b^8 - c^8}{2b^4} = \frac{(a^4 - b^4)^2}{4a^4}, \text{ and } a^8 - 4a^4b^8 - 2a^4c^8 + b^{12} = 0.$$

9. $\frac{m}{n} \cdot \frac{a^2 + b^2}{b^2 - a^2} = 1.$

10. From the first and third equations find x and y, and substitute the values in the second.

XXII.

1. Divide the first equation by the second, the result is $6y = 5x$.

2. $x^2 = 6xy - 5y^2$, $x^2 = 4xy + 5y^2$, whence $6x - 5y = 4x + 5y$.

4. Here $\frac{y^2}{x} = \frac{ax - 1}{a - x}$, $\frac{x^2}{y} = \frac{by - 1}{b - y}$, $\therefore xy = \frac{(ax - 1)(by - 1)}{(a - x)(b - y)}$, whence

$$x^2y^2 - xy(ay + bx) = -(ax + by) + 1.$$

Again $a(y^2 - x^2) = x(y^2 - 1)$, $b(y^2 - x^2) = -y(x^2 - 1)$, $\therefore \frac{a}{b} = \frac{-x(y^2 - 1)}{y(x^2 - 1)}$,

whence $xy = \frac{bx + ay}{ax + by}$. Substitute the value of xy in the equation

$$x^2y^2 - xy(ay + bx) = -(ax + by) + 1,$$

and the resulting equation is $(ax + by)(ax + by - 1) = 0$.

5. Eliminate $x^2 + y^2$ between the two equations, and $(a^2 - b^2)x^2y^2 + 2a^2b^2xy = 0$.

6. Divide the first by the second equation :

$$\frac{x - y}{b^3} = \frac{a^3}{x + y}, \therefore x - y = \frac{a^3b^3}{x + y}, \text{ and } (x - y)^2 = \frac{a^6b^6}{(x + y)^2},$$

but $(x - y)^2 = a^3(x + y)$, the first equation,

$$\therefore a^3(x + y) = \frac{a^6b^6}{(x + y)^2}, \therefore (x + y)^3 = a^3b^6, \text{ and } x + y = ab^2.$$

$$\text{Also } (x - y)^2 = a^3(x + y) = a^4b^2, \therefore x - y = a^2b.$$

7. From the first equation $x - y = \frac{6}{x + y + 1}$, from the second $(x - y)^2 = \frac{5}{x + y}$.

8. Dividing the first equation by the second, $x^2y^2 = 36$.

9. $x = \pm \left\{\frac{a + b}{a - b}\right\}^{\frac{1}{2}}$, $y = \pm \left\{\frac{a - b}{a + b}\right\}^{\frac{1}{2}}$.

11. Divide the square of the first equation by the second, and $(x - y)^2 = \frac{x^2 + y^2}{5}$;

from the second equation $(x - y)^2 = \frac{3xy}{x + y}$, $\therefore \frac{x^2 + y^2}{5} = \frac{3xy}{x + y}$,

and $x^3 + x^2y + xy^2 + y^3 = 15xy$, and from the first equation

$$x^3 - x^2y - xy^2 + y^3 = 3xy, \text{ whence } x + y = 6.$$

12. The left side of the first equation may be resolved into factors.

$$(a + b)(x - y)\{(ax - by)^2 - (ax - by)(bx - ay) + (bx - ay)^2\} = (a + b)^3, \text{ but } x - y = 1,$$

$$\therefore (ax - by)^2 - (ax - by)(bx - ay) + (bx - ay)^2 = (a + b)^2,$$

$$\text{or } (a^2 - ab + b^2)x^2 - (a^2 - 2ab + b^2)xy + (a^2 - ab + b^2)y^2 = (a + b)^2,$$

and $(a^2-ab+b^2)x^2-2(a^2-ab+b^2)xy+(a^2-ab+b^2)y^2=a^2-ab+b^2$,
by multiplying the square of the second equation by a^2-ab+b^2,
$\therefore (a^2+b^2)xy=3ab$, and $x-y=1$; from which x and y may be found.

13. $x=\dfrac{\pm(m+n)c}{\{2(m^2+n^2)\}^{\frac{1}{2}}}$, $y=\dfrac{\pm(m-n)c}{\{2(m^2+n^2)\}^{\frac{1}{2}}}$.

14. Add the first equation to twice the second and $(x+2y)^2=196$.

15. $2(x^2+y^2)=5xy$, $2(x^2-y^2)=3$. Square each equation and subtract,

$$\therefore 16x^2y^2=25y^2-9.$$

16. Here dividing the first by the second equation—

$$\frac{x^2}{y^2}=\frac{ax+by}{bx+ay},\ \therefore bx^3+ax^2y=axy^2+by^3, \text{ and}$$

$$b(x^3-y^3)+axy(x-y)=0, \text{ or } (x-y)\{b(x^2+xy+y^2)+axy\}=0,$$

$$\therefore x-y=0, \text{ and } b(x^2+xy+y^2)+axy=0.$$

18 $\dfrac{x}{y}-\dfrac{y}{x}=\dfrac{5}{6}$, $\therefore 6x^2-5xy=6y^2$, whence $x=\dfrac{3y}{2}$ and $\dfrac{-2y}{3}$. By substitution in the first equation may be found the values of x and y.

19. Here $\dfrac{x^2}{a^2}+2+\dfrac{a^2}{x^2}+\dfrac{y^2}{b^2}+2+\dfrac{b^2}{y^2}=m+4$ or $\left\{\dfrac{x}{a}+\dfrac{a}{x}\right\}^2+\left\{\dfrac{y}{b}+\dfrac{b}{y}\right\}^2=m+4$,

and $\left\{\dfrac{x}{a}+\dfrac{a}{x}\right\}+\left\{\dfrac{y}{b}+\dfrac{b}{y}\right\}=n$. Let $\dfrac{x}{a}+\dfrac{a}{x}=p$, and $\dfrac{y}{b}+\dfrac{b}{y}=q$.

The equations then become $p^2+q^2=m+4$ and $p+q=n$.

20. Here $\dfrac{x^3}{y}=a^2-xy$, and $\dfrac{y^3}{x}=b^2-xy$, adding these equations

$$\frac{x^3}{y}+\frac{y^3}{x}=a^2+b^2-2xy, \text{ and } \frac{x^4+2x^2y^2+y^4}{xy}=a^2+b^2.$$

Next multiplying the given equations

$$x^2y^2=(a^2-xy)(b^2-xy)=a^2b^2-(a^2+b^2)xy+x^2y^2,\ \therefore (a^2+b^2)xy=a^2b^2$$

22. Multiply the equations, $(x+y)^2=25$, divide the equations, $x^2y^2=16$.

23. $x^2y+xy^2=ab$, $x^3+y^3=a^2-ab+b^2$. Divide the second by the first equation,

$$\frac{x^2-xy+y^2}{xy}=\frac{a^2-ab+b^2}{ab} \text{ or } \frac{x^2+y^2}{xy}-1=\frac{a^2+b^2}{ab}-1,$$

$$\therefore x^2+y^2=\frac{(a^2+b^2)xy}{ab}. \quad \text{But } x+y=\frac{ab}{xy},\ \therefore x^2+y^2=\frac{a^2b^2}{x^2y^2}-2xy;$$

$$\therefore \frac{(a^2+b^2)xy}{ab}=\frac{a^2b^2}{x^2y^2}-2xy,\ \therefore (a^2+b^2)x^3y^3=a^3b^3-2x^3y^3ab,\ \therefore x^3y^3=\frac{a^3b^3}{(a+b)^2}.$$

24. From the first equation $\dfrac{x^2}{y}+\dfrac{y^2}{x}=\dfrac{35}{3}$, $\therefore x^3+y^3=\dfrac{35xy}{3}$, and $x+y=10$, then

$$\frac{x^3+y^3}{x+y}=\frac{35xy}{30}=\frac{7xy}{6}.$$

25. Subtracting the square of the first equation from twice the second and extracting the square root, $xy^2-y=\pm 44$.

26. Multiply the second equation by 3 and add to the first equation, and

$$(x+y)^2-9(x+y)=0.$$

27. Divide the first by the second equation, and there results

$$a^3x^3-b^3y^3+3abxy(ax-by)=0.$$

XXIII.

1. From $x^2+a^2=(x+y)^2+(a-b)^2$, $y(2x+y)=b(2a-b)$;

$$\text{from } y^2+b^2=(x+y)^2+(a-b)^2,\ x(x+2y)=a(2b-a),$$

$$\therefore x^2-y^2=a^2-b^2.$$

2. Subtract the first equation from the second, and $5xy-y^2=3y$.

3. From the first equation $x=(24-3y)^{\frac{1}{2}}$. Add 1 to each side of the second,

$$x^2+2x+1=1+3(y-1)(3y-1)$$
$$x+1=\{1+3(y-1)(3y-1)\}^{\frac{1}{2}},$$
$$\text{whence } (24-3y)^{\frac{1}{2}}=-1+\{1+3(y-1)(3y+1)\}^{\frac{1}{2}}.$$

4. From the second equation find y in terms of x, and substitute in the first equation.

5. Find y from the first, and substitute in the second equation.

6. Equate the values of xy from the equations.

7. Divide the second equation by the first.

8. The first equation can be reduced to $(x-y)^2-2(a-b)(x-y)+(a-b)^2=0$.

9. Eliminate x^2 between the given equations.

10. $x^2-xy+y^2=7$, $x^3+y^3=(x+y)(x^2-xy+y^2)=6xy-1$,

$$\therefore\ x+y=\frac{6xy-1}{7},\ \text{and } x^2+2xy+y^2=\frac{36x^2y^2-12xy+1}{49};$$

$$\text{but } x^2-xy+y^2=7,\ \therefore\ 3xy=\frac{36x^2y^2-12xy+1}{49}-7,$$

from which xy may be found.

11. The first equation may be arranged thus: $x^2+2xy+y^2=x^2y^2-2xy+1$.

13. Divide the first by the second equation,

$$\frac{1}{\sqrt{2}}=\frac{(1-xy)\sqrt{3}}{x-y},\quad y=\frac{x-\sqrt{6}}{1-x\sqrt{6}}.$$

Substitute this value of y in either equation.

16. Subtract the square of the first equation from the quotient arising from dividing the second by the first, and $(ax)^2+(by)^2=\frac{nxy}{m}-abxy$.

Also from the square of the first equation $(ax)^2+(by)^2=m^2+2abxy$.

$$\text{Hence } m^2+2abxy=\frac{nxy}{m}-abxy.$$

20. From the first equation $x^2+y^2=\frac{a-nxy}{m}$, from the second $x^2+y^2=b^2-2xy$.

21. Divide the first by the second equation,

$$\frac{a^2b^2x^2-y^2}{(a^2+b^2)y^2-2a^2b^2xy}=\frac{m}{n},\ \text{and } \frac{a^2b^2.\frac{x^2}{y^2}-1}{a^2+b^2-2a^2b^2\frac{x}{y}}=\frac{m}{n},$$

$$a^2b^2.\frac{x^2}{y^2}-1=\frac{m(a^2+b^2)}{n}-\frac{2ma^2b^2}{n}.\frac{x}{y},$$

$$\therefore\ \frac{x^2}{y^2}+\frac{2m}{n}.\frac{x}{y}=\frac{m(a^2+b^2)+n}{na^2b^2}.$$

22. $x^2+y^2-5(x+y)=2$, $2xy+2(x+y)=68$, $\therefore (x+y)^2-3(x+y)=70$.

23. Divide the first by the second equation, and $2x=-y$.

24. The first equation divided by the second gives $(x+y)^2=25$.

25. From the first equation $abx\{2(a+b-x)\}=(2ab-y)y$, from the second $abx=\{x-(a+b)\}y$. Dividing one by the other $2(a+b)-x=\frac{2ab-y}{x-(a+b)}$: find y and substitute in either equation.

26. Here $x^2-x=y$ and $x=3y-y^2$, $\therefore x-1=\frac{1}{3-y}$ and $y-xy+3x-3=1$, $y(1-x)=4-3x$, and $y=\frac{4-3x}{1-2}$. Substitute in the first equation, and

$$x+\frac{4-3x}{1-x}=x^2,\ \therefore\ x^3-2x^2-2x-4=0,\ \text{or } (x-2)(x^2-2)=0.$$

XXIV.

1. Eliminate y, and $7x^2-4x+9=0$.

2. By equating values of y from each equation, $(a+b)x^2-2(ab-1)x+a+b=0$.

3. Eliminate x, $aby=(c-by)(cby-a)$.

4. Here $x^2y^2-1=\frac{3y}{2}$, $x^2y^2+1=\frac{5x^2y}{2}$, $\therefore\ 2=\frac{5x^2y}{2}-\frac{3y}{2}$,

$$\text{and } x^2=\frac{3y+4}{5y},\ \therefore\ x^2y^2=\frac{3y^2+4y}{5} \text{ and } x^2y^2-1=\frac{3y^2+4y}{5}-1=\frac{3y}{2}.$$

5. Substitute in the first equation the value of y from the second.

7. Here $\frac{1}{x}+\frac{1}{y}=\frac{x+y}{6}$, $\therefore\ \frac{x+y}{xy}=\frac{x+y}{6}$, and $xy=6$,

$$\text{and } \frac{x+y}{6}=\frac{5}{x+y+1},\ \therefore\ (x+y)^2+(x+y)=30.$$

8. Divide the second equation by the first, and

$$\frac{x^2+y^2}{xy}=\frac{(x-y)(x^2+y^2)}{xy(x+y)},\ \therefore\ (x^2+y^2)(x+y)=(x^2+y^2)(x-y), \text{ or } 2(x^2+y^2)y=0.$$

9. Add 2 to the first equation, and it becomes

$$\left\{\frac{x^2+y^2}{xy}\right\}^2+\left\{\frac{x^2+y^2}{xy}\right\}=\frac{25}{4}.\quad \text{And } 27x^2y^2+48xy-64=0.$$

10. Eliminating x, $by^2-cy+ac^2=0$.

11. The first equation may be reduced to $a^2y+b^2x=c^2x+c^2y-x^2y-xy^2$, and the second equation to $a^2b+b^2x=c^2x-bx^2-x^2y+bc^2$. Then subtracting the latter from the former, $a^2(y-b)=c^2(y-b)-x^2(y-b)$, or $(y-b)(x^2+a^2-c^2)=0$.

13. Eliminating y, $9x^2+2bx=ap(q-b)$.

14. Eliminate $x-\frac{1}{x}$ between the equations, first putting them into the form

$$a\left(x-\frac{1}{x}\right)+b\left(y-\frac{1}{y}\right)=m,\quad a'\left(x-\frac{1}{x}\right)+b'\left(y-\frac{1}{y}\right)=m'.$$

15. The first equation is $\frac{2-x^2-y^2}{(1-y)(1-x)}=\frac{9}{13}$, the second $\frac{2-x^2-y^2}{(1+y)(1-x)}=\frac{4}{13}$,

$\therefore\ \frac{(1+y)(1-x)}{(1-y)(1-x)}=\frac{9}{4}$, and $y=\frac{5}{13}$. Substitute this value of y in the first equation.

XXV.

1. $x^2+xy=14-x$, $y^2+xy=28-y$, $\therefore\ \frac{x(x+y)}{y(y+x)}=\frac{14-x}{28-y}$,

$$28x-xy=14y-xy,\ \therefore\ y=2x.$$

2. From the second equation, $x^4=6xy+12$, subtract 3 times the first, and $x^4-9x^2=-24$.

3. From the second equation $(x^2+y^2)^2=\frac{84100}{x^2y^2}$, and $x^4+y^4=641$,

$$\therefore\ 2x^4y^4+64x^2y^2=84100.$$

4. From $x^4+6y^2=5y^2x$, $x^2=3y$ and $2y$, and from $y^4+6x^2=5xy^2$,

$$y^2=3x \text{ and } 2x.$$

5. Add $2x^2y^2$ to each side of the first equation, $(x^2+y^2)^2+2xy(x^2+y^2)=8x^2y^2$, completing the square $(x^2+y^2)^2+2xy(x^2+y^2)+x^2y^2=9x^2y^2$.

6. Transpose $-x^2y^2$ and divide $x^4+x^2y^2+y^4=931$, by $x^2+xy+y^2=49$.

7. Adding the equations $(x^2-x)^2+(y^2-y)^2=a+b$, and $(x^2-x)+(y^2-y)=a$.

8. Eliminate x^2, and $3y^4-28y^2=207$.

9. Here $x^4+y^4-(x^2+y^2)=84$, and $2x^2y^2+2(x^2+y^2)=98$,

$$\therefore\ (x^2+y^2)^2+(x^2+y^2)=182.$$

11. Add $4x^2y^2$ to each side of the first equation,

$2(x^2+y^2)^2+7xy(x^2+y^2)=740+4x^2y^2$. From the second $x^2+y^2=10-\frac{xy}{2}$. Substitute this value of x^2+y^2 and reduce, the result becomes $xy=6$.

13. From the first equation $x^4y^4+6x^2y^2+1=313y^4$. Subtract from this equation the square of the second, and $8x^2y^2=288y^4$.

14. From the second equation $x^4+x^2y^2+y^4=4x^2+10xy+4y^2+21$. Divide this equation by the first,

$$\text{then } x^2-xy+y^2=\frac{4x^2+10xy+4y^2+21}{19}, \text{ and } 15x^2-29xy+15y^2=21.$$

$$\text{Also } 15x^2+15xy+15y^2=285, \therefore 44xy=264, \text{ and } xy=6.$$

15. From the first equation $(x^2-y^2)^2=(1+xy)^2$, $\therefore x^2-y^2=1+xy$. From the second $\frac{x^3+y^3}{x+1}=2y^2+1$. Hence $\frac{x^3+y^3}{x+1}=x^2-xy+y^2$, and

$$(x^2-xy+y^2)\left\{\frac{x+y}{x+1}-1\right\}=0, \therefore x^2-xy+y^2=0, \text{ and } \frac{x+y}{x+1}-1=0.$$

16. Multiply the first equation by 2, and add to the second,

$$\text{and } 2(x+y)^2+2xy(x+y)=30.$$

XXVI.

1. Divide the first equation by the second, $\frac{ax}{by}=\frac{x^2y+y^3}{x^3-xy^2}=\frac{\frac{x^2}{y^2}+1}{\frac{x^3}{y^3}-\frac{x}{y}}$.

$$\text{Let } \frac{x}{y}=z, \text{ then } \frac{z^2+1}{z^3-z}=\frac{az}{b}:\ bz^2+b=az^4-az^2, \text{ and } az^4-(a+b)z^2-b=0.$$

2. Find x from the first equation, and substitute in the second.

4. Subtract the second equation from the first,

$$\text{and } x^3-y^3=31(x^2-y^2)+4(x^2-y^2)=35(x^2-y^2).$$

5. Squaring the first equation, $x^2+y^2+12(x+y)+2\{x^2y^2+12(x^2+y^2)+144xy\}^{\frac{1}{2}}=1089$;

$$\text{but } x+y=23, \therefore x^2+y^2=529-2xy, \text{ and } 3(x^2+y^2)-35xy=5041.$$

6. Find y from the second equation, and substitute its value in the first.

7. From the second equation $4y^2+3x^2=12$.

8. Equate the values of $y^{\frac{1}{2}}$ from each equation, and $x-x^{\frac{1}{2}}=2$.

9. The first equation reduced is $(x^2-y^2)-(x^2-y^2)^{\frac{1}{2}}=6$.

10. The first equation reduced is $\frac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}}+\frac{y^{\frac{1}{2}}}{2^{\frac{1}{2}}x^{\frac{1}{2}}}=3$.

11. First eliminate x^2+y^2 between the two equations.

12. The first equation becomes $x^2-y^2=34xy$.

13. Raising each side of the first equation to the third power,

$$a^2x^2+b^2y^2+3(abxy)^{\frac{2}{3}}\{(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}\}=(a^2-b^2)^2,$$

$$\text{or } a^2x^2+b^2y^2+3(abxy)^{\frac{2}{3}}.(a^2-b^2)^{\frac{2}{3}}=(a^2-b^2)^2,$$

$$\therefore a^2b^2x^2+b^4y^2+3(abxy)^{\frac{2}{3}}(a^2-b^2)^{\frac{2}{3}}b^2=(a^2-b^2)^2.b^2,$$

$$\text{and } a^2b^2x^2+a^4y^2 \qquad =a^4b^2 \text{ from second equation,}$$

$$(a^4-b^4)y^2-3(abxy)^{\frac{2}{3}}(a^2-b^2)^{\frac{2}{3}}b^2=a^4b^2-(a^2-b^2)^2.b^2$$

$$\text{Similarly } (a^4-b^4)x^2+3(abxy)^{\frac{2}{3}}(a^2-b^2)a^2=-a^2b^4+(a^2-b^2)^2a^2.$$

$$\text{Hence } \frac{(a^4-b^4)x^2-\{(a^2-b^2)^2-b^4\}a^2}{(a^4-b^4)y^2+\{(a^2-b^2)^2-a^4\}b^2}=\frac{-a^2}{b^2}, \text{ and } b^2x^2+a^2y^2=a^2b^2$$

are sufficient to determine the values of x and y.

14. The first equation reduced gives $(x^2-y^2)=\frac{4}{5}(x^4+x^2y^2+y^4)$,

and the second, $(x+y)^3+(x-y)^3+2(x^2-y^2)^{\frac{3}{2}}=a^3$.

XXVII.

1. Divide the second equation by the first.
5. The equations may be put into the form, $x^{\frac12}(x^{\frac12}+y^{\frac12})=10$, and $y^{\frac12}(y^{\frac12}+x^{\frac12})$ 15.
9. Subtract the square of the second equation from twice the first, and

$$(x+y)^{\frac12}-2(x^2-y^2)^{\frac14}+(x-y)^{\frac12}=2a-b^2.$$

10. From the first equation $x^{\frac13}=2y^{\frac13}$.
11. The first equation becomes $16x=25y$.
13. The first equation gives $3x=5y$, and the second $\frac{y(x^2-y^2)^{\frac12}}{x^2-2y^2}=\frac{24}{7}$.
15. Multiply the first equation by $x^{\frac13}y^{\frac13}$,

$$\therefore\ x^{\frac43}-x^{\frac23}+2x^{\frac23}y^{\frac23}-y^{\frac23}=4y+4y^{\frac23}x^{\frac13}+3y^{\frac23}+2x^{\frac13}y^{\frac13},$$

$$\therefore\ x^{\frac43}+2x^{\frac23}y^{\frac23}+y^{\frac43}=x^{\frac23}+y^{\frac23}+4y^{\frac23}x^{\frac13}+4y^{\frac43}+2x^{\frac13}y^{\frac13}+4y,$$

$$(x^{\frac23}+y^{\frac23})^2+4y^{\frac23}(x^{\frac23}+y^{\frac23})+4y^{\frac43}=(x^{\frac13}+y^{\frac13}+2y^{\frac23})^2,$$

$$\therefore\ x^{\frac23}+y^{\frac23}=x^{\frac13}+y^{\frac13}+2y^{\frac23},$$

$$\therefore\ x^{\frac23}-y^{\frac23}=x^{\frac13}+y^{\frac13},\ \therefore\ x^{\frac23}-x^{\frac13}=y^{\frac23}+y^{\frac13},\ \therefore\ x^{\frac23}-x^{\frac13}+\tfrac14=y^{\frac23}+y^{\frac13}+\tfrac14,$$

$$\therefore\ x^{\frac13}-\tfrac12=y^{\frac13}+\tfrac12,\ \therefore\ x^{\frac13}=y^{\frac13}+1\ (1).$$

Multiply the second equation by $y^{\frac43}x^{\frac23}$, $\therefore\ x^2-2x^{\frac43}y^{\frac13}-2xy=\frac{133x^{\frac23}y^{\frac23}}{36}-2y^{\frac43}x^{\frac13}-y^2$,

$$x^2-2xy+y^2=2x^{\frac43}y^{\frac13}-2x^{\frac13}y^{\frac43}+\frac{133x^{\frac23}y^{\frac23}}{36},$$

$$(x-y)^2-2x^{\frac13}y^{\frac13}(x-y)+x^{\frac23}y^{\frac23}=\frac{169x^{\frac23}y^{\frac23}}{36},$$

$$\therefore\ (x-y)-x^{\frac13}y^{\frac13}=\frac{13x^{\frac13}y^{\frac13}}{6},\ \therefore\ x-y=\frac{19x^{\frac13}y^{\frac13}}{6}\ (2).$$

Hence the values of x and y can be determined from $x^{\frac13}=y^{\frac13}+1$

and $x-y=\frac{19x^{\frac13}y^{\frac13}}{6}$.

XXVIII.

1. The equations under this number may be solved by assuming $y=vx$, or $x=vy$, where v is some unknown quantity.

2. Taking the second example.

Let $y=vx$, then $x^2+6x^2v=136$, $4x^2v-3v^2x^2=5$.

Divide the first by the second equation,

$$\therefore\ \frac{x^2(1+6v)}{x^2(4v-3v^2)}=\frac{136}{5}.\ \text{or}\ \frac{1+6v}{4v-3v^2}=\frac{136}{5},$$

$$\therefore\ 5+30v=544v-408v^2,\ \text{and}\ 408v^2-514v=-5,$$

from which v may be found, and the relation between x and y determined.

Otherwise, eliminate y between the equations, and $3x^4-59x^2=-288$.

XXIX.

1. This transformation offers no difficulty.

2. Let $2x+ay+b$, $x+cy+d$ denote the simple factors, then $(2x+ay+b)(x+cy+d)$ when expanded is identical to the given expression. Equate the coefficients of corresponding terms, and find the values of a, b, c, d. The factors are $x-11y+1$, and $2x+y-3$.

3. From the first equation $x^3+y^3=(2y^2+1)(x+1)$, and from the second, $x^2-xy+y^2=2y^2+1$.

4. Subtract the second from the first equation, and there results

$$(y-x)^2-2a(y-x)+a^2=0.$$

5. Each equation when reduced and the terms arranged, may be exhibited in the form $(xy'-x'y)^2=a^2(y-y')^2+b^2(x-x')^2$.

6. The equation is $x^2-y^2+2xy\sqrt{-1}=a+b\sqrt{-1}$, $\therefore x^2-y^2=a$, and $2xy=b$.

7. $x=\pm\sqrt{\pm 2}$, $y=\pm\sqrt{(\pm 3\sqrt{-1})}$.

9. Let $y+mx+p$, $y+nx+q$, denote the two factors, in which m, p, n, q are to be determined; the product of those factors is identical to the given expression. Equate the coefficients of corresponding powers of x^2, xy, x, y, and eliminate m, p, n, q from these equations.

XXX.

1. Eliminating $x+y$ there results $m^2-m(a+b)=0$.

2. Here $ae^{\frac{y}{a}}-x=(x^2-a^2)^{\frac{1}{2}}$, $\therefore\ a^2e^{\frac{2y}{a}}-2axe^{\frac{y}{a}}+x^2=x^2-a^2$,

$$\therefore\ 2axe^{\frac{y}{a}}=a^2e^{\frac{2y}{a}}+a^2;\ \therefore\ 2x=a(e^{\frac{y}{a}}+e^{\frac{-y}{a}}),$$

next $x=(a^2+s^2)^{\frac{1}{2}}$, $\therefore\ x^2=a^2+s^2$, and $x^2-a^2=s^2$, and $(x^2-a^2)^{\frac{1}{2}}=s$,

but $ae^{\frac{y}{a}}=x+(x^2-a^2)^{\frac{1}{2}}=x+s$, and $2x=a(e^{\frac{y}{a}}+e^{\frac{-y}{a}})$, $\therefore\ 2s=a(e^{\frac{y}{a}}-e^{\frac{-y}{a}})$.

3. From the second and third equations, $\alpha=\dfrac{ax}{(x^2+y^2)^{\frac{1}{2}}}$, $\beta=\dfrac{ay}{(x^2+y^2)^{\frac{1}{2}}}$. Substituting these values of α and β in the first equation, and reducing the terms,

$$(x^2+y^2)-2a(x^2+y^2)^{\frac{1}{2}}+a^2=c^2,\ \text{whence } x^2+y^2=(a\pm c)^2.$$

4. $\dfrac{y^2-x^2}{2xy}=\dfrac{y-\beta}{x-\alpha}$, then $\dfrac{(y-\beta)^2}{(x-\alpha)^2}=\dfrac{(y^2-x^2)^2}{4x^2y^2}$,

$$\frac{(y-\beta)^2+(x-\alpha)}{(x-\alpha)^2}=\frac{(y^2-x^2)^2+4x^2y^2}{4x^2y^2}=\frac{(y^2+x^2)^2}{4x^2y^2},\ \text{and}\ \frac{c^2}{(x-\alpha)^2}=\frac{(y^2+x^2)^2}{4x^2y^2},$$

$$\therefore\ \frac{y^2+x^2}{2xy}=\frac{\pm c}{x-\alpha},\ \text{and}\ \frac{y^2-x^2}{2xy}=\frac{y-\beta}{x-\alpha},\ \therefore\ \frac{2y^2}{2xy}\ \text{or}\ \frac{y}{x}=\frac{\pm c+y-\beta}{x-\alpha},$$

$$\text{and } xy-\alpha y=\pm cx+xy-\beta x,\ \therefore\ \beta x-\alpha y=\pm cx.$$

5. $\dfrac{4y^2}{b^2}+\dfrac{4x^2}{a^2}=\dfrac{m^2}{b^2}+\dfrac{n^2}{a^2}+\dfrac{a^2}{n^2}+\dfrac{b^2}{m^2}+4$, and $2x.2y=mn+\dfrac{nb^2}{m}+\dfrac{ma^2}{n}+\dfrac{a^2b^2}{mn}$;

but $4xy=mn+\dfrac{a^2b^2}{mn}$. Hence $\dfrac{nb^2}{m}+\dfrac{ma^2}{n}=0$, also $\dfrac{b^2}{m^2}+\dfrac{a^2}{n^2}=0$, and $\dfrac{m^2}{b^2}+\dfrac{n^2}{a^2}=0$,

$$\therefore\ 4\left(\frac{y^2}{b^2}+\frac{x^2}{a^2}\right)=4,\ \text{and}\ \frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

7. Find y' from the third equation, substitute its value in the first equation, and $x'(y^2+x^2)=a^2x$; next in the second equation, and $x'^2(b^2y^2+a^2x^2)=a^4x^2$.

From the third, fourth, and fifth equations of the second set,

$b^2(x_1+x_2)^2+a^2(y_1+y_2)^2=a^2b^2$; and from the first and second equations,

$(x_1+x_2)(x^2+y^2)=a^2x$; also from $b^2(x_1+x_2)^2+a^2(y_1+y_2)^2=a^2b^2$,

and the first equation, $(x_1+x_2)^2(a^2x^2+b^2y^2)=a^4x^2$.

9. From the first and second equations

$$\frac{x^2}{b^2}=\frac{n^2-y^2}{m^2-a^2}\cdot\frac{m^4}{b^4},\ \text{and from the third}\ \frac{x^2}{b^2}=\frac{m^4y^2}{n^4a^2};$$

by equating these equals there results $m^2y^2=n^2a^2$.

10. Here $\dfrac{ny_2^2}{a^2}=1-\dfrac{mx_1^2}{a^2}$, and $\dfrac{ny_1^2}{a^2}=1-\dfrac{mx_2^2}{a^2}$;

$$\therefore\ \frac{n^2y_1^2y_2^2}{a^4}=1-\frac{mx_1^2}{a^2}-\frac{mx_2^2}{a^2}+\frac{m^2x_1^2x_2^2}{a^4},$$

but since $mx_1x_2+ny_1y_2=0$, $\therefore\ \frac{m^2x_1^2x_2^2}{a^4}=\frac{n^2y_1^2y_2^2}{a^4}$, whence $m(x_1^2+x_2^2)=a^2$.

Similarly, it may be shewn that $n(y_1^2+y_2^2)=a^2$.

11. Here $x-\alpha=\frac{x(a^4y^2+b^4x^2)}{a^4b^2}$, and $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Now $a^2y^2=a^2b^2-b^2x^2$, $\therefore\ a^4y^2=a^4b^2-a^2b^2x^2$, and
$a^4y^2+b^4x^2=a^4b^2-a^2b^2x^2+b^4x^2=b^2\{a^4-x^2(a^2-b^2)\}$,

$$\therefore\ x-\alpha=\frac{b^2\{a^4-x^2(a^2-b^2)\}x}{a^4b^2},\ \text{and}\ \alpha=\frac{x^3(a^2-b^2)}{a^4}.$$

By a similar process, $\beta=-\frac{y^3(a^2-b^2)}{b^4}$,

$\therefore\ x=\frac{a^{\frac{4}{3}}\alpha^{\frac{1}{3}}}{(a^2-b^2)^{\frac{1}{3}}}$, $y=\frac{-b^{\frac{4}{3}}\beta^{\frac{1}{3}}}{(a^2-b^2)^{\frac{1}{3}}}$. Substitute in $b^2y^2+a^2x^2=a^2b^2$,

and reduce the expression, and $(b\beta)^{\frac{2}{3}}+(a\alpha)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$.

12. From the first and second equation, $\left\{\frac{ma}{a}\right\}^2=\frac{b'a}{aa'}$, and

$\left\{\frac{ma'}{a'}\right\}^2=\frac{ba'}{aa'}$, $\therefore\ \left\{\frac{ma}{a}\right\}^2+\left\{\frac{ma'}{a'}\right\}^2=\frac{b'a}{aa'}+\frac{ba'}{aa'}=1$, $\therefore\ b'a+a'b=aa'$.

13. Eliminate b^2-y^2 from the two equations, and $x=ab^2y$; substitute the resulting value of y in either equation.

14. Since m, n are roots of $ax^2+b^2x+c^2=0$, then
$am^2+b^2m+c^2=0$, and $ax^2+b^2x+c^2=0$, $\therefore\ a(m+n)+b^2=0$.

Similarly since p, q are roots of $cy^2+b^2y+a^2=0$,

$$c(p+q)+b^2=0,\ \therefore\ \frac{m+n}{p+q}=\frac{c}{a}.$$

Again, because m, n are roots of $x^2+\frac{b^2x}{a}+\frac{c^2}{a}=0$, $\therefore\ mn=\frac{c^2}{a}$,

also because p, q are roots $y^2+\frac{b^2y}{c}+\frac{a^2}{c}=0$, $\therefore\ pq=\frac{a^2}{c}$,

Hence $\frac{mn}{pq}=\frac{c^4}{a^4}$, and $\frac{c}{a}=\left\{\frac{mn}{pq}\right\}^{\frac{1}{4}}$.

15. By adding the two equations, $(a+c)x^2+2bxy+(a+c)y^2=(a+c)b^2$, and $(a+c)(x^2+y^2-b^2)=-2bxy$, $\therefore\ \frac{x^2+y^2-b^2}{2xy}=\frac{-b}{a+c}$, whence $\frac{a+c+b}{a+c-b}=\frac{b^2-(x-y)^2}{(x+y)^2-b^2}$.

Next by subtracting the equations, $(a-c)x^2-(a-c)y^2=(a-c)b^2$, $\therefore\ x^2-y^2=b^2$,

$$\text{and}\ x+y=d,\ \therefore\ x-y=\frac{b}{d};\ \therefore\ \frac{a+c+b}{a+c-b}=\frac{b^2-\frac{b^4}{d^2}}{d^2-b^2}=\frac{b^2(d^2-b^2)}{d^2(d^2-b^2)}=\frac{b^2}{d^2}.$$

17. Here $\frac{qy^2+p}{y}=\frac{px^2+q}{x}$, or $qy+\frac{p}{y}=px+\frac{q}{x}$, $\therefore\ q\left(y-\frac{1}{x}\right)=p\left(x-\frac{1}{y}\right)$,

or $\frac{q(xy-1)}{x}=\frac{p(xy-1)}{x}$, $\therefore\ xy-1=0$, and $\frac{q}{x}=\frac{p}{y}$. But $x^2(y-p)=q$,

$\therefore\ \frac{1}{y^2}(y-p)=q$, and $qy^2-y+p=0$, whence $y=\frac{1\pm(1-4pq)^{\frac{1}{2}}}{2q}$, and in order that y may be rational $(1-4pq)^{\frac{1}{2}}$ must $=0$.

18. Here $ax+by=0$, and $x+y+xy=0$, then $ax+ay+axy=0$,

and $(a-b)y+axy=0$, and $\therefore\ x=\frac{-(a-b)}{a}$. Also $bx+by+bxy=0$,

and $(a-b)y-bxy=0$, $\therefore\ y=\frac{a-b}{b}$, $\therefore\ x^2+y^2=\frac{(a-b)^2}{a^2}+\frac{(a-b)^2}{b^2}=1$,

$$\text{and}\ \frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{(a-b)^2}.$$

19. Expanding the terms on the left side of the equation, adding the results and dividing each side by 2,

$$(x^2+y^2)+2=x(x^2+4)^{\frac{1}{2}}+y(xy^2+4)^{\frac{1}{2}}.\quad \text{Squaring this equation,}$$

$$(x^2+y^2)^2+4(x^2+y^2)+4=x^4+4x^2+y^4+4y^2+2xy(x^2+4)^{\frac{1}{2}}(y^2+4)^{\frac{1}{2}}.$$

$$\text{Whence } x^2y^2+2=xy(x^2+4)^{\frac{1}{2}}(y^2+4)^{\frac{1}{2}}.\quad \text{Squaring this equation,}$$

$$\therefore\ x^4y^4+4x^2y^2+4=x^4y^4+4x^4y^2+4x^2y^4+16x^2y^2.$$

$$\text{Whence } x^4y^2+x^2y^4+3x^2y^2=1,$$

$$\text{from which } (x^2y)^{\frac{2}{3}}+(xy^2)^{\frac{2}{3}}=1 \text{ is to be deduced.}$$

If each side of this equation be cubed,

$$\text{Then } x^4y^2+x^2y^4+3x^2y^2\left\{(x^2y)^{\frac{2}{3}}+(xy^2)^{\frac{2}{3}}\right\}=1,$$

$$\text{but } (x^2y)^{\frac{2}{3}}+(xy^2)^{\frac{2}{3}}=1,$$

$$\therefore\ x^4y^2+x^2y^4+3x^2y^2=1.$$

It is sometimes not obvious how the final result of a series of operations may be reduced to a given form. A suggestion may sometimes arise from the performance of the converse process, for discovering what necessary steps are required for reducing the expression obtained to the required form.

XXXI.

1. Let x and y be the parts, $x+y=41$, and $x^2-y^2=1$.
2. The digits are 4 and 3; and the number 43 or 34.
3. The numbers are 11 and 7.
4. The number is 36.
5. If y, x denote the digits, then $x=2$, $y=5$.
6. The numbers are 8 and 4.
7. The number is 72.
8. $x=\frac{1}{2}(3\pm\sqrt{5})$, $y=1$.
9. $x=\frac{1}{4}(5\pm\sqrt{5})$, $y=\pm\frac{1}{2}\sqrt{5}$.
10. Eliminate y between the equations, and $(3x-2)^2-3(3x-2)-2=0$.

XXXII.

1. If x denote the length and y the breadth, then

$$2x+2y=680, \text{ and } (x^2+y^2)^{\frac{1}{2}}=260.$$

2. If x, y denote the sides, $2x+2y=n(x-y)$ and $xy=m(x-y)$.
3. If x, y be the sides, $x^3+y^3=109(x+y)$, $x^3-y^3=229(x-y)$,

$$\text{find } xy \text{ and } (x^2+y^2)^{\frac{1}{2}}.$$

5. Let $ABCD$ be a section of the square tower in the same plane as the line of rail, C the angle nearest to, A the angle farthest from the line. Let the sides AB, BC, OD, DE of the square be produced to meet the line in the points E, F, G, H respectively, and from C let CK be drawn perpendicular to the line EH. Then CK is the distance of the nearest corner from the rail. Let $AB=x$, $CH=y$, also $EG=av$, $GF=bv$, $FH=cv$.

Then equations may be formed which will give $\dfrac{acv}{(a^2+c^2)^{\frac{1}{2}}}$ the length of a side of a horizontal section of the tower, and $\dfrac{abv}{(a^2+c^2)^{\frac{1}{2}}}$ the distance of the nearest corner to the line.

6. Let x, x, y denote the edges, then $x^2y=a^3$, $2(x^2+2xy)=b^2$, $4(2x+y)=a$.
7. Let $x-y$, x, $x+y$ denote the edges of the cubes,

$$\text{then } x-y+x+x+y=a, \text{ and } (x-y)^3+x^3+(x+y)^3=c^3$$

8. Let x, x, y denote the dimensions of the chest,

Then x^2y denotes the external content, $(x-2)^2(y-2)$ the internal,

$$\therefore\ x^2y-(x-2)^2(y-2)=\text{content of the boards}=x^2y.$$

Also external surface $=2(x^2+2xy)$, and internal $=2\{(x-2)^2+2(x-2)(y-2)\}$,

$$\text{and } 2(x^2+2xy)=\frac{88}{57}\{2(x-2)^2+2(x-2)(y-2\}.$$

9. Let x denote the circumference of the fore, y of the hind wheel,

$$\text{Then } \frac{120}{x}=\frac{120}{y}+6,\quad \frac{120}{x+3}=\frac{120}{y+3}+4.$$

10. *A*'s stock was £600 and *B*'s £400. Also *A*'s gain was £60 and *B*'s £40.

11. Suppose x oranges sold for y pence, then $\frac{y}{x+5}=\frac{y}{x}-\frac{1}{2}$ and $\frac{y}{x-5}=\frac{y}{x}+\frac{1}{2}$.

The first equation reduced gives $x^2+5x=10y$, the second $x^2-5x=10y$. The equations are therefore incompatible.

XXXIII.

1. Here $x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-xz-yz)$,

$$(x+y+z)^2-(x^2+y^2+z^2)=2xy+2xz+2yz=a^2-b^2,$$

$$\therefore\ xy+xz+yz=\frac{a^2-b^2}{2},$$

$$\therefore\ x^2+y^2+z^2-xy-xz-yz=b^2-\frac{a^2-b^2}{2}=\frac{3b^2-a^2}{2}.$$

$$\text{Hence } \frac{a(3b^2-a^2)}{2}=c^3, \text{ or } a^3-3ab^2+2c^3=0.$$

2. $a^3-2ab^2+c^3-3d^3=0$.

3. $a^2-2cd+b^2=0$.

4. Since $\frac{x}{a}+\frac{a}{x}=\frac{y}{b}+\frac{b}{y},\ \frac{x}{a}+\frac{a}{x}=\frac{z}{c}+\frac{c}{z},\ \frac{y}{b}+\frac{b}{y}=\frac{z}{c}+\frac{c}{z}$,

$$\text{then } \frac{x}{a}-\frac{y}{b}=\frac{b}{y}-\frac{a}{x},\ \frac{x}{a}-\frac{z}{c}=\frac{c}{z}-\frac{a}{x},\ \frac{y}{b}-\frac{z}{c}=\frac{c}{z}-\frac{b}{y},$$

$$\text{or } \frac{bx-ay}{ab}=\frac{bx-ay}{xy},\ \frac{cx-az}{ac}=\frac{cx-az}{xz},\ \frac{cy-bz}{bc}=\frac{cy-bz}{yz}.$$

Hence $ab=xy,\ ac=xz,\ bc=yz,\ \therefore\ x^2y^2z^2=a^2b^2c^2$, and $xyz=abc$,

also $x=a,\ y=b,\ z=c$.

Hence $(x+y+z)^2=(a+b+c)^2=0$, and $a+b+c=0$.

5. Here $x^3+y^3+z^3-3xyz=0$,

$$\text{or } (x+y+z)(x^2+y^2+z^2-xy-xz-yz)=0,$$

$$\therefore\ x+y+z=0, \text{ and } 3a-x+z=3b-y+x=3c-z+y.$$

From these equations $x=a-b,\ y=b-c,\ z=c-a$,

$$\therefore\ x+y+z=(a-b)+(b-c)+(c-a)=0.$$

6. $\frac{a^2b^2c^2}{b^2c^2+a^2c^2+a^2b^2}=\frac{m^2bc+n^2ac+p^2ab}{bc+ac+ab}$.

7. $\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+1=\alpha+1$, and $\frac{x}{z}+\frac{y}{x}+\frac{z}{y}+1=\beta+1$.

$$\therefore\ (\alpha+1)+(\beta+1)=\frac{(x+y)(y+z)(z+x)}{xyz}=\frac{c^3}{a^3},\ \therefore\ c^3=a^3\{(\alpha+1)+(\beta+1)\}.$$

8. From the third and fourth equations $y=\frac{b^2}{a^2}\cdot\frac{y'}{x'}\cdot x,\quad z=\frac{c^2}{a^2}\cdot\frac{z'}{x'}\,x$.

Substitute these values of y and z in the first and second, and when the equations are reduced, from the first, $x^2=\frac{a^4x'^2}{a^2x'^2+b^2y'^2+c^2z'^2}$,

$$\text{and from the second } x^2=a^4x'^2\left\{\frac{a^4x'^2}{a^2+k^2}+\frac{b^4y'^2}{b^2+k^2}+\frac{c^4z'^2}{c^2+k^2}\right\}^{-1}.$$

$$\text{Whence } a^2x'^2+b^2y'^2+c^2z'^2=\frac{a^4x'^2}{a^2+k^2}+\frac{b^4y'^2}{b^2+k^2}+\frac{c^4z'^2}{c^2+k^2}.$$

9. Add the first and second equations,

then $x^2+y^2=x^2+y^2+2z^2-2ayz-2bxz$, $\therefore z=ay+bx$ (1).

Similarly, $y=az+cx$ (2), from the first and third,

and $x=bz+cy$ (3), from the second and third.

From (1) and (2) $\frac{x}{y}=\frac{1-a^2}{ab+c}$, from (1) and (3) $\frac{x}{y}=\frac{c+ab}{1-b^2}$.

Whence $a^2+b^2+c^2+2abc=1$.

10. Since $x^2(y+z)=a^3$, $y^2(x+z)=b^3$, $z^2(x+y)=c^3$,

$\therefore x^2y^2z^2(x+y)(x+z)(y+z)=a^3b^3c^3$,

and $x^3y^3z^3=a^3b^3c^3$, $\therefore (x+y)(x+z)(y+z)=xyz$,

or $x^2y+y^2x+y^2z+z^2x+yz^2+x^2z+2xyz=xyz$,

and $x^2y+xy^2+y^2z+zy^2+z^2x+x^2z=-xyz$, $\therefore \frac{x^2y+xy^2+y^2z+yz^2+xz^4+x^2z}{xyz}=-1$.

But $\frac{x^2(y+z)}{xyz}=\frac{a^3}{abc}=\frac{a^2}{bc}$, or $\frac{x}{z}+\frac{x}{y}=\frac{a^2}{bc}$. Similarly $\frac{x}{z}+\frac{x}{y}=\frac{b^2}{ac}$, $\frac{z}{y}+\frac{z}{x}=\frac{c^2}{ab}$,

$\therefore \frac{x+z}{y}+\frac{x+y}{z}+\frac{y+z}{x}=\frac{a^2}{bc}+\frac{b^2}{ac}+\frac{c^2}{ab}=\frac{a^3+b^3+c^3}{abc}$,

or $\frac{x^2y+xy^2+y^2z+yz^2+xz^2+x^2z}{xyz}=\frac{a^3+b^3+c^3}{abc}$.

Hence $\frac{a^3+b^3+c^3}{abc}=-1$, or $a^3+b^3+c^3+abc=0$.

11. $a^2+b^2+c^2-2ab=1$.

12. $\frac{3}{(c-b)(c-a)}-\frac{1}{(b-c)(b-a)}-\frac{1}{(a-c)(a-b)}=0$.

13. $a^2+c^2=0$.

15. Multiply the first equation by x, the second by y, and the third by z; then since $xyz=abc$, the first equation becomes $ax^2+abc=bcx$; similarly, $by^2+abc=acy$, $cz^2+abc=abz$, from which x, y, z can be found and substituted in $xyz=abc$.

16. From the first and second equations $\frac{1-a^2}{ab+c}=\frac{x}{y}$, from the second and third $\frac{x}{y}=\frac{b+ac}{bc+a}$.

17. Since $\frac{x}{y}+\frac{y}{x}=c$ $\therefore \frac{x^2}{y^2}+2+\frac{y^2}{x^2}=c^2$, and $\frac{x^2}{y^2}-2+\frac{y^2}{x^2}=c^2-4$,

$\therefore \frac{x}{y}-\frac{y}{x}=\pm(c^2-4)^{\frac{1}{2}}$, but $\frac{x}{y}+\frac{y}{x}=c$, $\therefore \frac{2x}{y}=c\pm(c^2-4)^{\frac{1}{2}}$.

Similarly $\frac{2y}{z}=a\pm(a^2-4)^{\frac{1}{2}}$, and $\frac{2z}{x}=b\pm(b^2-4)^{\frac{1}{2}}$,

$\therefore \left\{c\pm(c^2-4)^{\frac{1}{2}}\right\}.\left\{b\pm(b^2-4)^{\frac{1}{2}}\right\}.\left\{a\pm(b^2-4)^{\frac{1}{2}}\right\}=\frac{2x}{y}.\frac{2y}{z}.\frac{2z}{x}=8$.

XXXIV.

1. $x=\frac{4\sqrt{7}}{5}$, $y=\frac{10}{\sqrt{7}}$, $z=\frac{35}{2\sqrt{7}}$.

2. From the equations are deducible $xy=12$, $xz=5$, $yz=20$, and $x=3$, $y=4$, $z=5$. This is one of the problems of Diophantus.

3. Multiply the equations, and $x^4y^4z^4=a^3b^3c^3$.

4. Here $xyz=\frac{a^3}{x}=\frac{b^3}{y}=\frac{c^3}{z}$; find y, z in terms of x, and substitute in the first equation, and $x=\pm\left\{\frac{a^9}{b^3c^3}\right\}^{\frac{1}{4}}$.

5. Eliminating y and z, $(a-1)x^2-(a-1)x=a-a^2$.

6. Since $x+y=axy$, $\therefore\ \frac{1}{y}+\frac{1}{x}=\frac{1}{a}$. Similarly for the other two equations.

8. Add the three equations, $2xy+2xz+2yz=a^2+b^2+c^2$, and $2xy+2xz=2a^2$, $\therefore\ 2yz=b^2+c^2-a^2$. Similarly, $2xy=a^2+b^2-c^2$, $2xz=a^2+c^2-b^2$.

9. Perhaps the simplest mode of solving this equation is to assume $y=ux$, and $z=vx$.

Then $\frac{x^2+xy}{y^2+yz}=\frac{x^2(1+u)}{x^2(u^2+uv)}=\frac{10}{21}$, and $\frac{y^2+yz}{z^2+xz}=\frac{x^2(u^2+uv)}{x^2(v^2+v)}=\frac{21}{24}=\frac{7}{8}$.

Next find v from the former in terms of u, and substitute in the latter equation.

10. $xyz=a^2(x+y)$ $\therefore\ \frac{1}{yz}+\frac{1}{xz}=\frac{1}{a^2}$. Similarly for the other two equations.

12. $a^2y+b^2x=xyz$, $\therefore\ \frac{a^2}{xz}+\frac{b^2}{yz}=1$.

13. It will be found that the third equation is the same as that found by eliminating z^2 between the first and second equations. Hence the solution is indeterminate.

14. By equating the values of y from the first and second equations and the second and third, $xz=(a+c)x-ac$.

15. By eliminating a and b, $x+z+2y=0$.

16. From the first equation, $y=\frac{az+p}{z-a}=a+\frac{a^2+p}{z-a}$, from the third, $y=a+\frac{a^2+r}{x-a}$.

Hence $z=a+\frac{(x-a)(a^2+p)}{a^2+r}$, but from the second $z=a+\frac{a^2+q}{x-a}$,

$\therefore\ \frac{a^2+q}{x-a}=\frac{(x-a)(a^2+p)}{a^2+r}$, and $(x-a)^2=\frac{(a^2+q)(a^2+r)}{a^2+p}$.

17. The equations are homogeneous, let $y=ux$, and $z=vx$.

18. Eliminate x and y, and then find the value of z.

19. Multiply the equations, and $(x+y)^2(y+z)^2(z+x)^2=a^2b^2c^2$.

Extract the square root $\therefore\ (x+y)(y+z)(z+x)=abc$.

Divide this equation by each of the given equations respectively,

then $x+y=\frac{ab}{c}$, $z+x=\frac{ac}{b}$, $y+z=\frac{bc}{a}$, and $2x+2y+2z=\frac{b^2c^2+a^2c^2+a^2b^2}{abc}$.

Subtract $2x+2y=\frac{2ab}{c}$, $\therefore\ 2z=\frac{b^2c^2+a^2c^2-a^2b^2}{abc}$, and $z=\frac{b^2c^2+a^2c^2-a^2b^2}{2abc}$,

and similarly for y and x.

20. The three equations may be put into the forms, $a^2\{(y+z)^2-y^2z^2\}=x^2y^2z^2$, $b^2\{(z+x)^2-z^2x^2\}=x^2y^2z^2$, $c^2\{(x+y)^2-x^2y^2\}=x^2y^2z^2$. Or the first equation may be reduced to the form $\left(\frac{1}{z}+\frac{1}{y}\right)^2-\frac{1}{x^2}=\frac{1}{a^2}$, also the second and third to the same form, and the sum of the three equations will give $\left\{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right\}^2=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$.

XXXV.

1. Subtract the third from the first equation, and $(1-x)(1-y)=12$, and the third from the second, and $(1-x)(1-z)=7$. Divide the former by the latter and $\frac{1-y}{1-z}=\frac{12}{7}$.

2. Add together the square of the first equation, and twice the second, and $x^2+y^2+z^2+4xy=22$. Subtract the third from this sum, $2x^2+4xy+2y^2=18$.

3. Subtract the second from the square of the first equation, and

$$2(xy+xz+yz)=a^2-b^2,\ x+z=a-y, \text{ and } xz=c^2.$$

5. From the second equation $y+z=18-x$, substitute in the third and

$$x^2-8x=-65.$$

6. $x^2+z^2+2y=14$, but $14=x^2+y^2+z^2$, $\therefore\ y^2=2y$.

7. The second equation is $x(y+z)+yz=3c^2$, from the first $y+z=3c-x$, from the third $yz=\frac{c^3}{x}$, substituting in the second equation $x^3-3cx^2+3c^2x-c^3=0$,

or $(x-c)^3=0$, which indicates 3 roots each equal to c.

8. From the first and second equations $2xyz=3y$.

9. From the second equation $x+y=x\sqrt{2}$, and from the third

$$z^2+2(x+y)z+(x+y)^2=x^2+(x+y)^2=x^2+2x^2=3x^2,$$

$$\therefore\ z+x+y=\pm x\sqrt{3}=a,\ \therefore\ x=\frac{\pm a}{\sqrt{3}}.$$

11. Find the values of xy, xz, yz, by elimination.

12. From the first and third equations $3y^2=75$.

13. By adding the three equations, $(a+b+c)(x+y+z)=3(x+y+z)^2$.

14. Here $yz+xz+xy=a^2-x^2=b^2-y^2=c^2-z^2$,

$$\therefore\ x^2+(y+z)x+yz=a^2, \text{ or } (x+y(x+z)=a^2\ ;$$

also $(y+x)(y+z)=b^2$, $(z-x)(z-y)=c^2$; whence $(x+y)(y+z)(z+x)=a^2b^2c^2$.

15. These equations are homogeneous, or the solution may be effected otherwise by first adding them together.

16. From the first and second equations by eliminating x, $\frac{y}{z}+\frac{z}{x}=-yz-1$.

$$\text{Similarly, } \frac{x}{z}+\frac{z}{x}=-xz-1,\ \frac{x}{y}+\frac{y}{x}=-xy-1.$$

$$\therefore\ \frac{x}{z}+\frac{z}{x}+\frac{y}{z}+\frac{z}{y}+\frac{x}{y}+\frac{y}{x}=-(xy+xz+yz)-3=-3,$$

$$\text{and } \frac{x}{z}+\frac{z}{x}+\frac{y}{z}+\frac{z}{y}+\frac{x}{y}+\frac{y}{x}+2=-1\ ;$$

$$\therefore\ \left\{\frac{x}{z}+\frac{z}{x}+\frac{y}{z}+\frac{z}{y}+\frac{x}{y}+\frac{y}{x}+2\right\}^2=1\ ;$$

$$\text{but } 2xyz=(1+x^2)(1+y^2)(1+z^2)=\left(\frac{x}{z}+\frac{z}{x}+\frac{y}{z}+\frac{z}{y}+\frac{x}{y}+\frac{y}{x}+2\right)^2.$$

Hence $2xyz=1$, and $yz=\frac{1}{2x}$; substitute this in the second equation,

and $x(y+z)=-yz=-\frac{1}{2x}$; $\therefore\ y+z=-\frac{1}{2x^2}$. Next substitute in the first equation,

$$\text{and } x+y+z=xyz=x\times\frac{1}{2x}=\frac{1}{2},\ \therefore\ y+z=\frac{1}{2}-x.$$

Hence $-\frac{1}{2x^2}=\frac{1}{2}-x$, and $2x^3-x^2=1=0$, or $x^3-x^2+x^3-1=0$,

$$\text{and } x^2(x-1)+(x^3-1)=0,\ \therefore\ (x-1)\{x^2-(x^2+x+1)\}=0.$$

18. These equations are homogeneous with respect to x, y, z. Let $y=ux$ and $z=vx$.

19. From the second and third equations $x^2y+xy^2+x^2z+xz^2+y^2z+yz^2=3a^3$,

$$\text{but } (x+y+z)^3=x^3+y^3+z^3+3(x+y)(x+z)(y+z),$$

$$\text{whence } 3(x+y)(y+z)(z+x)=-30a^3.$$

20. These equations are homogeneous with respect to x, y, z. Or otherwise, they can be put into the form $(x-y)(z-1)=-x^2$, $(x-y)(z-x)=-b^2$, $(y-z)(z-x)=-c^2$.

22. Add x^2 to the second equation, $\therefore\ (x+y)(x+z)=33+x^2$. Substitute this in the third equation, and $(33+x^2)(y+z)=294$, but $y+z=10-x$,

$$\therefore\ (33+x^2)(10-x)=294, \text{ whence } x^3-10x^2+33x-36=0, \text{ and}$$

$$x^3-3x^2-7x^2+21x+12x-36=0, \text{ or } (x-3)x^2-7(x-3)x+12(x-3)=0.$$

XXXVI.

1. Here $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{7}{2}$, or $\frac{xy+xz+yz}{xyz}=\frac{7}{2}$, and $xy+xz+yz=\frac{7}{2}$, or $x(y+z)+yz=\frac{7}{2}$

But since $x+y+z=\frac{7}{2}$, $y+z=\frac{7}{2}-x$, and $xyz=1$, $yz=\frac{1}{x}$,

make these substitutions in the first equation,

and $x\left(\frac{7}{2}-x\right)+\frac{1}{x}=\frac{7}{2}$, $\therefore\ 7x^2-2x^3+2=7x$,

or $2x^3-7x^2+7x-2=0$, or $2(x^3-1)-7x(x-1)=0$,

$\therefore\ (x-1)\{2(x^2+x+1)-7x\}=0$.

2. From the first equation $yz-xz-xy=0$. Add twice this equation to the second, and $(y+z-x)^2=36$, and $y+z-x=\pm 6$, also $x+2y+3z=12$.

Hence $3y+4z=18$ and 6.

3. By eliminating x and y, $z^2+c(p-q)z-c^2=0$.

4. Here $y-b=\frac{m(x-a)}{l}$, $z-c=\frac{n(x-a)}{l}$, by substitution

$$(x-a)^2\left\{1+\frac{m^2}{l^2}+\frac{n^2}{l^2}\right\}=r^2,\ \therefore\ (x-a)^2=\frac{l^2r^2}{(l^2+m^2+n^2)},\ \text{and}$$

$$x-a=\frac{\pm lr}{(l^2+m^2+n^2)^{\frac{1}{2}}},\quad y-b=\frac{\pm mr}{(l^2+m^2+n^2)^{\frac{1}{2}}},\quad z-c=\frac{\pm nr}{(l^2+m^2+n^2)^{\frac{1}{2}}}.$$

5. From the second equation $\frac{x}{y}-1=\frac{y}{z}-1$, and $\therefore\ \frac{x}{y}=\frac{y}{z}$, and $y^2=xz$,

but $y^2=bz$, $\therefore\ xz=bz$, and $x=b$.

6. Find the values of y and z in terms of x from the first two equations, and substitute these values in the third.

8. The equations may be written $\frac{1}{xz}+\frac{1}{yz}=\frac{1}{c^2}$, $\frac{1}{xy}+\frac{1}{yz}=\frac{1}{b^2}$, $\frac{1}{xy}+\frac{1}{xz}=\frac{1}{a^2}$,

from which may be found $\frac{2}{xy}=\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{c^2}$, $\frac{2}{xz}=\frac{1}{a^2}+\frac{1}{c^2}-\frac{1}{b^2}$, $\frac{2}{yz}=\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{a^2}$,

and $z^2=\frac{4a^2b^2c^2(b^2c^2+a^2c^2-a^2b^2)}{(b^2c^2+a^2b^2-a^2c^2)(a^2b^2+a^2c^2-b^2c^2)}$, similarly for y^2 and x^2.

9. Add the equations and $(x+y+z)^2=a^2+b^2+c^2$.

10. $x+y+z=\frac{x}{a^2}=\frac{y}{b^2}=\frac{z}{c^2}$, is a form in which the equations can be expressed. By adding the proposed equations there arises $a^2+b^2+c^2=1$. Explain this.

12. In the second equation, substitute the values of xz, x, y, $1-x$, $1-z$ in terms of y, and the resulting equation for finding the value of y is

$$y=\frac{aby^2}{\{(1-b)(1-y)-by\}\{(1-a)(1-y)-ay\}}.$$

XXXVII.

1. $x=\pm a$, $y=\pm b$, $z=\pm c$.

2. From the first equation $ax^2=b\{(y-x)z-y^2\}$, and from the second

$a\{(x-z)y-y^2\}=bx^2$. Dividing the former by the latter,

$\frac{x^2}{(x-z)y-y^2}=\frac{(y-z)z-y^2}{x^2}$. Reduce this expression to its simplest form.

3. Since $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, $\therefore\ bcx+acy+abz=abc$, squaring this equation,

$$b^2c^2x^2+a^2c^2y^2+a^2b^2z^2+2abc(cxy+bxz+ayz)=a^2b^2c^2.$$

But since $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=0$, $\therefore\ cxy+bxz+ayz=0$.

Hence $b^2c^2x^2+a^2c^2y^2+a^2b^2z^2=a^2b^2c^2$, and $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

4. Eliminate x between the given equations, and the result is reducible to $2(c-a)(a-b)yz=(a-b)^2z^2+(a-b)z^2$.

5. The equations give $x=\frac{a^2+b^2-c^2}{2ab}$, $y=\frac{a^2+c^2-b^2}{2ac}$, $z=\frac{b^2+c^2-a^2}{2bc}$;

whence $x+1=\frac{(c+b)^2-a^2}{2bc}$, $x-1=\frac{(c-b)^2-a^2}{2bc}$,

and $\frac{x+1}{x-1}=\frac{(c+b-a)(a+b+c)}{(c-b-a)(c+a-b)}$, and the rest in like manner.

6. From the proposed equations $\frac{a^2+x^2}{x}=\frac{a^2+y^2}{2y}=\frac{a^2+z^2}{4z}$.

From the equality of the first and second, $a^2=\frac{(y-2x)xy}{2y-x}$,

and from the equality of the second and third $a^2=\frac{(z-2y)yz}{2z-y}$.

8. First, equating the first and third expressions, $b=\frac{(ax-z)y}{n^2-y^2}$.

Next, equating the first and fourth expressions, and substituting the value of b; $a=\frac{zxy^2}{(x^2-m^2)n^2+m^2y^2}$; thirdly, substituting this value of a in the equation between the third and fourth expressions, and $b=\frac{m^2}{n^2}\cdot\frac{zy^2}{(n^2-m^2)n^2+n^2y^2}$. Lastly, substitute a and b in the equality of the third and fourth expressions, and reduce the result to its simplest form.

9. Take the sum of the squares of the first and second equations, and

$$(a^6+b^6)+2x(a^4+b^4)+(a^2+b^2)x^2=y^2+x^2.$$

But since $a^2+b^2=1$, $\therefore\ a^4+b^4=1-2a^2b^2$, $a^6+b^6=1-3a^2b^2$.

Make these substitutions, and $a^2b^2=\frac{(1+x)^2-y^2-z^2}{4x+3}$;

then combining this equation with $a^2+b^2=1$, a^2 and b^2 can be found in terms of x, y, z, and then lastly substituting in $a^2+b^2=1$, an equation is determined involving only x, y, z.

12. To simplify the work put m, n, p for $(a^2-yz)^{\frac{1}{2}}$, $(b^2-xz)^{\frac{1}{2}}$, $(c^2-xy)^{\frac{1}{2}}$, respectively, and since every square root is both positive and negative,

$$xyz=(a+m)(b+n)(c+p)=abc+bcm+acn+abp+anp+bmp+cmn+mnp,$$
$$xyz=(a-m)(b-n)(c-p)=abc-bcm-acn-abp+anp+bmp+cmn-mnp,$$
$$\therefore\ 2xyz=2(abc+anp+bmp+cmn), \text{ and } anp+bmp+cmn+abc=xyz,$$
$$\text{and } 0=2(bcm+acn+abp+mnp),$$
$$\therefore\ bcm+acn+abp=-mnp,$$
$$\text{and } b^2c^2m^2+a^2c^2n^2+a^2b^2p^2+2abc(cmn+bmp+anp)=m^2n^2p^2,$$
$$\text{or } b^2c^2m^2+a^2c^2n^2+a^2b^2p^2+2abc(xyz-abc)=m^2n^2p^2.$$

Restoring the values of m, n, p,

$$\therefore\ b^2c^2(a^2-yz)+a^2c^2(b^2-xz)+a^2b^2(c^2-xy)+2abc(xyz-abc)$$
$$=(c^2-xy)(b^2-xz)(a^2-yz)=a^2b^2c^2-a^2c^2xz+b^2c^2yz+a^2b^2xy+xyz(c^2z+a^2x+b^2y)$$
$$-x^2y^2z^2.$$

Hence rejecting common quantities on each side, and reducing the result,

$$a^2x+b^2y+c^2z=xyz+2abc.$$

13. From the first equation $y^2=\frac{a^2}{a-z^2}$, and from the second $y^2=\frac{ax^2-a^2}{x^2}$.

XXXVIII.

1. Let x, y, z denote the three numbers, then $z-x=5$, $(x+y+z)x=48$, $(x+y+z)z=128$.

2. Since the number is greater than 100 and less than 1,000, it consists of three digits. Let x, y, z denote the digits in the place of units, tens, hundreds; then $y=2x$, $100x+10y+z+99=100z+10y+x$, $x^2+y^2+z^2=21$.

3. Let x, y, z denote the numbers, then $x+y+z=15$, $x^2+y^2+z^2=495$, and and $x-y=y-z$.

4. If x, y, z denote the numbers, then $x-y$, $y-z$ are the differences, and the difference of the differences is $x-2y+z=5$, also $x+y+z=20$, and $xyz=130$.

5. The three numbers are respectively $\pm(bc)^{\frac{1}{2}}$, $\pm(ac)^{\frac{1}{2}}$, $\pm(ab)^{\frac{1}{2}}$.

6. If x, y, z denote the digits, then $z=y+x$, $yx=x+y+z+2$,

$$\frac{100z+10y+x}{z+y+x}=53.$$

7. Let x, y, z denote the three numbers, then $x+y+xy=a$, $x+z+xy=b$, $y+z+yz=c$.

If 1 be respectively added to each of these equations, they become $1+x+y+xy=1+a$, &c., or $(1+x)(1+y)=1+a$, $(1+x)(1+z)=b$, $(1+y)(1+z)=c$.

Taking the product $(1+x)^2(1+y)^2(1+z)^2=(1+a)(1+b)(1+c)$, and $(1+x)(1+y)(1+z)=(1+a)^{\frac{1}{2}}(1+b)^{\frac{1}{2}}(1+c)^{\frac{1}{2}}$. Whence x, y, z are readily found.

Otherwise, by successively eliminating x, y, and finding the value of z.

8. v, x, y, z denote the four numbers, and let a, b, c denote the three products respectively. Then $(v-x)(y-z)=a$, $(v-y)(x-z)=b$, $(v-z)(x-y)=c$. From the first and second equality $(v-z)(x-y)=a-b$; $\therefore a=b+c$, the relation of the three products.

9. Let $\frac{u}{v}$, $\frac{x}{y}$ denote the fractions, then $\frac{u+x}{v+y}=\frac{8u}{9v}$, $\frac{u-x}{v-y}=\frac{1}{2}$, $u+x=2(v-y)$.

10. Let x, y, z denote the sides of the triangle, then $x+y+z=56$, $x^2+y^2=z^2$, and $xyz=4{,}200$.

11. First, let £810 11s. 8d. $=m$, £841 1s. $=n$, £586 14s. 4d. $=p$, $q=$ £3 17s. 10½d. And suppose x ounces of gold and y ounces of silver, and that z is the price of an ounce of silver, then $qx+zy=m$, $q(x+y)=n$, $zx+qy=n$. From these equations, x, y, z may be found in terms of p, q, m, n, and by substitution, the required numerical results.

12. By taking successively each of the first three equations with the fourth, may be found $ux=180$ and 110, $xy=330$ and 60, $yz=110$ and 180, $zu=60$ and 330, $uy=264$ and 75, $xz=75$ and 264. Whence the values of u, x, y, z may be determined, and these numbers point out the four letters of the alphabet required.

XXXIX.

1. The equation when reduced becomes $\left\{\frac{a+x}{a-x}\right\}^{\frac{2}{m}}-4\left\{\frac{a+x}{a-x}\right\}^{\frac{1}{m}}=-2$.

2. First add one to each side of the equation, next subtract one from each side, thirdly divide the former of these results by the latter, and $x^{(m-n)^2}=\frac{a^m+b^n}{a^m-b^n}$.

3. (1) Since $x^my^n=a$, and $x^py^q=b$, $\therefore xy^{\frac{n}{m}}=a^{\frac{1}{m}}$ and $xy^{\frac{q}{p}}=b^{\frac{1}{p}}$.

Divide the former by the latter, $y^{\frac{n}{m}-\frac{q}{p}}=\frac{a^{\frac{1}{m}}}{b^{\frac{1}{p}}}$, and $\therefore y=\left\{\frac{a^p}{b^m}\right\}^{\frac{1}{np-mq}}$; x may be found by the same process. The other three equations may be solved in the same manner.

4. These equations may be solved by the same method of elimination at those in 3.

5. Since $x^y=y^x$ and $x^m=y^n$, $x=y^{\frac{x}{y}}$ and $x=y^{\frac{n}{m}}$, $\therefore\ y^{\frac{x}{y}}=y^{\frac{n}{m}}$, and $\therefore$ also $\frac{x}{y}=\frac{n}{m}$, and $x=\frac{ny}{m}$. Substitute this value of x in the second equation,

$$\text{then } \frac{n^m y^m}{m^m}=y^n,\ \therefore\ \frac{y^m}{y^n}=\left\{\frac{m}{n}\right\}^m,\ \text{or } y^{m-n}=\left\{\frac{m}{n}\right\}^m,$$

$$\therefore\ y=\left\{\frac{m}{n}\right\}^{\frac{m}{m-n}},\ \text{and } x=\frac{ny}{m}=\frac{n}{m}\cdot\left\{\frac{m}{n}\right\}^{\frac{m}{m-n}}=\left\{\frac{m}{n}\right\}^{\frac{n}{m-n}}.$$

6. Divide each side of the equation by x, then $x^{\frac{2q}{p-q}}=2-x^{\frac{q}{p-q}}$,

$$\text{and } x^{\frac{2q}{p-q}}+x^{\frac{q}{p-q}}=2,$$

which can be solved by a quadratic equation, from which $x^{\frac{q}{p-q}}=1$ and -2.

7. Divide each side of the equation by $x^{\frac{p+q}{2pq}}$, and

$$1=\frac{1}{2}\cdot\frac{a^2-b^2}{a^2+b^2}\cdot\left\{x^{\frac{p-q}{2pq}}+x^{\frac{q-p}{2pq}}\right\}=\frac{1}{2}\cdot\frac{a^2-b^2}{a^2+b^2}\cdot\left\{x^{\frac{p-q}{2pq}}+\frac{1}{x^{\frac{p-q}{2pq}}}\right\}.$$

Multiply each side by $x^{\frac{p-q}{2pq}}$, $\therefore\ x^{\frac{p-q}{2pq}}=\frac{1}{2}\cdot\frac{a^2-b^2}{a^2+b^2}\cdot x^{\frac{p-q}{pq}}+\frac{1}{2}\cdot\frac{a^2-b^2}{a^2+b^2}$, which can be solved by a quadratic equation.

XL.

1. (1) Let $x+a=m^2$, then $x-a=m^2-2a$. Suppose $m^2-2a=(m-a)^2$, then $m=\frac{a+2}{2}$, and $m^2=\frac{1}{4}(a^2+4a+4)$, $\therefore\ x=m^2-a=\frac{a^2+4}{4}$, in which a may be any number. If a be odd, x will be fractional, if a be even, x will be integral.

(2) Let $x^2+a=(x+p)^2=x^2+2px+p^2$, $\therefore\ x=\frac{a-p^2}{2p}$, where p must be a fraction in order to obtain positive values of x; if $p=\frac{m}{n}$, then $x=\frac{an^2-p^2m^2}{2mn}$, and if $a=1$, then $x^2+1=\frac{x^4+2m^2n^2+m^4}{4m^2n^2}$, which is a complete square.

(3) If $a=1$, $2x^2-1$ is a square when $x=1$, 5, &c.

(4) Here $a+bx+cx^2$ is a square, if $4a\times cx^2=(bx)^2$. Also when b^2-4ac is a square, $a+bx+cx^2$ may be resolved into two factors.

Let $m+nx$ and $p+qx$ denote the factors,

$$\text{then } (a+bx+cx^2)^{\frac{1}{2}}=\{(m+nx)(p+qx\}^{\frac{1}{2}}=\frac{r(m+nx)}{s}.\ \text{Suppose,}$$

$$\text{then } (m+nx)(p+qx)=\frac{r^2(m+nx)^2}{s^2},\ \text{and } p+qx=\frac{r^2(m+nx)}{s^2},$$

$$\text{whence } x=\frac{mr^2-ps^2}{qs^2-nr^2}.$$

(5) Is a particular example of this case, for $a=6$, $b=13$, $c=6$, $b^2-4ac=25$.

Here the factors are $2+3x$ and $3+2x$, and $x=\frac{3s^2-2r^2}{3r^2-2s^2}$, $x=\frac{3s^2-2r^2}{3r^2-2s^2}=\frac{\frac{3}{2}-\frac{r^2}{s^2}}{\frac{r^2}{s^2}-\frac{2}{3}}$.

In order that x may be positive $\frac{r^2}{s^2}$ must be less than $\frac{3}{2}$, but greater than $\frac{2}{3}$.

There are other cases which may be considered when a is a square number, also when c is a square number, and the cases in which the expression cannot be made a square.

2. (1) Since $5x+7y-2y=xy$, $\therefore\ x=\frac{7y-2q}{y-5}$, and any assumed value of y will give a corresponding value of x. If the equation be restricted to positive values of x and y. Let $y=1, 2, 3, 4, 5, 6, 7$, &c.

It will be found that the corresponding values of x are

$$\frac{11}{2},\ 5,\ 4,\ 1,\ \frac{6}{0},\ 13,\ 10,\ \&c.$$

When $y=1$, $x=\frac{11}{2}$; $y=5$, $x=\frac{6}{0}$, these values of x and y must be rejected, as not being positive integers.

(2) From $3y-xy+4x+2=0$, $x=\frac{3y+2}{y-4}$, here y must be greater than 4 in order to obtain positive values of x and y.

(7) The equation $2x(x+y)+y^2-b(y-3)=0$ may be reduced to the form $y^2+2(x-3)y=-18-2x^2$.

(8) $x=1, 3, 4, 7$; $y=39, 19, 15, 9$. (9) $x=4, 5, 6$; $y=27, 11, 5$.

(10) The least values are $x=70$, $y=30$.

3. Let x, y denote the numbers, then x^2+y^2 is to be a square number. Assume $x^2+y^2=(nx-y)^2=n^2x^2-2nxy+y^2$, then $x=\frac{2xy}{x^2-1}$; and for all values of x, corresponding values of y may be found. If integral values of x and y only be required. If $y=x^2-1$, then $x=2n$, and by taking $x=2, 3$, &c., a series of corresponding values of y will be found.

Next let $x^2-y^2=(x-ny)^2$, and by a similar process, if $y=2n$, then $x=a^2+1$.

8. Let the numbers be x^2, y^2, z^2, then $x^2+z^2=2y^2$. Let $x=m+n$, and $y=m-n$, then $x^2+z^2=2(m^2+n^2)=2y^2$, and $m^2+n^2=y^2$, the question is reduced to find m and n, the sum of whose squares shall be a square. See example 4. The numbers 1, 25, 49, and 4, 100, 196, are two integral solutions of the problem.

9. Let x denote the number of men in the side of the first of the five hollow squares, then $x+5$, $x+10$, $x+15$, $x+20$ are respectively the numbers of men in a side of the other four hollow squares.

Then $x^2-(x-10)^2=$ number of men in the first hollow square.
$(x+5)^2-(x-5)^2=$,, ,, second ,,
$(x+10)^2-x^2=$,, ,, third ,,
$(x+15)^2-(x+5)^2=$,, ,, fourth ,,
$(x+20)^2-(x+10)^2=$,, ,, fifth ,,

The whole number of men is $100x+500$. If y denote the number in the side of the solid square, then $y^2=100x+500$, and it is required to determine x, so that $100(x+5)$ may be a complete square.

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RATIO, PROPORTION, AND VARIATION.

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RATIO, PROPORTION, AND VARIATION.

Art. 1. Two unequal magnitudes of the same kind may be compared by considering how much (*quantum*) one of them is greater than the other; or how many times (*quot*) one contains the other. The former has been named their arithmetical, the latter their geometrical relation; and it is in this latter view the subject of ratio and proportion of quantities is considered.

Ratio has been defined to be the relation which one quantity bears to another of the same kind with respect to *quotity*, and the comparison is made by considering how many times one quantity is contained in the other, or what multiple part or parts, one is of the other.

If a, b denote two quantities of the same kind, a may be compared with b, or b with a. If a be compared with b, the ratio of a to b is denoted by $a : b$, and is represented by the quotient $\frac{a}{b}$, which is called the measure of the ratio $a : b$, and indicates how many times b is contained in a, or what multiple or fractional part a is of b, and the ratio $a : b$ may always be changed into a fraction $\frac{a}{b}$, or the fraction into a ratio.

It is also evident that the properties of ratios are dependent on the principles of fractions. The two numbers a, b are called the terms of the ratio $a : b$, the first is called the antecedent of the ratio, and the second the consequent. If the antecedent be greater than the consequent, the ratio is said to be one of greater inequality; but if the antecedent be less than the consequent, the ratio is named a ratio of less inequality.

The ratio of two concrete numbers of the same kind is the same as the ratio of two other concrete numbers of a different kind, when the antecedent of the former two contains the consequent or the same part or parts of it, as the antecedent of the latter two, contains the consequent or the like part or parts of it.

Two or more ratios may be compared by reducing their corresponding fractions to a common denominator, and comparing the numerators.

If the antecedents of two or more ratios be multiplied together for a new antecedent, and their consequents for a new consequent, then the resulting ratio is said to be compounded of these ratios, as if $a : b$, $c : d$ be any two ratios, then the ratio of $ac : bd$ is said to be compounded of the ratios $a : b$ and $c : d$. And similarly for any number of ratios.

If two equal ratios $a:b$ and $a:b$ be compounded, the resulting ratio is that of $a^2:b^2$, which is called the duplicate ratio of a to b.

In the same manner, if three equal ratios be compounded, the resulting ratio is $a^3:b^3$, and is named the triplicate ratio of a to b.

In a series of ratios, if the antecedent of a succeeding ratio be always equal to the consequent of a preceding ratio, then the ratio compounded of these ratios will be equal to the ratio of the first antecedent to the last consequent.

When two quantities of the same kind are compared by means of numbers, the numbers must express units of the same magnitude, and the ratio of the two quantities will also be the same as the ratio of two abstract numbers which denote the number of units in the two quantities.

It will often happen that one of the two quantities does not contain the other a certain number of times exactly; in that case the ratio between them is the ratio of the numbers which express how many times each quantity contains exactly some third quantity of the same kind.

And sometimes the strict meaning of the definition of a ratio is not admissible, as in cases in which the measure of the ratio of two quantities is neither an integer nor a rational fraction.*

---

* If $ABCD$ be a square whose side AB contains a units, the diagonal AC is equal to $a\sqrt{2}$ (Euc. I., 47); and the ratio of AC to AB is as $\sqrt{2}$ to 1; also if AB be the edge of a cube $ABCDE$, the diagonal AE is equal to $a\sqrt{3}$; and the ratio AE to AB is as $\sqrt{3}$ to 1; both of these ratios are incommensurable, or such that there is no integer or rational fraction which can exactly measure the side and diagonal of a square, or the side and diagonal of a cube. In Euc. X., 117, is demonstrated the proposition, that the side and diagonal of a square are incommensurable.

Although the ratio of the diagonal to the side of a square is incommensurable, approximations may be made to the true ratio. Since $\sqrt{2} = 1 \cdot 4142136$ . . . . . If two, three, four, &c., figures of this number be taken, these will be successive approximations to, but *each less than* $\sqrt{2}$. But if the last figures on the right of each of these numbers be increased by unity, then these numbers will be successive approximations to, but *each greater than* $\sqrt{2}$. So that $\frac{14}{10}$, $\frac{141}{100}$, $\frac{1414}{1000}$, &c., are successive approximations, but *each less than*, and $\frac{15}{10}$, $\frac{142}{100}$, $\frac{1415}{1000}$, &c., are also approximations, but *each greater than* the true value of the ratio.

These approximations may be continued to any extent, and therefore the value of the ratio $\sqrt{2}$ can be expressed to any degree of accuracy required.

It may be seen that the difference of each pair of the successive approximations becomes less and less, according as the number of figures becomes greater in each successive approximation.

The ratio of the circumference of a circle to its diameter, or the semi-circumference to the radius, is an irrational number.

The numbers $3\frac{1}{7}$, $\frac{355}{113}$, and $3 \cdot 14159$ . . . . are approximations to the value of the ratio of the circumference of a circle to the diameter.

In the volume of the Berlin Memoirs for 1761, M. Lambert has printed a paper entitled "Mémoire sur quelques propriétés remarquables des quantités

A variable quantity is one which admits of gradual change of its magnitude, and a quantity which can be made greater than any assignable magnitude, is said to increase without limit; and a quantity which can be made less than any assignable magnitude, is said to decrease without limit.

The limit of a variable quantity is defined to be that constant quantity towards which it may be made to approach nearer than by any difference that can be assigned.*

Two variable quantities may be supposed to approach towards equality or to some constant quantity, either by the continued diminution of their difference, or by the approach of their ratio either to a ratio of equality or some constant quantity.

The limiting ratio of two variable quantities is some constant quantity towards which their ratio may be made to approach nearer than by any other ratio which can be assigned.†

---

transcendantes Circulaires et Logarithmiques," in which he has proved that both the numerical values of the semi-circumference of a circle and the square of it are irrational numbers. The proof he has given depends on continued fractions.

Dr. Brewster, in 1824, published a translation of the Geometry of Legendre, and in the note 4, page 239, he has given in English Lambert's demonstration.

* The notion of a limit is necessary to understand what is called the sum of a *converging* series of quantities in geometrical progression indefinitely continued.

A *convergent series* is one in which each successive term becomes less and less than the preceding term, and is such as to admit of a limit.

A *divergent series* is one in which each successive term in general, becomes greater than the preceding, and is of such a nature as not to admit of a limit.

A geometrical series is *convergent* when the common ratio is less, but *divergent* when it is greater than unity.

The words *infinite* and *indefinite* are *negative* terms, and the exact meanings they have in mathematical reasonings must be determined by the sense of the positive terms *finite* and *definite*, which respectively express an opposite or contrary relation. A finite number is one which can be assigned, but an infinite number is one which cannot be assigned, and which can bear no conceivable relation to a finite number.

In the reasonings on prime and ultimate ratios, the symbol ∞ has been assumed to denote a number greater than can be assigned. If the symbol 0 be assumed to denote a number less than can be assigned, and not absolutely nothing (as it is employed in the numerical scale to denote absence of number), some confusion of ideas might be avoided by the learner.

A number or quantity may be considered to become, by continual increase or decrease, indefinitely great or indefinitely small, so that at length it may be conceived to become greater or less than any number or quantity that can be assigned, and in that case, it is said to be greater or less than any assignable number or quantity.

If two quantities increase or decrease without limit, their ratio does not necessarily increase or decrease without limit, but may ultimately have a finite limit which cannot be exceeded.

† The following are illustrations of the definition.

Let x and y be two variable quantities connected by the equation $y=x^2$, and if x receive an increment h, and x', y' be the corresponding values of x and y, then $x=x+h$ and $y'=(x+h)^2=x^2+2xh+h^2$.

2. Prop. *To ascertain the effect produced on a ratio by adding the same quantity to both its terms.*

Let $a : b$ be the given ratio, and let x be added to both its terms, so that it becomes the ratio of $a+x : b+x$.

Then the ratio of $a+x : b+x >$ or $<$ ratio of $a : b$,

according as $\frac{a+x}{b+x} >$ or $< \frac{a}{b}$,

$\therefore y' - y = 2xh + h^2$, the increment of y, and $\frac{y'-y}{h}$ or $\frac{y'-y}{x'-x} = 2x + h$.

Now, if the increment h be supposed continually to diminish, so as at length to become less than any assignable quantity, then the limit to which $\frac{y'-y}{x'-x}$ tends, as h is diminished, will be $2x$, which it can never exceed; or the ultimate ratio of the corresponding increments of y and x will be $2x$,

or the limit of $\frac{y'-y}{x'-x} = 2x$, when $y = x^2$.

This result admits of a geometrical interpretation. If x denote the side or edge, and y the area of a square, the limiting ratio of the corresponding increments of the area and the side of a square is equal to twice the length of the side.

In the same manner, if $y = x^3$, and if y', x' be the corresponding values of y and x when x receives an increment h, so that $x' - x = h$,

$$y' = (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3,$$

and $y' - y = 3x^2h + 3xh^2 + h^3$, the increment of y,

and $\frac{y'-y}{h}$ or $\frac{y'-y}{x'-x} = 3x^2 + 3xh + h^2$,

and the limit of $\frac{y'-y}{x'-x} = 3x^2$ when the increment h is diminished indefinitely.

This result also admits of a geometrical interpretation.

If x denote the edge, and y the volume of a cube, the limiting ratio of the corresponding increments of the volume and the edge of the cube whose edge is x, will be 3 times the area of one of the faces of the cube.

Bishop Berkeley, perhaps, was not mistaken in the objections he raised against the mathematicians of his time, when, in 1734, he wrote in the *Analyst:* "I have no controversy about your conclusions, but only about your logic; and it must be remembered that I am not concerned about the truth of your theorems, but only about the way of coming at them." The bishop, in full persuasion of the truth of the old aphorism "*ex nihilo nihil fit,*" could not admit that, when a finite quantity was assumed to vanish or become zero, any intelligible finite result as a consequence, could logically follow from such a change made in the original hypothesis.

In the former example, if $y = x^2$, then $\frac{y'-y}{x'-x} = 2x + h$, is true so long as the increment h is finite; but if the increment h be made, or supposed to become absolutely zero, then $x' - x = 0$ and $y' - y = 0$, and $\frac{y'-y}{x'-x} = 2x + h$ becomes $\frac{0}{0} = 2x + 0$.

And the bishop could not admit that the quotient arising from the division of zero by zero could ever be equal to any definite number or quantity.

But if it be admitted that the symbol 0 may denote a number less than any number that can be assigned, the expression $\frac{0}{0}$ may mean that the quotient or the ratio of two such quantities, each of which is less than any assignable quantity, may be equal to a finite quantity.

that is as $\frac{ab+bx}{b(b+x)} >$ or $< \frac{ab+ax}{b(b+x)}$,

as $ab+bx >$ or $< ab+ax$,

as $bx >$ or $< ax$,

as $b >$ or $< a$.

If $b > a$, then the ratio $a+x : b+x$ is $>$ the ratio of $a : b$.

If $b < a$, then ratio of $a+x : b+x <$ the ratio of $a : b$.

Hence a ratio of *less* inequality *is increased*, and of *greater* inequality *is diminished*, by adding the same quantity to both terms of the ratio.*

In a similar manner, it may be shewn that a ratio of *greater* inequality is increased, and of *less* inequality is diminished, by subtracting the same quantity from both its terms.

3. Prop. *If the terms of a ratio $a : b$ be multiplied by the same quantity, the ratio remains unaltered.*

The ratio of $a : b$ is represented by the fraction $\frac{a}{b}$; and $\frac{a}{b} = \frac{ma}{mb}$, for any fraction is unaltered by multiplying the numerator and denominator by the same quantity.

Hence the ratio of $ma : na$ is the same as the ratio of $a : b$.

Conversely. If the terms of a ratio $ma : mb$ be divided by the same quantity, the ratio remains unaltered.

The fraction $\frac{ma}{mb} = \frac{a}{b}$, for any fraction is unaltered in value when the numerator and denominator are divided by the same quantity. Hence the ratio $a : b$ is the same as the ratio of $ma : mb$.

Hence, the two numbers which express the ratio or relative magnitude of two quantities, do not always express the actual magnitude of the quantities compared, as it appears that different pairs of numbers can denote the same ratio.

4. "Four numbers are proportionals when the first is the same multiple of the second, or the same part or parts of it, as the third is of the fourth." (Euc. VII., Def. 20.)

Or. A proportion may be defined to be the equality of two ratios, or when the first of four quantities divided by the second gives the same quotient as the third divided by the fourth.

As, if a, b, c, d be four quantities such that $\frac{a}{b} = \frac{c}{d}$, then a, b, c, d are proportionals; and if a, b, c, d be proportionals, then $\frac{a}{b}=\frac{c}{d}$.

---

* It appears that the successive addition of the same quantity to both terms of a ratio, tends to make it to approach continually to a ratio of equality.

If the terms of the ratio 5 to 3 be successively increased by unity, the values of the successive ratios are $\frac{5}{3}, \frac{6}{4}, \frac{7}{5}, \frac{8}{6}, \frac{9}{7}, \frac{10}{8}$, &c., or $1\frac{2}{3}, 1\frac{2}{4}, 1\frac{2}{5}, 1\frac{2}{6}, 1\frac{2}{7}, 1\frac{2}{8}$, &c., and the successive differences are $\frac{1}{6}, \frac{1}{10}, \frac{1}{15}, \frac{1}{21}, \frac{1}{28}$, &c., each of which obviously becomes less and less, and therefore the successive values of the ratios tend to equality.

This tendency to equality is illustrated by the fact, that the ages of two persons, one older than the other, in successive years, continually tend to become relatively equal, while the difference between their ages remains constant.

A proportion is sometimes written $a : b :: c : d$, and is expressed by saying that a is to b as c is to d; the first and fourth terms are called the extremes, and the second and third the means, of the proportion.

If the third term be equal to the second, the proportion consists of three terms, a, b, d, and the second term b is called a mean proportional between a and d, the first and the third.

If the successive terms of four or more quantities, as a, b, c, d, have the same ratios, a to b, as b to c, as c to d; they are said to be in continued proportion.

There is a distinction between *direct* and *inverse* proportion. Four quantities a, b, c, d are said to be in direct proportion when the first and second terms are directly proportional to the third and fourth terms, as $\frac{a}{b}=\frac{c}{d}$; and it is obvious from the definition that if the first of four proportionals has the same ratio to the second, as the third has to the fourth; then if the first term be greater than the second, the third is greater than the fourth; if equal, equal; and if less, less.

Four quantities a, b, c, d are said to be in *inverse* proportion when the first and second terms are *directly* proportional to the *reciprocals* of the third and fourth terms, as $a : b :: \frac{1}{c} : \frac{1}{d}$, or as $a : b :: d : c$.

And in this case, if the first term be greater than the second, the reciprocal of the third is greater than the reciprocal of the fourth; if equal, equal; and if less, less.*

---

* The distinction between a *direct* and *inverse* proportion has a real existence in the forces of nature, as shewn by the following examples:—

(1) If a agents produce an effect e in time t; and a' agents produce the effect e' in time t'; find the relation between the effects, times, and agents, the units being the same in each case.

Since a agents produce effect e in time t; 1 agent produces effect $\frac{e}{a}$ in time t, and 1 agent produces effect $\frac{e}{at}$ in an unit of time.

Similarly, in the second case, 1 agent produces effect $\frac{e'}{a't'}$ in an unit of time. But as the units of agent and time are equal, $\therefore \frac{e}{at}=\frac{e'}{a't'}$, and $ea't'=e'at$, dividing each by $a'e't'$, $\therefore \frac{e}{e'}=\frac{at}{a't'}$, or $e : e' :: at : a't'$; that is, the effects produced are directly proportional to the product of the numbers which denote the agents and times in each case.

(2) If the times be the same in each case, then $t'=t$ and $ea't'=e'at$, becomes $ea'=e'a$, and $\frac{e}{e'}=\frac{a}{a'}$, or $e : e' :: a : a'$.

When four quantities are said to be directly proportional, it is to be understood that the third and fourth terms increase or diminish in the same ratio as the first and second terms increase or diminish.

But when the first and second terms increase or diminish in the same ratio as the third or fourth diminish or increase, the first is said to have directly to the second the same ratio as the inverse of the third has to the inverse of the fourth.

5. Prop. *If four quantities be proportionals, the product of the extremes is equal to the product of the means.*

Conversely. If the product of two quantities be equal to the product of two others, the four quantities may form a proportion of which the terms of either product may be the extremes, and of the other the means. (Euc. VII. 19.)

That is, if an effect e is produced by a agents, and the effect e' is produced by a' agents; then the effects produced are directly proportional to the number of agents in each case.

(3) If the number of agents be the same in each case, then $a'=a$, and $ea't'=e'at$ becomes $et'=e't$, or $e:e'::t:t'$.

That is, the effects produced, where the agents are the same in both cases, are directly proportional to the times.

(4) If the effects are the same in each case, then $e'=e$, and $ea't'=e'at$ becomes $a't'=at$, and $\frac{a}{a'}=\frac{t'}{t}=\frac{\frac{1}{t}}{\frac{1}{t'}}$, or $a:a'::\frac{1}{t}:\frac{1}{t'}$.

That is, when the effects are the same, the number of agents in each case is directly proportional to the reciprocals of the times, or the agents are inversely as the times.

An inverse proportion is illustrated by the condition for the equilibrium of a lever. If two weights a, b, at the extremities of the arms c, d, of a lever, balance each other about a fulcrum, the weights are directly proportional to the reciprocals of the lengths of the arms; or $a:b::\frac{1}{c}:\frac{1}{d}$.

The common rules for the interest and discount of money both at simple and compound interest may be expressed by Algebraical formulæ, the principal and time being considered *the cause*, and the interest *the effect*.

1. Find general expressions for the interest, discount, and amount at a sum of money for any given time at any given rate at simple interest.

Let I denote the interest for n years of the principal P; r being the interest of one pound for one year, and M the amount.

As the interest is directly proportional to the principal for any given time; since nr is the interest of £1 for n years,

Then $1:P::nr:I$, $\therefore I=Pnr$.

And $I=Pnr=\frac{P\times(100r)\times n}{100}$, which expressed in words gives the common rule for simple interest.

The simple interest of any sum of money for any number of years is found by multiplying the principal by the rate per cent., and this product by the number of years, and dividing the result by 100.

And the amount $M=P+I=P+Pnr=P(1+nr)$:

Also, if D denote the discount of M pounds due n years hence, if P be the present

Let a, b, c, d be proportionals, then $a : b :: c : a$, and $\therefore \frac{a}{b}=\frac{c}{d}$.

Multiply these equals by bd, $\therefore \frac{abd}{b}=\frac{cbd}{d}$, or $ad=bc$, that is the product of d, a the extremes is equal to the product of b, c the means.

Conversely. Let a, b, c, d be four quantities, such that $ad=bc$.

Divide these equals by bd, $\therefore \frac{ad}{bd}=\frac{bc}{bd}$, or $\frac{a}{b}=\frac{c}{d}$, and $\therefore a : b :: c : d$ or the four quantities are proportionals.

Since the product of the extremes is equal to the product of the means of four proportional quantities; it is obvious, that the product of the means divided by either of the extremes will give the other extreme; and the product of the extremes divided by either of the means will give the other mean:—thus

$$a=\frac{bc}{d},\ d=\frac{bc}{a},\ \text{and}\ b=\frac{ad}{c},\ c=\frac{ad}{b};$$

---

worth, the discount is equal to the difference between the given sum and its present worth; or, $D=M-P=M-\frac{M}{1+nr}=\frac{Mnr}{1+nr}$, the discount.

The ordinary rule for discount is deducible from this expression $D=\frac{Mnr}{1+nr}$.

For $\frac{1+nr}{Mnr}=\frac{1}{D}$, $\therefore \frac{1+nr}{M}=\frac{nr}{D}$, and $\frac{100+n(100r)}{M}=\frac{n(100r)}{D}$, which expressed in words gives the rule for discount.

As the amount of £100 at the given rate for the given time is to the given sum, so is the interest of £100 for the given time to the discount required.

In practice, however, it is better first to find the present worth and deduct it from the given sum, and the remainder is the discount.

$M=P(1+nr)$, $\therefore \frac{1}{P}=\frac{1+nr}{M}$, $\therefore \frac{100+n(100r)}{M}=\frac{100}{P}$, which gives the rule.

As the amount of £100 for the given time at the given rate is to the given sum, so is £100 to the present worth.

In calculating by compound interest, the amount at the end of the first year becomes the principal for the second year, the amount at the end of the second year the principal for the third year, and so on for successive years.

If P denote the original principal, r the interest of one pound for one year, and $P_1, P_2, P_3 \ldots P_n$ the amounts at the end of the first, second, third, ... nth year respectively.

Then $1 : P :: r : Pr$ the interest of P for the first year, and $P_1=P+Pr=P(1+r)$, is the principal for the second year. Next $1 : P_1 :: r : P_1r$ the interest of P_1 for the second year, and $P_2=P_1+P_1r=P_1(1+r)=P(1+r)^2$ is the principal for the third year. Similarly $P_3=P(1+r)^3$, and generally, $P_n=P(1+r)^n$, the amount of P in n years, and $I=P_n-P=P\{(1+r)^n-1\}$ the interest for n years.

Also, if D denote the discount of P_n for n years,

$$\text{Since } P_n=P(1+r)^n,\ \therefore P=\frac{P_n}{(1+r)^n} \text{ the present worth of } P_n,$$

$\therefore D=P_n-P=P_n-\frac{P_n}{(1+r)^n}=\frac{P_n\{(1+r)^n-1\}}{(1+r)^n}$, the discount of P_n for n years at compound interest.

this property is universally true, whether the proportion be direct or inverse.

When $a : b :: c : d$, these are proportionals when taken inversely as $b : a :: d : c$, and $\frac{b}{a}=\frac{d}{c}$.

Also when the four quantities are of the same kind, or abstract numbers, they are proportionals when taken alternately, as

$$a : c :: b : d, \text{ and } \frac{a}{c}=\frac{b}{d}. \quad \text{(Euc. VII. 13.)}$$

6. Prop. *When four quantities are proportionals,* (1) *the sum of the first and second is to the second, as the sum of the third and fourth is to the fourth.* (2) *The difference of the first and second, is to the second, as the difference of the third and fourth is to the fourth.* (3) *The sum of the first and second is to their difference, as the sum of the third and fourth is to their difference.* (Euc. V. 17, 18.)

(1) Since $a : b :: c : d$,

$$\therefore \frac{a}{b}=\frac{c}{d}, \text{ add 1 to these equals;}$$

$$\text{then } \frac{a}{b}+1=\frac{c}{d}+1 \text{ or } \frac{a+b}{b}=\frac{c+d}{d},$$

$$\text{and } a+b : b :: c+d : d,$$

or the sum of the first and second is to the second, &c.

(2) Since $a : b :: c : d$

$$\therefore \frac{a}{b}=\frac{c}{d}, \text{ subtract 1 from these equals;}$$

$$\text{then } \frac{a}{b}-1=\frac{c}{d}-1, \text{ or } \frac{a-b}{b}=\frac{c-d}{d}.$$

$$\text{and } a-b : b :: c-d : d,$$

or the difference of the first and second is to the second, &c.

(3) Since $\frac{a+b}{b}=\frac{c+d}{d}$, and $\frac{a-b}{b}=\frac{c-d}{d}$.

Divide the former of these equals by the latter;

$$\therefore \frac{a+b}{b}\div\frac{a-b}{d}=\frac{c+d}{d}, \div\frac{c-d}{d}, \text{ or } \frac{a+b}{a-b}=\frac{c+d}{c-d},$$

$$\text{and } a+b : a-b :: c+d : c-d$$

or the sum of the first and second is to their difference, as the sum of the third and fourth is to their difference.

7. Prop. *If four quantities of the same kind be proportionals, the first quantity being the greatest, the sum of the extremes is greater than the sum of the means.* (Euc. V. 25.)

Let a, b, c, d be proportionals, then $ad=bc$, and $\frac{a}{c}=\frac{b}{d}$.

Subtract 1 from these equals,

$$\therefore \frac{a}{c}-1-1=\frac{b}{d}-1 \text{ or } \frac{a-c}{c}=\frac{b-d}{d},$$

and $\frac{a-c}{b-d}=\frac{c}{d}$, but $\frac{a}{b}=\frac{c}{d}$,

$\therefore \frac{a-c}{b-d}=\frac{a}{b}$,

but a being the greatest of the four quantities, is greater then b,

$\therefore$ also $a-c>b-d$,

and adding $c+d$ to these unequals,

$\therefore a+d>b+c$,

or the sum of the extremes is greater then the sum of the means.

8. Prop. *If a, b, c, d be proportionals, there is no number which can be added to each term, so that the four terms so increased shall be proportionals.*

Since a, b, c, d are proportionals,

$\therefore ad=bc$.

Let any number x be added to each term.

And if possible, let $(a+x)(b+x)=(b+x)(c+x)$,

or $ad+(a+d)x+x^2=bc+(b+c)x+x^2$,

whence $(a+d)x=(b+c)x$,

and $a+d=b+c$,

that is, the sum of the first and fourth terms of a proportion is equal to that of the second and third.

But it has been shewn that $a+d$ is greater than $b+c$.

Hence it follows, that there is no number which can be added to the terms of a proportion, so that the terms so increased shall be proportional.

9. Prop. *If a and b are prime to each other, they are the least quantities in the proportion $\frac{a}{b}=\frac{c}{d}$.*

For if not, let $\frac{a}{b}=\frac{c}{d}$, in which c and d are less than a and b.

Then $c=\frac{ad}{b}$, and therefore ad is divisible by b, but a is prime to b, $\therefore d$ is divisible by b, which is impossible, since d is less than b; therefore $\frac{a}{b}$ cannot be equal to $\frac{c}{d}$; whence c and d are respectively less than a and b; that is, a and b are the least in every proportion in which they appear as antecedent and consequent.

10. Prop. *If any number of quantities are proportionals, as any antecedent is to its consequent, so is the sum of all the antecedents to all the consequents.* (Euc. VII. 12.)

Suppose these quantities a, b, c, d, e, f to be proportionals so that $a:b::c:d::e:f$.

Then since $a:b::c:d$, $\therefore ad=bc$,

and $a:b:e:f$, $\therefore af=be$,

and $ab=ba$.

By addition, $ab+ad+af=ba+bc+be$,

or $a(b+d+f)=b(a+c+e)$,

$$\therefore \frac{a}{b}=\frac{a+c+e}{b+d+f},$$

and $a : b :: a+c+e : b+d+f$.

The same is true for four, five, or any number of proportions.

11. Prop. *If the first of six magnitudes has the same ratio to the second as the third to the fourth, and the third to the fourth the same ratio as the fifth to the sixth; then the first shall have to the second the same ratio as the fifth has to the sixth.* (Euc. V. 22.)

Let a, b, c, d, e, f be six quantities,

such that $a : b :: c : d$, and $c : d :: e : f$,

since $a : b :: c : d$, $\therefore \frac{a}{b}=\frac{c}{d}$,

and $c : d :: e : f$, $\therefore \frac{c}{d}=\frac{e}{f}$,

$\therefore \frac{a}{b}=\frac{e}{f}$, and $a : b :: e : f$.

Or, If the first, &c.

12. Prop. *If the first of six quantities has the same ratio to the second as the third to the fourth, and the second to the fifth the same ratio as the fourth to the sixth; then shall the first have the same ratio to the fifth as the third has to the sixth.*

For since $a : b :: c : d$, $\therefore \frac{a}{b}=\frac{c}{d}$,

and $b : e :: d : f$, $\therefore \frac{b}{e}=\frac{d}{f}$,

$\therefore \frac{a}{b}\times\frac{b}{e}=\frac{c}{d}\times\frac{d}{f}$, or $\frac{a}{e}=\frac{c}{f}$,

and $a : e :: c : f$.

or the first is to the fifth as the third is to the sixth.

13. Prop. *If four quantities be proportionals, then any equimultiples or equisubmultiples of the first and second, shall also be proportionals to any other equimultiples or equisubmultiples of the third and fourth.*

If $a : b :: c : d$, then $\frac{a}{b}=\frac{c}{d}$,

and $\frac{a}{b}=\frac{ma}{mb}$, also $\frac{c}{d}=\frac{nc}{nd}$,

m and n being either integral or fractional;

then $\frac{ma}{mb}=\frac{na}{nd}$,

and $ma : mb :: nc : na$.

Or, The equimultiples or equisubmultiples of the first and second, &c.

14. Prop. *If four quantities be proportionals, and if equimultiples or equisubmultiples of the first and third be taken, and any other equimultiples or*

equisubmultiples of the second and fourth be also taken, then these four quantities shall be proportionals.

If $a : b :: c : d$, then $\frac{a}{b}=\frac{c}{d}$.

Multiply these equals by $\frac{m}{n}$, m and n being integral or fractional;

$$\therefore \frac{ma}{nb}=\frac{mc}{nd},$$

and $ma : nb :: mc : nd$.

Or, If equimultiples or equisubmultiples, &c.

It is also manifest from Art. 6

$$\begin{aligned}\text{that } ma+nb : b &:: mc+nd : d,\\ ma+nb : b &:: mc-nd : d,\\ ma+nb : ma-nb &:: mc+nd : mc-nd.\end{aligned}$$

15. Prop. *If four quantities be proportionals, the same powers and the same roots of these quantities are also proportionals.*

Since a, b, c, d are proportionals,

$$\text{then } \frac{a}{b}=\frac{c}{d},$$

and $\frac{a^m}{b^m}=\frac{c^m}{d^m}$, by raising these equals to the mth power,

or $a^m : b^m :: c^m : d^m$.

Also $\frac{a^{\frac{1}{m}}}{b^{\frac{1}{m}}}=\frac{c^{\frac{1}{m}}}{d^{\frac{1}{m}}}$, by taking the mth root of the equals,

or $a^{\frac{1}{m}} : b^{\frac{1}{m}} :: c^{\frac{1}{m}} : d^{\frac{1}{m}}$.

That is, equal powers and equal roots of proportionals, are also proportionals.

$$\begin{aligned}\text{Also } a^m+b^m : b^m &:: c^m+d^m : d^m,\\ a^m-b^m : b^m &:: c^m-d^m : d^m,\\ a^m+b^m : a^m-b^m &:: c^m+d^m : c^m-d^m.\end{aligned}$$

The following may also be shewn to be true:—

$$\begin{aligned}a^{\frac{1}{m}}+b^{\frac{1}{m}} : b^{\frac{1}{m}} &:: c^{\frac{1}{m}}+d^{\frac{1}{m}} : d^{\frac{1}{m}},\\ a^{\frac{1}{m}}-b^{\frac{1}{m}} : b^{\frac{1}{m}} &:: c^{\frac{1}{m}}-d^{\frac{1}{m}} : d^{\frac{1}{m}},\\ a^{\frac{1}{m}}+b^{\frac{1}{m}} : a^{\frac{1}{m}}-b^{\frac{1}{m}} &:: c^{\frac{1}{m}}+d^{\frac{1}{m}} : c^{\frac{1}{m}}-d^{\frac{1}{m}}.\end{aligned}$$

16. Prop. *If there be two or any number of sets of proportionals, the products of their corresponding terms are proportionals.*

For let $a : b :: c : d$, $\therefore \frac{a}{b}=\frac{c}{d}$,

$$e : f :: g : h, \therefore \frac{e}{f}=\frac{g}{h},$$

$$k : l :: m : n, \therefore \frac{k}{l} = \frac{m}{n}.$$

Then $\frac{a}{b} \times \frac{e}{f} = \frac{c}{d} \times \frac{g}{h}$, or $\frac{ae}{bf} = \frac{cg}{dh}$,

and $ae : bf :: cg : dh$.

Also $\frac{a}{b} \times \frac{e}{f} \times \frac{k}{l} = \frac{c}{d} \times \frac{g}{h} \times \frac{m}{n}$, or $\frac{aek}{bfl} = \frac{cgm}{dhn}$,

and $aek : bfl :: cgm : dhn$.

And so, for any number of proportionals.

17. Prop. *If the corresponding terms of two sets of proportionals be divided, the quotients will form a proportion.*

For if $a : b :: c : d$, $\frac{a}{b} = \frac{c}{d}$,

and $e : f :: g : h$, $\frac{e}{f} = \frac{g}{h}$.

Then $\frac{a}{b} \div \frac{e}{f} = \frac{c}{d} \div \frac{g}{h}$, or $\dfrac{\frac{a}{e}}{\frac{b}{f}} = \dfrac{\frac{c}{g}}{\frac{d}{h}}$,

and $\frac{a}{e} : \frac{b}{f} :: \frac{c}{g} : \frac{d}{h}$.

18. Prop. *If three quantities a, b, c be in continued proportion, the first is to the third as the square of the first is to the square of the second.* (Euc. VIII. 11, 26.)

Since a, b, c are proportionals

$\frac{a}{b} = \frac{b}{c}$, and $\frac{b}{c} = \frac{b}{c}$

$\therefore \frac{a}{b} \times \frac{b}{c} = \frac{b}{c} \times \frac{b}{c}$, or $\frac{a}{c} = \frac{b^2}{b^2}$,

that is $a : c :: a^2 : b^2$.

19. Prop. *If four quantities a, b, c, d be in continued proportion, the first is to the fourth as the cube of the first is to the cube of the second.*[*] (Euc. VIII. 13, 27.)

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[*] From the following illustration, it will be seen that the *duplicate* and *triplicate* ratios in Geometry correspond to the ratios of the squares and cubes in Algebra :—

1. Similar triangles are to one another in the duplicate ratio of their homologous sides. (Euc. VI. 19.)

Let a, b, c ; a', b', c' represent the sides of two similar triangles ;

Then, because the triangles are similar, their homologous sides are proportional, that is $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$;

And if p, p' be the perpendiculars from the vertices on the sides a, a', then $\frac{p}{p'} = \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$.

Also if A, A' denote the areas of the triangles, $A = \frac{1}{2}ap$, and $A' = \frac{1}{2}a'p'$. (Euc. I. 41.)

Since a, b, c, d are proportionals,

$$\frac{a}{b}=\frac{b}{c}=\frac{c}{d};$$

then $\frac{a}{b}=\frac{b}{c}$, $\frac{b}{c}=\frac{c}{d}$, and $\frac{a}{b}=\frac{a}{b}$,

and $\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}=\frac{a}{b}\times\frac{a}{b}\times\frac{a}{b}$,

or $\frac{a}{d}=\frac{a^3}{b^3}$,

that is $a : d :: a^3 : b^3$.

20. Prop. *If a and b be incommensurable quantities, and also c and d; and if when $\frac{a}{b}$ lies between $\frac{m}{n}$ and $\frac{m+1}{n}$, $\frac{c}{d}$ also lies between $\frac{m}{n}$ and $\frac{m+1}{n}$; however much the magnitudes m and n be increased, then $\frac{a}{b}=\frac{c}{d}$.*

Suppose a to be a measure of b, and let $b=nx$; also let a be greater than mx but less than $(m+1)n$; then $\frac{a}{b}$ is greater than $\frac{m}{n}$ but less than $\frac{m+1}{n}$, or the difference between $\frac{m}{n}$ and $\frac{a}{b}$ is less than $\frac{1}{n}$; and as a is diminished, since $na=b$, n is increased, and $\frac{1}{n}$ diminished. Therefore by diminishing a, the difference between $\frac{m}{n}$ and $\frac{a}{b}$ may be made less than any number that can be assigned.

$$\therefore \frac{A}{A'}=\frac{a}{a'}\cdot\frac{p}{p'}=\frac{a}{a'}\cdot\frac{a}{a'}=\frac{a^2}{a'^2}=\frac{b^2}{b'^2}=\frac{c^2}{c'^2},$$

or $A : A' :: a^2 : a'^2 :: b^2 : b'^2 :: c^2 : c'^2$.

2. Similar solid parallelopipeds are to one another in the triplicate ratio of their homologous sides. (Euc. XI. 33.)

Let a, b, c; a', b', c' be the corresponding edges (or homologous sides) of two similar rectangular parallelopipeds, and V, V' their volumes respectively.

Then since the parallelopipeds are similar

$$\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}, \text{ and } V=abc,\ V'=a'b'c',$$

$$\therefore \frac{V}{V'}=\frac{abc}{a'b'c'}=\frac{a}{a'}\cdot\frac{b}{b'}\cdot\frac{c}{c'}=\frac{a}{a'}\cdot\frac{a}{a'}\cdot\frac{a}{a'}=\frac{a^3}{a'^3}=\frac{b^3}{b'^3}=\frac{c^3}{c'^3}.$$

or $V : V' :: a^3 : a'^3 :: b^3 : b'^3 :: c^3 : c'^3$.

The weights of similar volumes of substances of the same density are also proportional to the cubes of the homologous sides of the volumes.

As an example may be taken the two similar rectangular parallelopipeds.

Let W, W' denote respectively the weights of the volumes V, V', and let s denote the weight of one cubic unit of the substance,

Then $W=sV$ and $W'=sV'$,

Therefore $\frac{W}{W'}=\frac{sV}{sV'}=\frac{V}{V'}=\frac{a^3}{a'^3}=\frac{b^3}{b'^3}=\frac{c^3}{c'^3}$.

If c and d as well as a and b be incommensurable, and if when $\frac{a}{b}$ lies between $\frac{m}{n}$ and $\frac{m+1}{n}$, $\frac{c}{d}$ lies also between $\frac{m}{n}$ and $\frac{m+1}{n}$, however the numbers m and n may be increased; then $\frac{a}{b}=\frac{c}{d}$.

For, if they are not equal, they must have some assignable difference, and because each of them lies between $\frac{m}{n}$ and $\frac{m+1}{n}$, this difference is less than $\frac{1}{n}$; but since n may, by the supposition, be increased without limit, $\frac{1}{n}$ may be diminished without limit; that is $\frac{1}{n}$ may become less than any number that can be assigned, and therefore the difference between $\frac{a}{b}$ and $\frac{c}{d}$ is less than can be assigned; that is $\frac{a}{b}=\frac{c}{d}$.

And all the preceding propositions respecting proportionals are true of four quantities which are not commensurable.

The criterion of the proportionality of four magnitudes expressed in Euc. V., def. 5, is in exact accordance with the criterion of the proportion of four numbers, as expressed in Euc. VII., def. 20.

In other words—

21. Prop. *The geometrical criterion of the proportionality of four magnitudes may be shewn to follow from the criterion of the proportionality of four numbers; and conversely.*

If a, b, c, d be four magnitudes expressed by numbers, of which the first is the same multiple part or parts of the second, as the third is of the fourth, then $\frac{a}{b}=\frac{c}{d}$, and if m, n be any integral numbers, then $\frac{ma}{nb}=\frac{mc}{nd}$, in which ma, mc are any equimultiples whatever of a and c the first and third magnitudes, and nb, nd, any equimultiples whatever of b and d the second and fourth.

If ma be greater than nb, then mc is greater than nd; or if ma be equal to nb, then mc is equal to nd; or if ma be less than nb, then mc is less than nd.

And a, b, c, d are assumed to be proportionals by Euc. VII., def. 20.

Hence it follows that a, b, c, d are proportionals by Euc. V., def. 5, namely:—

The first of four magnitudes is said to have the same ratio to the second, which the third has to the fourth, when any equimultiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth; if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the fourth; or, if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth; or, if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth.

Conversely. If four magnitudes be proportional according to Euc. V., def. 5, their numerical values shall also be proportional according to Euc. VII., def. 20.

Let a, b, c, d be four magnitudes which are proportional by Euc. V., def. 5, where any equimultiples whatever ma and mc of a, c the first and third magnitudes being taken, and any equimultiples whatever nb and nd of b, d the second and fourth, such that if ma be greater than, equal to, or less than nb, then mc is greater than, equal to, or less than nd; then also a, b, c, d shall be proportional according to Euc. VII., def. 20;

that is $a : b :: c : d$, or $\frac{a}{b} = \frac{c}{d}$.

For if not, if possible, let a, b, c, d' be proportionals,

then $a : b :: c : d'$ or $\frac{a}{b} = \frac{c}{d'}$, and $\frac{ma}{nb} = \frac{mc}{nd'}$,

and if ma be greater than, equal to, or less than nb, then mc is also greater than, equal to, or less than cd'; but these conditions are assumed of ma, nb, mc, nd; $\therefore nd' = nd$ and $d' = d$; therefore $\frac{a}{b} = \frac{c}{d}$ and a, b, c, d are proportionals according to Euc. VII., def. 20.

22. In the proportion $x : b :: y : d$, or $\frac{x}{b} = \frac{y}{d}$, if b, d remain constant while x and y change their values so that always $\frac{x}{y} = \frac{b}{d}$, or $x = \frac{b}{d}.y$, then x is said to vary *directly* as y, and thus a proportion may be expressed by two terms. The mark $\propto$ is used for the words "varies as," and the proportion is written $x \propto y$, and is read "x varies as y."

In the proportion $x : b : \frac{1}{y} : \frac{1}{d}$, or $\frac{x}{b} = \frac{d}{y}$, if b and d remain constant while x and y change their values, then $x = \frac{bd}{y}$ and x is said to vary as y inversely.

In the proportion $z : b :: xy : d$, or $z = \frac{b}{d}.xy$, in which b and d are constant while x, y, z change their values, then in $z = \frac{b}{d}.xy$, z is said to vary jointly as x and y, or as the product of x and y.

It is obvious from the equations $x = \frac{b}{d}.y$, $x = bd.\frac{1}{y}$, and $z = \frac{b}{d}.xy$, that variable quantities can always be connected by some constant multiplier, either integral or fractional, and that every variation is convertible into an equation, and conversely.*

23. Prop. *If x vary as y, and y vary as z, then x varies as z.*

Since $x \propto y$ and $y \propto z$, let $x = py$ and $y = qz$, where p and q are some constants.

Then $x = pqz$, and $\therefore x \propto z$.

24. Prop. *If v vary as x and y vary as z, then vy varies as xz.*

Since $v \propto x$ and $y \propto z$, let $v = px$ and $y = qz$, where p and q are constant.

Then $vy = pqxz$, $\therefore vy \propto xz$.

25. Prop. *If x vary as y and z vary as y, then both the sum and the difference of x and z, varies as y, and the product of x and z varies as y^2, the square of y.*

For since $x \propto y$, $\therefore x = py$, and $z = qy$, where p and q are some constant quantities.

Then $x \pm z = py \pm qy = (p \pm q)y$, and $\therefore x \pm z \propto y$.

Also $xz = pqy^2$, and $\therefore xz \propto y^2$.

26. Prop. *If z vary as x when y is constant, and z vary as y when x is constant; then z varies as xy when all three are variable.*

Let z', x' be correspondent values of z and x when y is constant, then $z : z' :: x : x'$, and $z' = \frac{zx'}{x}$.

But this value z' of z, will like z vary as y.

Let z'', y' be correspondent values of z' and y when x is constant, then $z' : z'' :: y : y'$, and $z' = \frac{yz''}{y'}$.

Hence $\frac{zx'}{x} = \frac{yz''}{y'}$, and $z = \frac{z''}{x'y'} . xy$, but $\frac{z''}{x'y'}$ is constant $\therefore z \propto xy$, or z varies as the product of x and y when all three are variable.

It may also be shewn that $x \propto \frac{z}{y}$ and $y \propto \frac{z}{x}$.

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[*] These three definitions of variation may be illustrated by considering that the area of a triangle is equal to half the product of the base and the perpendicular from the vertex on the base.

If A, b, p denote the area, base, and perpendicular, then $A = \frac{1}{2}bp$.

(1) If b be constant, then A varies directly as p, and if p be constant, A varies directly as b.

(2) If A be constant, then p varies inversely as b, and b varies inversely as p.

(3) If all three A, b, p vary, then A varies as bp, or the area varies as b and p jointly.

EXERCISES.

I.

1. EXPLAIN why in Algebra a ratio is treated as a fraction, when it is not so treated in Geometry?

2. Define the ratio of a to b; what it is; what it equals; and what is measured by it.

3. Determine the ratio of x to y from the equation $2x+5y=4x+3y$.

4. Two quantities are in the ratio of $m:n$; what quantity must be added to each that they may be in the ratio of $p:q$?

5. What is the number, which, if added to each term of the ratio 5 : 3, makes it three fourths of what it would have become, if the same number had been subtracted from each term?

6. If $\frac{3a-4}{12(a-1)}=\frac{b+1}{4b+3}$; find the ratio of a to b.

7. If $\frac{a-b}{a+b}=\frac{b-c}{2b-3c}$, and $6(a+b)=18b-8c$; find the ratios of a to b, and of b to c.

8. Prove that $\left\{\frac{a}{b}\right\}^{a+b}$ is increased by adding the same quantity to a and b; a being less than b.

9. If the ratio $a:b$ be unchanged when c is added to the antecedent and $2c$ to the consequent; then tho value of the ratio is $\frac{1}{2}$.

10. If $\frac{mx+ny}{mx-ny}=a$, find the ratio of x to y.

11. If x be very small compared with a, shew that the ratio of $(a+x)^3 : a^3$, is very nearly that of $a+3x : a$, and the ratio of $(a+x)^{\frac{1}{3}} : a^{\frac{1}{3}}$ is very nearly that of $a+\frac{1}{3}x : a$.

Ex. Find the approximate ratio of $(403)^3 : (402)^3$; and of $(403)^{\frac{1}{3}} : (402)^{\frac{1}{3}}$.

12. If $x : 5 :: y : 8$, find the numerical ratio of $x+5$ to $y+8$.

13. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{ax+b}{cx+d}$ has always the same value, whatever may be the value of x.

14. If $\frac{a}{b}$ and $\frac{c}{d}$ be two unequal ratios, then the ratio $\frac{a+c}{b+d}$ is greater than one of them and less then the other.

15. A number of two figures is altered in the ratio of n to m, if its digits be interchanged; shew that the digits are to each other in the ratio of $10m-n$ to $10n-m$.

16. If $\frac{x+3z}{2y}=\frac{z}{x}=\frac{3x}{2x-y}$; find the ratios $\frac{x}{y}$ and $\frac{y}{z}$.

17. Explain what are the necessary or sufficient tests of the ultimate

ratio of two quantities which by continuous diminution become indefinitely small ultimately.

18. Shew that the ratio $\frac{(1+3x)^2.(x^4+1)^{\frac{1}{4}}}{1+x+x^2}$ is equal to $1+5x+3x^2-8x^3$ nearly, when x is small; and equal to $9x-3$ nearly, when x is large.

II.

1. Find a third proportional to 999 and 33·3

2. Find a mean proportional between ·016 and 2·704.

3. Find a fourth proportional to ·0004, 1·4 and ·02.

4. If $x^2+y^2 : xy :: 34 : 15$, then $x : y :: 5 : 3$: and conversely.

5. Find the value of the common ratio when a, b, c, d are in continued proportion, and $2(a+d) = 3(b+c)$.

6. If $\frac{a}{b}=\frac{x}{y}$, and $\frac{b}{c}=\frac{y}{z}$; then $\frac{a}{c}=\frac{x}{z}$; under what restriction is this proportion true if it involve concrete quantities?

7. Find the proportion deducible from the equation, $a^2+b^2=2ac$.

8. If a, b, c, d are proportionals; if a be prime to b, then c and d are equimultiples of a and b.

9. Can any number be added to each of four numbers in arithmetical progression, so that the sums are in geometrical proportion?

10. If a, b, c, d be proportionals, there is no number which can be added to, or subtracted from each, so that the four terms so increased or diminished shall be proportionals.

11. If a, b, c, d, are proportionals, deduce $(a+b)(c+d)$ from each of the following expressions:—

(1) $(a-d)^2=(b-c)^2+(c-a)^2+(d-b)^2$.
(2) $(a+b)^2+(d+c)^2+(a-b-c+d)^2=2(a+d)^2$.
(3) $(a-b+c-d)^2=(a-b)^2+(c-d)^2+2(b-c)^2$.

12. When are two quantities said to be commensurable or incommensurable? Find the approximate ratio of the diagonal to the side of a square, so that the error may be within one thousandth part of unity.

13. Shew that the geometrical definition of proportion (Euc. V., def. 5) is a consequence of the arithmetical definition (Euc. VII., def. 20), and the converse.

III.

Given $a : b :: c : d$, prove the following proportions:—

1. $a : b :: a\pm c : b\pm d$; and conversely.
2. $ac : bd :: c^2 : d^2$.
3. $a^2 : c^2 :: a^2+b^2 : c^2+d^2$.
4. $ab : cd :: (a\pm b)^2 : (c\pm d)^2 :: a^2-b^2 : c^2-d^2$.
5. $a^2+b^2 : c^2+d^2 :: (a-b)^2 : (c-d)^2$.

6. $(a^2+b^2)^2 : (c^2+d^2)^2 :: a^4+b^4 : c^4+d^4 :: ab^3 : cd^3$.
7. $(a^3+c^3)^2 : (b^3+d^3)^2 :: (a^2+c^2)^3 : (b^2+d^2)^3$.
8. $(a^5-b^5)^3 : (c^5-d^5)^3 :: (a^3-b^3)^5 : (c^3-d^3)^5$.
9. $a+(a^2+b^2)^{\frac{1}{2}} : a-(a^2+b^2)^{\frac{1}{2}} :: c+(c^2+d^2)^{\frac{1}{2}} : c-(c^2-d^2)^{\frac{1}{2}}$.

IV.

If a, b, c, d be proportionals; shew the truth of the following properties :—

1. $a(a+b+c+d)=(a+b)(a+c)$.
2. $(a+b+c+d)(a-b-c+d)=(a-b+c-d)(a+b-c-d)$.
3. $(a^2+c^2)(b^2+d^2)=(ab+dc)^2$.
4. $(a^4+b^4+c^4+d^4)(a+b+c+d)^4$
$$=\{(a+b)^4+(c+d)^4\}.\{(a+c)^4+(b+d)^4\}.$$
5. $\dfrac{a^2}{b^2}+\dfrac{c^2}{d^2}=\dfrac{2ac}{bd}$.
6. $(a+d)-(b+c)=\dfrac{(a-b)(a-c)}{a}$, a being the greatest.
7. $(a+b)-(c+d)=\dfrac{(a+b)(b-d)}{b}$.
8. $\left\{\dfrac{1}{a}+\dfrac{1}{d}\right\}-\left\{\dfrac{1}{b}+\dfrac{1}{c}\right\}=\dfrac{(a-b)(a-c)}{abc}$.
9. $\dfrac{(a^2+3ab+b^2)}{(c^2+3cd+d^2)}=\dfrac{2ab+3b^2}{2cd+3d^2}$.
10. $\dfrac{(a+c)(a^2+c^2)}{(a-c)(a^2-c^2)}=\dfrac{(b+d)(b^2+d^2)}{(b-d)(b^2-d^2)}$.
11. $\dfrac{a^4+b^4+c^4+d^4}{a^{-4}+b^{-4}+c^{-4}+d^{-4}}=(abcd)^2$.
12. $\dfrac{a^2-b^2}{b^2-d^2}\cdot\dfrac{a^2-c^2}{c^2-d^2}\cdot\dfrac{d^2}{bc}=1$.
13. $\dfrac{bcd+cda+dab+abc}{a+b+c+d}=(abcd)^{\frac{1}{2}}$.

V.

If $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}$, prove the following properties :—

1. $\dfrac{x+y}{a+b}=\dfrac{x+z}{a+c}=\dfrac{y+z}{b+c}$.
2. $\dfrac{mx+ny+pz}{ma+nb+pc}=\dfrac{x}{a}$, m, n, p, being any numbers whatever.
3. $\dfrac{x^2+y^2+z^2}{a^2+b^2+c^2}=\dfrac{(x+y+z)^2}{(a+b+c)^2}=\dfrac{xy+xz+yz}{ab+ac+bc}=\dfrac{x^2}{a^2}$.
4. $\left\{\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}\right\}\cdot\left\{\dfrac{a}{x}+\dfrac{b}{y}+\dfrac{c}{z}\right\}=\dfrac{(m+n+1)(mn+m+n)}{mn}$;

if $\dfrac{y}{b}=\dfrac{mx}{a}$, $\dfrac{z}{c}=\dfrac{nx}{a}$.

5. $(ax+by+cz)^2=(a^2+b^2+c^2)(x^2+y^2+z^2)$.
6. $(ax^2+by^2+cz^2)(a^2x+b^2y+c^2z)=(a^3+b^3+c^3)(x^3+y^3+z^3)$.
7. $27(x+a)(y+b)(z+c)=xyz\left\{\frac{x+a}{a}+\frac{y+b}{b}+\frac{z+c}{c}\right\}$.
8. $\frac{mxy+nyz}{xz}=\frac{mab+nbc}{ac}$.
9. $\left\{\frac{xyz}{abc}\right\}^2=\left\{\frac{xy+yz+xz}{ab+bc+ca}\right\}^3$.
10. $\frac{xyz}{abc}=\left\{\frac{x+y+z}{a+b+c}\right\}^3=\frac{x^3+y^3+z^3}{a^3+b^3+c^3}$.
11. $(ax)^{\frac{1}{2}}+(by)^{\frac{1}{2}}+(cz)^{\frac{1}{2}}=\{(a+b+c)(x+y+z)\}^{\frac{1}{2}}$.
12. $\left\{\frac{x^{\frac{1}{2}}+y^{\frac{1}{2}}+z^{\frac{1}{2}}}{a^{\frac{1}{2}}+b^{\frac{1}{2}}+c^{\frac{1}{2}}}\right\}^2=\left\{\frac{xy+xz+yz}{ab+bc+cd}\right\}^{\frac{1}{2}}=\frac{x}{a}$.
13. $\frac{x^3+y^3+z^3-3xyz}{a^3+b^3+c^3-3abc}=\frac{x^3}{a^3}$.
14. $abc\left\{\frac{x+a}{a}+\frac{y+b}{b}+\frac{z+c}{c}\right\}^3=6(x+a)(y+b)(z+c)$.

VI.

If $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}=\frac{e}{f}$; prove the following equivalents:—

1. $(b+c)^2=(a+b)(c+d)$.
2. $a^2+b^2+c^2=(a+b+c)(a-b+c)$.
3. $c(a+2c+e)=(b+d)^2$.
4. $(a^2+b^2)(b^2+c^2)=(ab+bc)^2$.
5. $2(ac-bd)(ab-cd)=(ad+bc)\{(a+d)^2-(b+c)^2\}$.
6. $(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$.
7. $a^3+b^3+c^3=a^2b^2c^2(a^{-3}+b^{-3}+c^{-3})$.
8. $(ab+bc+ca)^3d^3=(bc+cd+db)^3a^3$.
9. $\frac{1}{a^2-b^2}+\frac{1}{c^2-b^2}=-\frac{1}{b^2}$.
10. $\left\{\frac{a+b+c+d}{c+d+e+f}\right\}^2=\frac{a^2+b^2+c^2+d^2}{b^2+c^2+d^2+e^2}$.

VII.

1. If a, b, c are proportionals, then $\frac{a+b}{b+c}$, b, bc are proportionals.
2. If $\frac{a+c}{b+d}=\frac{b-d}{a-c}$, then shall $\frac{a+b}{c+d}=\frac{c-d}{a-b}$.
3. If $\frac{(a-c)^2}{(b-d)^2}=\frac{a-b}{c-d}$; then shall $(a-b)(c-d)=(b-c)^2$.
4. If a, b, c, d be a geometrical proportion, then shall $\frac{1}{a^2-b^2}$, $\frac{1}{b^2-c^2}$, $\frac{1}{c^2-d^2}$ and $\frac{1}{a^3+b^3}$, $\frac{1}{b^3+c^3}$, $\frac{1}{c^3+d^3}$ be in geometrical proportion.

5. If $c(a'b-ab')^2+b(a'b-ab')(ac'-a'c)+a(ac'-a'c)^2=0$; shew that $ac'-a'c$ is a mean proportional between $a'b-ab'$ and $b'c-bc'$.

6. If $\frac{b+c+d}{c+d+a}=\frac{d+a+c}{a+b+c}$, then $\frac{a^3-d^3}{a-d}=\frac{b^3-c^3}{b-c}$.

7. If $\frac{ad-bc}{a-b-c+d}=\frac{ac-bd}{a-b+c-d}$; then shall $a+b=c+d$, and each ratio is $\frac{1}{4}(a+b+c+d)$.

8. If $a_1, a_2, a_3, \ldots\ldots a_n$ be in continued proportion, then $a_1 : a_3 :: a_1^2 : a_2^2$ and $a_1 : a_4 :: a_1^3 : a^3_2$; and generally $a_1 : a_n :: a_1^{n-1} : a_2^{n-1}$.

9. If x and y be real quantities; what inference can be deduced from the proportion $\frac{1-xy}{x-y}=\frac{1+xy}{x+y}$?

10. If $x : y :: a^2 : b^2$ and $a : b :: (a+x)^{\frac{1}{2}} : (a-y)^{\frac{1}{2}}$; shew that $2x : a :: x-y : y$.

11. If $\frac{a^{\frac{1}{n}}-mb^{\frac{1}{n}}}{a^{\frac{1}{n}}+mb^{\frac{1}{n}}}=\frac{a^{\frac{1}{2n}}-m(a-b)^{\frac{1}{2n}}}{a^{\frac{1}{2n}}+m(a-b)^{\frac{1}{2n}}}$, then $\frac{a}{b}=\frac{1\pm\sqrt{5}}{2}$.

12. If $\frac{a+(a^2+b^2)^{\frac{1}{2}}}{c+(c^2+d^2)^{\frac{1}{2}}}=\frac{a-(a^2-b^2)^{\frac{1}{2}}}{c-(c^2-d^2)^{\frac{1}{2}}}$; then $\frac{a}{b}=\frac{c}{d}$, and conversely.

13. If $(1-x)^{\frac{1}{2}}+(1-x^2)^{\frac{1}{2}}=y^{-\frac{1}{2}}+y^{-\frac{1}{2}}$, and $(1+x)^{\frac{1}{2}}-(1-x^2)^{\frac{1}{2}}=y^{\frac{1}{2}}-y^{-\frac{1}{2}}$; then shall $1-x$, $1-2y$, and $1+x$ be in geometrical progression.

VIII.

1. If $ax+by+cz=0$ and $a'x+b'x+c'y=0$, determine the ratios of x to y, x to z, and y to z.

2. If $ax^2+by^2+cz^2=0$ and $a'x+b'y+c'z=0$; find the ratios of x to y, y to z, and x to z.

3. If $a+c$ have to $b+c$ the duplicate ratio of a to b; then shall $a=b$, or $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$.

4. If $\frac{x-y}{a}=\frac{y-z}{b}=\frac{z-x}{c}$, then shall $a+b+c=0$.

5. If $\frac{x+y}{z}=\frac{y}{x}=\frac{x}{y-z}$; then shall $\frac{x}{2}=\frac{y}{4}=\frac{z}{3}$.

6. If $\frac{ay-bx}{c}=\frac{cx-az}{b}=\frac{bz-cy}{a}$; then shall $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$.

7. If $\frac{ax+by}{cz}=\frac{cz-az}{by}=\frac{by+cz}{ax}$, then $ax+by+cz=0$ and $ax=by=cz$.

8. If $a(y-z)+b(z-x)+c(x-y)=0$, then the successive differences of a, b, c are proportional to the successive differences of x, y, z.

9. If $\frac{cy+bz}{b-c}=\frac{az+cx}{c-a}=\frac{bx+ay}{a-b}$; then $(a+b+c)(x+y+z)=ax+by+cz$.

10. If $\frac{a+b}{x+y}=\frac{b+c}{y+z}=\frac{c+a}{x+z}$; shew that each is equal to $\frac{a+b+c}{x+y+z}$, and that $\frac{a}{x}=\frac{b}{y}=\frac{c}{z}$.

11. If $\frac{x}{a}=\frac{b}{c}=\frac{d}{y}$; then $\frac{(x+a)(x+b)(x+d)}{(y+a)(y+c)(y+d)}=\frac{(a+c)b^2x}{(b+d)c^2y}$.

12. If $\frac{x^2-yz}{x-xyz}=\frac{y^2-xz}{y-xyz}=\frac{z^2-xy}{z-xyz}$; then shall $x+y+z=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$.

13. Given $\frac{x(y-z)}{a-x}+\frac{y(z-x)}{b-y}+\frac{z(x-y)}{c-z}=0$ and $\frac{a(b-c)}{a-x}+\frac{b(c-a)}{b-y}+\frac{c(a-b)}{c-z}=0$; shew that $\frac{a}{x}=\frac{b}{y}=\frac{c}{z}$, or $a-x=b-y=c-z$.

IX.

1. Explain what is meant by saying that one quantity varies *directly* or *inversely* as another; and shew that if $x \propto y$, then $x=cy$, in which c is some constant quantity.

2. If $ax+by=cx+dy$; then $x \propto y$.

3. If $ax^3+by \propto cx^3-dy$; shew that $x \propto y^{\frac{1}{3}}$, a, b, c, d being constant.

4. If $(x-a) \propto (y-b)$, and when $x=m$, $y=n$; find the constant which connects $x-a$ and $y-b$.

5. If $x+y \propto x-y$, then $x^2+y^2 \propto xy$, and conversely.

6. If $x^2 \propto \frac{1}{y^3}$, and if when $x=2$, $y=3$; find the equation between x and y.

7. If $x \propto y$; then $x^3+y^3 \propto x^3-y^3$.

8. If $y^2 \propto (a^2 \pm x^2)$, and if $x=0$ when $y=b$; shew that $\frac{y^2}{b^2} \mp \frac{x^2}{a^2}=1$ is the equation which connects the variables x and y.

9. If y varies as $\frac{x^2}{a+x}$, and if when $x=\frac{b^2}{a}$, $y=\frac{a^3}{a^2+b^2}$; find the equation which connects x and y.

10. A quantity is made up of the sum of three parts, the first part is constant, the second varies as x, and the third varies as x^2; having given that when $x=a$, $y=0$; $x=2a$, $y=a$; $x=3a$, $y=4a$; shew that when $x=na$, then $y=(n-1)^2.a$.

11. If $y=a+b+c$, and $a \propto x^3$, $b \propto y^2$, $c \propto x$; also when $x=1, 2, 3$; $y=3, 13, 39$ respectively. Determine the value of y when $x=4$.

12. If $y=a+b$, of which $a \propto \frac{1}{x}$ and $b \propto x^2$; and when $x=2, 3$; $y=10, 20$ respectively; find the relation between x and y.

X.

1. Define and explain what is meant when one quantity is said to vary jointly as two other quantities. If $z \propto xy$ and 10, 9, 6 be three corresponding values of z, x, y; find the constant multiplier which connects x, y, and z.

2. If $y \propto x+z$ and $y^2 \propto xz$; then shall $x \propto y$ and $y \propto z$.

3. If $y \propto \dfrac{1}{x}$ and $x \propto z+y$; if when $y=2, z=2$; what will be the value of z when $y=4$?

4. If $z=x+my$, and when $x=1, y=2, z=3$; also when $x=2, y=3, z=4$; find the equation which connects the three variables.

5. If $x \propto y$ and $xy \propto z^2$, then $x^3+y^3+z^3 \propto xyz$.

6. If $x^2 \propto yz$ and $y^2 \propto xz$, then $x^2+y^2 \propto (x+y)z$.

7. A quantity z is made up of two parts, one of which varies directly as x, and the other inversely as y; and if the value of z be 30 when $x=2$, $y=1$; and 15 when $x=3$, $y=\frac{1}{3}$; find the value of z when $x=3$, $y=4$.

8. If x vary as y and z jointly, and when $y=m-m^{-1}$ and $z=m^{\frac{1}{2}}-m^{-\frac{1}{2}}$, it is known that $x=m^{\frac{1}{2}}+m^{-\frac{1}{2}}$; shew that $(m-1)^2a=bcm$, a, b, c being contemporaneous values of x, y, z respectively.

9. If $s \propto t$, when v is constant; and $s \propto v$ when t is constant; prove that $s \propto tv$ when neither v nor t is constant. If $s^2 \propto vt^3$ and $v^4 \propto st^2$ when f is constant; also $s^3 \propto v^2f$ and $v^5 \propto sf^4$ when t is constant; shew that $v^2 \propto fs$ when all vary.

10. If the sum of two quantities vary jointly as the difference of their cubes, and a third quantity; and the difference vary jointly as the sum of the cubes, and the third quantity; shew that the one of the two quantities varies as the other.

XI.

1. Divide £1,000 among *A*, *B*, and *C*, so that *A's* share may be to *B's* as 5 to 7 and *B's* to *C's* as 11 to 9.

2. If a sheep be worth b oxen, and c oxen equivalent to d horses; how many horses are worth c sheep?

3. *A's* age is to *B's* as 4 to 3; 18 years ago, *A's* age was to *B's* as 3 to 2. What are their present ages?

4. If by selling merchandise for m pounds sterling, p per cent. be lost; what will be the gain or loss per cent. by selling them for n pounds?

5. If two merchants, *A* and *B*, enter into partnership in trade; *A* adventures a pounds for m years, and *B* b pounds for n years; they gain p pounds; determine the relation between their capitals, times, and shares of the gains.

6. The rent of a house in one year was p per cent. less than it was in the following year; what was the former year's rent if that of the following year was m pounds?

7. A debt of £21,000 is to be divided among four creditors, in proportion to their claims; the claim of the first to the second is as 3 to 2; that of the second to the third as 4 to 5; and that of the third to the fourth as 6 to 7. Find what sum each creditor has to receive.

8. The difference between the number of chapters in the Old and in the New Testament is 669; and if one were added to the number in each, the ratio would be that of 310 to 87; how many chapters are there in each?

9. The attendance on Professors' lectures varies directly as the Professors' power of exposition, and inversely as the square of the number of lectures delivered. If 52 students attend the lectures of Professor A, who delivers a course of 12 lectures, and 48 students attend the lectures of Professor B who delivers a course of 18 lectures; compare the powers of exposition of the Professors.

10. On a division of the House of Commons, if the number of members for a certain motion had been increased by 40 from the other side, the question would have been carried by 3 to 2; but if those against the motion had received 60 of the other party, the motion would have been lost by 2 to 1. Was the motion carried or lost, and how many members voted on the question?

XII.

1. Explain the distinction between interest and discount; and shew that the discount of a sum of money due at any future period, is equal to the interest of the present worth for that period.

2. Find in what time a sum of money will double and treble itself at any rate of simple and compound interest.

3. If the interest of A pounds for one year be equal to the discount of B pounds for the same time; find the rate of interest.

4. From the expression $D = \frac{Mnr}{(1+nr)}$, deduce the common rule for finding (1) the present worth (2) the discount of any sum of money due at any future time at any rate of simple interest. If compound interest be allowed, what modification of the rules is required?

5. Shew that the difference between the interest and discount of any sum for a given time is equal to the interest of the discount of that sum for the same time.

6. The ratio of the interest to the discount on a certain sum of money at a certain rate per cent. for 2 years, simple interest, is as 9 to 8; find the rate per cent. And if the difference of the interest and discount be thirty shillings, find the sum.

7. What would be the amount of one farthing for a thousand years at 5 per cent. per annum (1) at simple, (2) at compound interest?

8. If P pounds at compound interest at the rate r, double itself in m years, and at rate $2r$ in n years; what is the ratio of m and n?

What difference is made in the ratio when the interest is payable half-yearly and quarterly?

9. In the expression for the amount of P pounds for 5 years at compound interest, if the expression be expanded according to powers of r, the rate of interest; give an interpretation to each term of the expansion.

10. A mortgage is taken on an estate worth N acres of it; land rises n per cent. in price, and in consequence the mortgage is only worth N_1 acres, and it is then paid off. During the continuance of high prices another mortgage is taken, which is worth N acres as before; prices return to their former level, and the mortgage is worth N_2 acres; shew that $N-N_1 : N_2-N = 1 : 1+\frac{n}{100}$.

11. Determine the advantages in the payment of interest, when it is paid half-yearly and quarterly instead of yearly.

12. If a pension of £100 a year be unpaid for 6 years, what sum should the pensionary receive at the end of that period; compound interest being allowed at the rate of £5 per cent. per annum?

13. If £2,980 be invested partly in the 3 per cents. at 74, and partly in the 4 per cents. at 100, in such proportion that the income from each is the same; find the income.

14. If a_1, a_2, a_3 be the discounts of a sum of money for the same time at rates r_1, r_2, r_3 respectively; shew that

$$a_1r_2r_3(a_2-a_3)+a_2r_1r_3(a_3-a_1)+a_3r_1r_2(a_1-a_2)=0.$$

XIII.

1. When a body falls freely by the action of gravity, the space fallen through from rest in a given time varies as the square of the time. Find the relation between the space and the time if a body fall from rest through 64 feet in two seconds.

2. When a pendulum l inches long vibrates once in t seconds, then $l \propto t^2$; find the length of a pendulum which shall vibrate once in two seconds, if the pendulum which vibrates in one second be 39·4 inches long.

3. A boat is rowed 5 miles an hour in still water; in what time can it be rowed 10 miles and back again on a stream which runs at the rate of 2 miles an hour.

4. A person rows a distance of a miles on a river and back again in t hours, and finds that he can row b miles with the stream in the same time that he can row c miles against it; determine the times of going and returning, and the velocity of the stream.

5. If a or b agents produce an effect e in time t, and a' and b' agents together produce the effect e' in time t'; find the relation between the effects, agents, and times; a, a' consisting of the same units, as also b, b', but the latter different from the former.

6. If a_1 oxen in the time t_1 eat up a meadow of b_1 acres; and a_2 oxen eat up a like pasture of b_2 acres in time t_2; how many oxen will eat up a like pasture of b_3 acres in time t_3, the grass being supposed to grow uniformly? (*Newtoni Arithmetica Universalis, Prob. XI., p.* 90.)

7. Two towns, *A* and *B*, are distant *c* miles from each other; a traveller sets out from *A* for *B*, and another at the same time sets out from *B* for *A*. They both travel uniformly, and in such proportion that the former arrives at *B*, *a* hours after their meeting on the road, and the latter *b* hours after at *A*. In how many hours did each perform the journey?

8. There are two roads from *A* to *B*, one of them is 14 miles longer than the other, and two roads from *B* to *C*, one of them 8 miles longer than the other. The distances from *A* to *B* and from *B* to *C* along the shorter roads are in the ratio of 1 to 2; and the distances along the longer roads are in the ratio of 2 to 3; determine the distances.

9. If two lamps of different intensity of brightness be *a* feet apart; find the point between them at which the illumination received from one of them shall be double of that received from the other; supposing that the illumination varies inversely as the square of the distance from the lamps.

10. If the period of the earth and of the planet Venus in their orbits round the sun, be 365 and 224 days respectively; determine at what intervals these planets are in conjunction with the sun.

11. Find a formula by which it can be determined when the hour and minute hands of a clock are together, and when they are at any given angle, during one revolution of the hour hand.

12. If the sun moves every day one degree, and the moon thirteen degrees, and at a certain time the sun be at the beginning of Cancer, and in three days after, the moon in the beginning of Aries, the place of their next following conjunction is required (*Newt. Arith. Univ., p.* 84).

XIV.

1. A straight line is 3ft. 4in. long; divide it into two parts, one of which shall be to the other as 5 to 7.

2. How many inches must be added to or subtracted from each of the lengths, a foot, a yard, a furlong, and a mile, that the four sums or differences may form a proportion?

3. The sides of a triangular field are *a*, *b*, *c* yards respectively, *a* being the shortest side; what is the length of a hedge made parallel to the shortest side that will divide the field into two equal parts, and where do the ends of the hedge meet the other two sides of the field?

4. If the weights of two different substances be in the ratio of *m* to *n*, and together a compound *a* pounds; how much of one must be added,

or how much of the other taken away, in order that the ratio of the two substances may be changed into the ratio of m' to n'?

5. If a body weigh m pounds whose length is a, find the length of a similar body of the same density which weighs n pounds.

6. One vessel contains a mixture of wine and water in the ratio of 2 to 3, and another in the ratio of 3 to 7; what quantity must be drawn from each to make a mixture consisting of 5 gallons of water and 11 of wine?

7. If p pounds of a certain metal A when weighed in water loses a pounds, and p pounds of another B loses b pounds when weighed in water; find how much of each metal is contained in a compound mass of A and B, which weighs p pounds, and loses c pounds when weighed in water.

8. In what proportion must platinum and silver be mixed that the mixture may be equal in magnitude to an equal weight of gold, the specific gravity of gold, silver, and platinum being respectively 19·25, 10·47, 21·47?

9. Lead weighs 11·324 times as heavy as water; cork weighs $\frac{6}{25}$ and fir $\frac{9}{20}$ respectively of the weight of an equal volume of water. How much cork and lead must be combined together, so that the mass may be equal to 80 pounds, the weight of a beam of fir timber of the same magnitude?

10. A person has three pieces of metal, each of which consists of gold, silver, and copper. The first contains 5 ounces of gold, 15 of silver, and 30 of copper; the second contains 20 ounces of gold, 28 of silver, and 48 of copper; the third contains 12 ounces of gold, 39 of silver, and 24 of copper. He wishes to take some from each, and to melt all into one mass, in order to obtain a composition consisting of 10 ounces of gold, 23 of silver, and 26 of copper. How much must he take of each?

11. There are two globes of gold whose radii are a, b inches respectively; if they be melted and formed into a single globe; find its radius having given that the volume of a globe varies at the cube of its radius.

12. Prove that the volume of a sphere whose radius is 6 inches is equal to the sum of the volumes of three spheres whose radii are respectively 3, 4, 5 inches.

13. If a body weigh a ton at the surface of the earth, what would be its weight ten miles above the surface?

14. The weight of a given body at the surface of a planet varies directly as the mass of the planet, and inversely as the square of its radius. The mass of the planet Jupiter is 338 times the mass of the earth, and the diameter of Jupiter is 11 times the earth's diameter. What would be the weight at the surface of Jupiter of a man who weighs 181·5lb. at the surface of the earth?

RESULTS, HINTS, ETC., FOR THE EXERCISES ON RATIO AND PROPORTION.

I.

1. The ratio of two magnitudes in geometry is denoted by the magnitudes themselves; and there being no unit of magnitude assumed in geometry, there is no mode of expressing the value of a ratio, as in algebra, by the division of the number of units in the antecedent by the number of units in the consequent of the ratio.

2. See Art. I., p. 1.

3. $2x+5y=4x+3y$, then $5y-3y=4x-2x$, or $2x=2y$ and $x=y$.
The ratio of x to y is a ratio of equality.

4. Let x be the quantity required, then $\frac{m+x}{n+x}=\frac{p}{q}$, find the value of x.

5. If x be the number, then $\frac{5+x}{3+x}=\frac{3}{4}\cdot\frac{5-x}{3-x}$; find the value of x.

8. Let x be added to a and b, then $\left\{\frac{a+x}{b+x}\right\}^{a+b+2x}$ is greater than $\left(\frac{a}{b}\right)^{a+b}$.

Since b is greater than a, then $\frac{a}{b}$ is a proper fraction, and $\frac{a+x}{b+x}$ is also a proper fraction, but greater than $\frac{a}{b}$. Art. 2, p. 4.

$$\therefore \left\{\frac{a+x}{b+x}\right\}^{a+b} > \left\{\frac{a}{b}\right\}^{a+b}, \text{ and a fortiori is } \left\{\frac{a+x}{b+x}\right\}^{a+b+2x} > \left\{\frac{a}{b}\right\}^{a+b}.$$

11. The ratio of $(a+x)^3 : a^3$ is represented by $\frac{(a+x)^3}{a^3}$;

And $\frac{(a+x)^3}{a^3}=1+\frac{3x}{a}+\frac{3x^2}{a^2}+\frac{x^3}{a^3}$; but x is very small when compared with a, $\therefore \frac{x}{a}$ is a very small fraction, and a fortiori are $\frac{3x^2}{a^2}$ and $\frac{x^3}{a^3}$ much smaller fractions, and may be rejected without materially affecting the result,
and $\frac{(a+x)^3}{a^3}=1+\frac{3x}{a}$ nearly $=\frac{a+3x}{a}$.

The ratio of $(403)^2 : (402)^3 =$ ratio of $802 : 806$ nearly;
and the ratio $(403)^{\frac{1}{2}} : (402)^{\frac{1}{2}} =$ ratio of $803 : 802$ nearly.

12. The ratio of $x+5$ to $y+8$ is the same as the ratio 5 to 8.

14. See Exercises in Fractions. I. Example 5.

15. Let x the digit in the place of tens, y in the place of units, then $10x+y$ is the number, and $10y+x$ is the number with the digits interchanged.

16. $\frac{x}{y}=\frac{7}{11}$ and $\frac{y}{z}=\frac{11}{49}$. 17. See Art. 1, p. 3.

II.

1. 1·11. 2. ·208. 3. 70.

4. Since $\frac{x^2+y^2}{xy}=\frac{34}{15}$, add 2 to these equals, and $\frac{x^2+2xy+y^2}{xy}=\frac{64}{15}$; next subtract 2, then $\frac{x^2-2xy+y^2}{xy}=\frac{4}{15}$. Divide the former by the latter, $\therefore \frac{x^2+2xy+y^2}{x^2-2xy+y^2}=16$, and $\frac{x+y}{x-y}=4$, whence $x:y::5:3$.

5. Let x denote the common ratio, then $b=ax$, $c=ax^2$, $d=ax^3$; but since $2(a+d)=3(b+c)$,

$\therefore 2a(1+x^3)=3ax(1+x)$, and $2(1-x+x^2)=3x$, from which $x=\frac{1}{4}(5\pm\sqrt{17})$.

6. See Art. 1, p. 1; and Art. 5, p. 8.

7. Since $a^2+b^2=2ac \therefore b^2=a(2c-a)$ and $\frac{2c-a}{b}=\frac{b}{a}$.

8. See Example 3 under I., Exercises in Fractions.

9. Let a, $a+b$, $a+2b$, $a+3b$, four quantities in arithmetical progression.

Let x be added to each of them; then $\frac{a+x}{a+b+x}=\frac{a+2b+x}{a+3b+x}$.

The possibility of the answer will be determined by the value of x.

10. See Art. 8, p. 9.

11. Here (1) $(a-d)^2=(b-c)^2+(c-a)^2+(a-b)^2$, by expanding the terms there results $a^2-2ad+d^2=b^2-2bc+c^2+c^2-2ac+c^2+d^2-2bd+b^2$,

$$\text{and reducing } o=2b^2+2c^2-2ac-2bd$$
$$\text{and } \therefore b^2+c^2=ac+bd$$
$$\text{but } 2bc=ad+bc$$
$$\therefore b^2+2bc+c^2=ac+ad+bc+bd=a(c+d)+b(c+d)$$
$$\therefore (b+c)^2=(a+b)(c+d).$$

12. See Art. 1, p. 2, note. 13. See Art. 20, p. 14.

III.

1. See Art. 6, p. 8.

2. If $a:b::c:d$; then $ac:bd::c^2:d^2$

Here $ad=bc$, $\therefore \frac{a}{c}=\frac{b}{d}$, and $\frac{a}{c}=\frac{ac}{c^2}$, and $\frac{b}{d}=\frac{bd}{d^2}$, $\therefore \frac{ac}{c^2}=\frac{bd}{d^2}$, and $\frac{ac}{ba}=\frac{c^2}{d^2}$.

Conversely. If $ac:bd::c^2:d^2$, then $a:b::c:d$.

Since $ac:bd::c^2:d^2$, $\therefore acd^2=bc^2d$, $\therefore ad=bc$, and $\frac{a}{b}=\frac{c}{d}$.

3. If the learner may sometimes not perceive the necessary steps for the direct solution of a question, in cases where the result is known, he is recommended to try the converse by simply retracing the steps from the predicate to the hypothesis, or from the quæsita to the data, and reversing the process.

Given $a:b::c:d$, prove $a^2:c^2::a^2+b^2:c^2+d^2$.

$$\text{Here } \frac{a^2}{c^2}=\frac{a^2+b^2}{c^2+d^2}, \therefore a^2c^2+a^2d^2=a^2d^2+b^2c^2,$$

and $a^2d^2=b^2c^2$, by subtracting a^2c^2 from these equals,

$$\therefore ad=bc, \text{ and } \frac{a}{b}=\frac{c}{d}$$

4. Here $\frac{a}{b}=\frac{c}{d}$, $\therefore \frac{ab}{b^2}=\frac{cd}{d^2}$; and $\frac{ab}{cd}=\frac{b^2}{d^2}$;

$$\text{next } \frac{a}{b}=\frac{c}{d}, \therefore \frac{a\pm b}{b}=\frac{c\pm d}{d} \text{ and } \frac{a\pm b}{c\pm d}=\frac{b}{d}, \text{ also } \frac{(a\pm b)^2}{(c\pm d)^2}=\frac{b^2}{d^2};$$

$$\text{also } \frac{a^2}{b^2}=\frac{c^2}{d^2}, \therefore \frac{a^2-b^2}{b^2}=\frac{c^2-d^2}{d^2}, \text{ and } \frac{a^2-b^2}{c^2-d^2}=\frac{b^2}{d^2}.$$

$$\text{Hence } \frac{ab}{cd}=\frac{b^2}{d^2}=\frac{(a\pm b)^2}{(c\pm d)^2}=\frac{a^2-b^2}{c^2-d^2}.$$

5. Since $a:b::c:d$,

$$\text{then } \frac{a}{b}=\frac{c}{d}, \frac{a^2}{b^2}=\frac{c^2}{d^2}, \text{ and } \frac{a^2+b^2}{b^2}=\frac{c^2+d^2}{d^2}, \therefore \frac{a^2+b^2}{c^2+d^2}=\frac{b^2}{d^2};$$

$$\text{and } \frac{a}{b}=\frac{c}{d}, \frac{a-b}{b}=\frac{c-d}{d}, \text{ and } \frac{(a-b)^2}{b^2}=\frac{(c-d)^2}{d^2}, \therefore \frac{(a-b)^2}{(c-d)^2}=\frac{b^2}{d^2}.$$

Hence $\frac{a^2+b^2}{c^2+d^2}=\frac{(a-b)^2}{(c-d)^2}$, and $a^2+b^2 : c^2+d^2 :: (a-b)^2 : (c-d)^2$.

6. Since $a:b::c:d$, then $\frac{a}{b}=\frac{c}{d}$, and $\frac{a^2}{b^2}=\frac{c^2}{d^2}$,

$$\therefore \frac{a^2+b^2}{b^2}=\frac{c^2+d^2}{d^2}, \therefore \frac{a^2+b^2}{c^2+d^2}=\frac{b^2}{d^2}, \therefore \frac{(a^2+b^2)^2}{(c^2+d^2)^2}=\frac{b^4}{d^4},$$

but since $\frac{a}{b}=\frac{c}{d}$, $\frac{a^4}{b^4}=\frac{c^4}{d^4}$, and $\frac{a^4+b^4}{b^4}=\frac{c^4+d^4}{d^4}$ $\therefore \frac{a^4+b^4}{c^4+d^4}=\frac{b^4}{d^4}$.

Hence $\frac{(a^2+b^2)^2}{(c^2+d^2)^2}=\frac{a^4+b^4}{c^4+d^4}$, and $(a^2+b^2)^2 : (c^2+d^2)^2 :: a^4+b^4 : c^4+d^4$.

IV.

1. It is not always obvious what steps are necessary to be taken to arrive at the quæsitum of a proposition. In such cases, by assuming the quæsitum, it may be more easy to retrace the steps to the datum, as in this example. The datum is $ad=bc$, the quæsitum is $a(a+b+c+d)=(a+c)(b+d)$, or $a^2+ab+ac+ad=a^2+ab+ac+bc$. Here it is obvious that $a^2+ab+ac$ is added to each side of $ad=bc$.

2. Multiply the factors, and reduce the expression, then the reverse operation.

3. Multiply the factors, and the identity is manifest. Reverse the process.

4. First $\frac{a}{b}=\frac{c}{d}$, $\frac{a+b}{b}=\frac{c+d}{d}$, $\frac{(a+b)^4}{b^4}=\frac{(c+d)^4}{d^4}$, $\frac{(a+b)^4+(c+d)^4}{(c+d)^4}=\frac{b^4+d^4}{d^4}$;

next $\frac{a}{c}=\frac{b}{d}$, $\frac{a+c}{c}=\frac{b+d}{d}$, $\frac{(a+c)^4}{c^4}=\frac{(b+d)^4}{d^4}$, $\frac{(a+c)^4+(b+d)^4}{(b+d)^4}=\frac{c^4+d^4}{d^4}$

$$\therefore \{(a+b)^4+(c+d)^4\}\cdot\{(a+c)^4+(b+d)^4\}=\frac{(c+d)^4(b^4+d^4)}{d^4}\times\frac{(b+d)^4(c^4+d^4)}{d^4}.$$

Again $\frac{a}{b}=\frac{c}{d}$, $\frac{a^4}{b^4}=\frac{c^4}{d^4}$, $\frac{a^4+b^4}{b^4}=\frac{c^4+d^4}{d^4}$, $\frac{a^4+b^4+c^4+d^4}{c^4+d^4}=\frac{b^4+d^4}{d^4}$,

and $\frac{a}{b}=\frac{c}{d}$, $\frac{a+b}{b}=\frac{c+d}{d}$, $\frac{a+b+c+d}{c+d}=\frac{b+d}{d}$, $\frac{(a+b+c+d)^4}{(c+d)^4}=\frac{(b+d)^4}{d^4}$

$$\therefore (a^4+b^4+c^4+d^4)(a+b+c+d)^4=\frac{(b^4+d^4)(c^4+d^4)}{d^4}\times\frac{(b+d)^4(c+d)^4}{d^4}.$$

Hence $(a^4+b^4+c^4+d^4)(a+b+c+d)^4=\{(a+b)^4+(c+d)^4\}\cdot\{(a+c)^4+(b+d)^4\}$.

5. $\frac{a}{b}=\frac{c}{d}$ $\therefore \frac{a^2}{b^2}=\frac{ac}{bd}$, and $\frac{c^2}{d^2}=\frac{ac}{bd}$ $\therefore \frac{a^2}{b^2}+\frac{c^2}{d^2}=\frac{2ac}{bd}$.

6. $ad=bc$, $d=\frac{bc}{a}$. Then $a+d-(b+c)=a+\frac{bc}{a}-(b+c)=\frac{a^2+bc-ab-ac}{a}$

$$=\frac{a^2-(b+c)a+bc}{a}=\frac{(a-b)(a-c)}{a}.$$

11. Here $ad=bc$, $\therefore a^4=\frac{b^4c^4}{d^4}$, $b^4=\frac{a^4d^4}{c^4}$, $c^4=\frac{a^4d^4}{c^4}$, $d^4=\frac{b^4c^4}{a^4}$

and $a^4+b^4+c^4+d^4=b^4c^4\left\{\frac{1}{a^4}+\frac{1}{d^4}\right\}+a^4d^4\left\{\frac{1}{b^4}+\frac{1}{c^4}\right\}$

$$=b^2c^2a^2d^2\left\{\frac{1}{a^4}+\frac{1}{d^4}\right\}+a^2d^2b^2c^2\left\{\frac{1}{b^4}+\frac{1}{c^4}\right\}=(abcd)^2.\left\{\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}+\frac{1}{d^4}\right\}$$

$$=(abcd)^2.(a^{-4}+b^{-4}+c^{-4}+d^{-4}), \therefore \frac{a^4+b^4+c^4+d^4}{a^{-4}+b^{-4}+c^{-4}+d^{-4}}=(abcd)^2$$

13. Since $ad=bc$; then $bcd=ad.a$, $cda=ad.c$, $dab=ad.b$, $abc=ad.a$;

$\therefore bcd+cda+dab+abc=(a+b+c+d)ad$, and

$$\frac{bcd+cda+dab+abc}{a+b+c+d}=ad=(ad)^{\frac{1}{2}}.(ad)^{\frac{1}{2}}=(ad)^{\frac{1}{2}}.(bc)^{\frac{1}{2}}=(abcd)^{\frac{1}{2}}.$$

V.

1. $\frac{x}{a}=\frac{y}{b}$, then $\frac{x}{y}=\frac{a}{b}$ $\therefore$ $\frac{x+y}{y}=\frac{a+c}{b}$ and $\frac{x+y}{a+b}=\frac{y}{b}$. Similarly $\frac{x+z}{a+c}=\frac{z}{c}$, $\frac{y+z}{b+c}=\frac{z}{c}$,

$$\text{then } \frac{x+y}{a+b}=\frac{x+z}{b+c}=\frac{y+z}{b+c}.$$

2. Let $\frac{x}{a}=\frac{mx}{ma}=\frac{ny}{nb}=\frac{pz}{pc}=R$, then $mx=Rma$, $ny=Rnb$, $pz=Rpc$,

$$\therefore R(ma+nb+pc)=mx+ny+pz, \text{ and } \frac{mx+ny+pz}{ma+ny+pz}=\frac{x}{a}.$$

5. The required expression is reducible to the form $(bx-ay)^2+(az-cx)^2+(cy-bz)^2=o$, which can be true only on the condition that each square is equal to zero.

$$\therefore bx-ay=o,\ az-cx=o,\ cy-bz=o, \text{ and } bx=ay,\ az=cx,\ cy=bz,$$

$$\therefore \frac{x}{a}=\frac{b}{y},\ \frac{z}{c}=\frac{x}{a},\ \frac{y}{b}=\frac{z}{c}. \quad \text{Reverse the process.}$$

6. Let $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=p$, then $x=pa$, $y=pb$, $z=pc$, and $a^2x=pa^3$, $b^2y=pb^3$, $c^2z=pc^3$,

$$\therefore a^2x+b^2y+c^2z=p(a^3+b^3+c^3).$$

Next $\frac{a}{x}=\frac{b}{y}=\frac{c}{z}=\frac{1}{p}$, then $a=\frac{x}{p}$, $b=\frac{y}{p}$, $c=\frac{z}{p}$, and $ax^2=\frac{x^3}{p}$, $by^2=\frac{y^3}{p}$, $cz^2=\frac{z^3}{p}$,

$$\therefore ax^2+by^2+cz^2=\frac{1}{p}(x^3+y^3+z^3), \text{ but } a^2x+b^2y+c^2z=p(a^3+b^3+c^3),$$

$$\therefore (ax^2+by^2+cz^2)(a^2x+b^2y+c^2z)=(a^3+b^3+c^3)(x^3+y^3+z^3).$$

8. $\frac{x}{a}=\frac{z}{c}$, $\frac{x}{z}=\frac{a}{c}$, $\frac{mx}{z}=\frac{ma}{c}$, $\frac{mx}{z}+n=\frac{ma}{c}+n$, or

$$\frac{mx+nz}{z}=\frac{ma+nc}{c}, \text{ but } \frac{y}{x}=\frac{b}{a} \therefore \frac{mxy+nyz}{xz}=\frac{mab+nbc}{ac}.$$

9. It may be shewn that $\frac{xyz}{abc}=\frac{x^3}{a^3}$, and $\frac{xy+yz+xz}{ab+bc+ca}=\frac{x^2}{a^2}$.

10. Let $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}=p$, then $p(a+b+c)=x+y+z$, and p or $\frac{x}{a}=\frac{x+y+z}{a+b+c}$.

$$\text{Also } \frac{xyz}{abc}=\frac{x^3}{a^3},\ \therefore \frac{x^3}{a^3}=\left\{\frac{x+y+z}{a+b+c}\right\}^3, \text{ but } \frac{xyz}{abc}=\frac{x^3}{a^3} \therefore \frac{xyz}{abc}=\left\{\frac{x+y+z}{a+b+c}\right\}^3.$$

$$\text{Again, let } q=\frac{x^3}{a^3}=\frac{y^3}{b^3}=\frac{z^3}{c^3}, \text{ then } q(a^3+b^3+c^3)=x^3+y^3+z^3,$$

$$\text{and } q \text{ or } \frac{x^3}{a^3}=\frac{x^3+y^3+z^3}{a^3+b^3+c^3}.$$

11. Since $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, $\therefore \frac{ax}{a^2}=\frac{by}{b^2}=\frac{cz}{c^2}$, and let $\frac{(ax)^{\frac{1}{2}}}{a}=\frac{(by)^{\frac{1}{2}}}{b}=\frac{(cz)^{\frac{1}{2}}}{c}=p$

then $(ax)^{\frac{1}{2}}+(by)^{\frac{1}{2}}+(cz)^{\frac{1}{2}}=p(a+b+c)$: similarly $(ax)^{\frac{1}{2}}+(by)^{\frac{1}{2}}+(cz)^{\frac{1}{2}}=(x+y+z).\frac{1}{p}$

$$\text{Hence } (ax)^{\frac{1}{2}}+(by)^{\frac{1}{2}}+(cz)^{\frac{1}{2}}=(a+b+c)(x+y+z).$$

13. Here $\frac{x}{y}=\frac{y}{b}$ and $\frac{x}{a}=\frac{z}{c}$ $\therefore \frac{xyz}{abc}=\frac{x^3}{a^3}=\frac{y^3}{b^3}=\frac{z^3}{c^3}$. Let p denote each ratio;

$$\text{then } 3xyz=3bcp,\ x^3=pa^3,\ y^3=pb^3,\ z^3=pc^3$$

$$\therefore x^3+y^3+z^3-3xyz=(a^3+b^3+c^3-3abc)p$$

$$\therefore p=\frac{x^3+y^3+z^3-3xyz}{a^3+b^3+c^3-3abc}=\frac{x^3}{a^3}.$$

14. $\frac{x+a}{a}=\frac{x+b}{b}$, $\frac{x+a}{a}=\frac{z+c}{c}$, and $\frac{x+a}{a}=\frac{x+a}{a}$, $\therefore \left\{\frac{x+a}{a}\right\}^3=\frac{(x+a)(y+b)(z+c)}{abc}$.

$$\text{Similarly } \left\{\frac{x+b}{b}\right\}^3=\frac{(x+a)(y+b)(z+c)}{abc} \text{ and } \left(\frac{z+c}{c}\right)^3=\frac{(x+a)(y+b)(z+c)}{abc}.$$

Now $\left\{\frac{x+a}{a}+\frac{y+b}{b}+\frac{z+c}{c}\right\}^3=\left\{\frac{x+a}{a}\right\}^3+\left\{\frac{y+b}{b}\right\}^3+\left\{\frac{z+c}{c}\right\}^3+\frac{3(x+a)(y+b)(z+c)}{abc}$

$$=\frac{3(x+a)(y+b)(z+c)}{abc}+\frac{3(x+a)(y+b)(z+c)}{abc}=\frac{6(x+a)(x+b)(x+c)}{abc}$$

$\therefore\ abc\left\{\frac{x+a}{a}+\frac{y+b}{b}+\frac{z+c}{c}\right\}^3=6(x+a)(x+b)(x+c)$.

VI.

1. Since $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, $\therefore\ b^2=ac$, $c^2=bd$, and $ad=bc$,

$\therefore\ b^2+c^2=ac+bd$, but $2bc=ad+bc$, $\therefore\ b^2+2bc+c^2=ac+ad+bd+bc$
or $(b+c)^2=a(c+d)+b(c+d)=(a+b)(c+d)$.

3. Here $c(a+2c+e)=ac+c^2+c^2+ce$: since $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e}$,

$\therefore\ ac+c^2=b^2+bd$ and $c^2+ce=bd+d^2$ $\therefore\ ac+2c^2+ce=b^2+2bd+d^2=(b+d)^2$.

4. Since $\frac{a}{b}=\frac{b}{c}$, $\therefore\ b^2=ac$, and $a^2+b^2=a^2+ac=a(a+c)$,

and $b^2+c^2=ac+c^2=c(a+c)$,
$\therefore\ (a^2+b^2)(b^2+c^2)=ac(a+c)^2=b^2(a+c)^2=(ab+bc)^2$.

5. The products form an identity, when $ad=bc$.

6. Let $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=p$, then $\frac{a^2}{ab}=\frac{b^2}{bc}=\frac{c^2}{cd}=p$

$\therefore\ a^2=pab$, $b^2=pbc$, $c^2=pcd$, and $a^2+b^2+c^2=p(ab+ac+bc)$.

Again $\frac{ba}{b^2}=\frac{bc}{c^2}=\frac{cd}{d^2}=p$ $\therefore\ ba=pb^2$, $bc=pc^2$, $cd=pd^2$,

and $p(b^2+c^2+d^2)=ab+bc+cd$; but $a^2+b^2+c^2=p(ab+ac+bc)$
$\therefore\ (a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$.

9. $b^2=ac$, then $a^2-b^2=a^2-ac=a(a-c)$; $c^2-b^2=c^2-ac=-(a-c)c$,

then $\frac{1}{a^2-b^2}+\frac{1}{c^2-b^2}=\frac{1}{a-c}\left\{\frac{1}{a}-\frac{1}{c}\right\}=\frac{-(a-c)}{(a-c)ac}=-\frac{1}{ac}=-\frac{1}{b^2}$.

VII.

1. Suppose the result that $\frac{a+b}{b+c}$, b, bc are proportionals,

then $\frac{(a+b)bc}{b+c}=b^2$, and $ac+bc=b^2+bc$, $\therefore\ ac=b^2$. The reverse offers no difficulty.

If also a, b, c are so related that $a+b+c=0$, and $a^{-1}+b^{-1}+c^{-1}=0$;
then $a^2=bc$, $b^2=ac$, $c^2=ab$, or that one of the three quantities is a mean proportional between the other two.

2. Since $\frac{a+c}{b+d}=\frac{b-d}{a-c}$, then $a^2-c^2=b^2-d^2$,

and $a^2-b^2=c^2-d^2$, or $(a+b)(a-b)=(c+d)(c-d)$.

3. $\frac{(a-c)^2}{(b-d)^2}=\frac{a-b}{c-d}$; subtract unity from each side,

then $\frac{(a-c)^2-(b-d)^2}{(b-d)^2}=\frac{a-b-c+d}{a-d}=\frac{(a-c)-(b-d)}{c-d}$,

and $\frac{(a-c)+(b-d)}{(b-d)^2}=\frac{1}{c-d}$, dividing by $(a-c)-(b-d)$,

then $\frac{a-c}{(b-d)^2}=\frac{1}{c-d}-\frac{1}{b-d}=\frac{b-c}{(c-d)(b-d)}$,

$$\text{and } \therefore (a-c)(c-d)=(b-c)(b-d),$$
$$\text{or } ac-ad-c^2+cd=b^2-bd-bc+cd,$$
$$\therefore -bc+ac-ad+bd=b^2-2bc+c^2,$$
$$\text{or } (a-b)(c-d)=(b-c)^2.$$

5. Here
$$\begin{aligned}-a(ac'-a'c)^2&=(a'b-ab')\{c(a'b-ab')+b(ac'-a'c)\}\\&=(a'b-ab')\{a'bc-ab'c+abc'-a'bc\}\\&=(a'b-ab')\{-a(b'c-bc')\}\\&=-a(a'b-ab')(b'c-bc')\end{aligned}$$
$$\therefore (ac'-a'c)^2=(a'b-ab')(b'c-bc').$$

6. If unity be subtracted from each of these equals, and the result reduced, it becomes $b^2+bc+c^2=a^2+ad+d^2$.

7. Let each expression be put equal to p, then $2p=c+d$.

8. See Arts. 18 and 19, p. 13.

9. The relation between the two quantities gives $x=-y$.

11. By reducing the expression it becomes $a^2-ab=b^2$, whence $\frac{a}{b}=\frac{1\pm\sqrt{5}}{2}$.

12. $\frac{a+(a^2+b^2)^{\frac{1}{2}}}{a-(a^2-b^2)^{\frac{1}{2}}}=\frac{c+(c^2+d^2)^{\frac{1}{2}}}{c-(c^2-d^2)^{\frac{1}{2}}}$; rationalise the denominators,

then $\frac{a+(a^2+b^2)^{\frac{1}{2}}}{c+(c^2+d^2)^{\frac{1}{2}}}=\frac{b}{d}$, from which by reduction may be found $(bc-ad)^2=0$.

13. By adding the two equations there results $(1-x)^{\frac{1}{2}}+(1+x)^{\frac{1}{2}}=2y^{\frac{1}{2}}$, from which may be deduced $(1-x)(1+x)=(1-2y)^2$.

VIII.

1. $ax+by=-cz,\ a'x+b'y=-c'z$, then $\frac{ax+by}{a'x+b'y}=\frac{c}{c'}$,

whence $\frac{x}{y}=\frac{b'x-bc'}{ac'-a'c}$, similarly $\frac{x}{z}=\frac{bc'-b'c}{ab'-a'b}$ and

$\frac{y}{z}=\frac{a'c-ac'}{ab'-a'b}$. $\therefore \frac{x}{b'c-bc'}=\frac{y}{ac'-a'c}=\frac{z}{ab'-a'b}$.

2. $\frac{ax^2+by^2}{(a'x+b'y)^2}=-\frac{c}{c'^2}$, from this may be found $\frac{x}{y}$, and so for the rest.

3. Since $\frac{a+c}{b+c}=\frac{a^2}{b^2}$, then $\frac{a^2-b^2}{b^2}=\frac{a-b}{b+c}$, $\therefore (a-b)\left\{\frac{a+b}{b^2}-\frac{1}{b+c}\right\}=o$,

$$\therefore a-b=o, \text{ and } \frac{a+b}{b^2}-\frac{1}{b+c}=o.$$

4. Let each ratio be put equal to p, then $p(a+b+c)=o$.

5. Equate values of z between the first and second ratios, then between the second and third, and $y=2x$.

6. By combining the first expression with the second, and the second with the third, and equating the values of y, there results $a(a^2+b^2+c^2)z=c(b^2+c^2+a^2)x$ and $cx=az$.

8. First shew that $\frac{a-b}{b-c}=\frac{x-y}{y-z}$:

$$\text{Here is given } a(y-z)+b(z-x)+c(x-y)=o$$
$$\text{then } a(y-z)+bz=bx-c(x-y)$$
$$\text{add } -by \text{ to these equals,}$$
$$\therefore a(y-z)-b(y-z)=b(x-y)-c(x-y)$$
$$\text{or } (a-b)(y-z)=(b-c)(x-y)$$
$$\text{and } \therefore \frac{a-b}{x-y}=\frac{b-c}{y-z}.$$

Similarly may be shewn that $\frac{b-c}{y-z}=\frac{c-a}{z-x}$.

9. Let p denote each ratio, then $cy+bz=p(b-c)$, $az+cx=p(c-a)$, $bx+ay=p(a-b)$,

$\therefore bx+cx+ay+cy+az+bz=o$, add $ax+by+cz$ to each side

and $(a+b+c)x+(a+b+c)y+(a+b+c)z=ax+by+cz$.

10. Let $\frac{a+b}{x+y}=p$, then $a+b=p(x+y)$, similarly $b+c=(y+z)p$ and $c+a=(x+z)p$, then adding these equalities, $2a+2b+2c=2p(x+y+z)$ and $p=\frac{a+b+c}{x+y+z}$.

11. $\frac{x}{a}=\frac{d}{y}$, $\frac{x+a}{a}=\frac{d+y}{y}$, $\frac{x+a}{d+y}=\frac{a}{y}$,

$$\frac{x}{d}=\frac{a}{y},\ \frac{x+d}{d}=\frac{a+y}{y},\ \frac{x+d}{a+y}=\frac{d}{y},$$

$$\frac{x}{a}=\frac{b}{c},\ \frac{x}{b}=\frac{a}{c},\ \frac{x+b}{b}=\frac{a+c}{c},\ \text{and } x+b=\frac{(a+c)b}{c},$$

$$\frac{b}{c}=\frac{d}{y},\ \frac{y}{c}=\frac{d}{b},\ \frac{y+c}{c}=\frac{d+b}{b}\ \text{and } y+c=\frac{(d+b)c}{b},$$

$$\therefore \frac{(x+a)(x+d)(x+b)}{(d+y)(a+y)(c+y)}=\frac{a}{y}.\frac{d}{y}.\frac{(a+c)b}{c}.\frac{b}{(d+b)c}$$

$$=\frac{a}{y}.\frac{x}{a}.\frac{(a+c)b^2}{(d+b)c^2}=\frac{(a+c)b^2x}{(d+b)c^2y}.$$

12. By equating any two of the ratios, the following expression results, $xy+xz+yz=xyz(x+y+z)=o$.

13. Assume the results, and from them form the given equations, and then reverse the process.

IX.

1. See Art. 22, p. 15.   2. $x=\frac{d-b}{a-c}.y$, and $\therefore x\propto y$.

3. Let $ax^2+by=p(cx^2-dy)$, where p is some constant quantity,

$$\text{then } x=\left\{\frac{b+pd}{pc-a}\right\}^{\frac{1}{2}}.y^{\frac{1}{2}},\ \therefore x\propto y^{\frac{1}{2}}.$$

4. Let $x-a=p(y-b)$, p being constant, when $x=m$ and $y=n$, then $p=\frac{m-a}{n-b}$, and $x-a=\frac{m-a}{n-b}(y-b)$ is the equation required.

5. Let $x+y=p(x-y)$, then $x^2+2xy+y^2=p^2(x^2-2xy+y^2)$ and $(p^2-1)(x^2+y^2)=2(p^2+1)xy$, $\therefore x^2+y^2=\frac{2(p^2+1)}{p^2-1}.xy$, $\therefore x^2+y^2\propto xy$.

Conversely. Let $x^2+y^2=pxy$, then $x^2+2xy+y^2=(p+2)xy$, and

$$x^2-2xy+y^2=(p-2)xy;$$

$$\therefore \frac{(x+y)^2}{(x-y)^2}=\frac{p+2}{p-2},\ \text{and } \frac{x+y}{x-y}=\left\{\frac{p+2}{p-2}\right\}^{\frac{1}{2}}\ \therefore x+y=\left\{\frac{p+2}{p-2}\right\}^{\frac{1}{2}}(x-y),$$

and $x+y\propto x-y$.

6. $xy^3=54$.

7. Since $x\propto y$, let $x=py$, then $x^3+y^3=(p^3+1)y^3$ and $x^3-y^3=(p^3-1)y^3$,

$$\therefore \frac{x^3+y^3}{x^3-y^3}=\frac{p^3+1}{p^3-1},\ \text{and } x^3+y^3=\frac{p^3+1}{p^3-1}.(x^3-y^3),\ \therefore x^3+y^3\propto x^3-y^3.$$

9. $y=\frac{a^4}{b^4}.\frac{x^2}{a+x}$.

11. The equation connecting x and y is $2y=3x^3+8x^2+x$, and the value of y is 162 when $x=4$.

12. $y=\frac{132}{13x}+\frac{8x^3}{13}$.

X.

1. See Art. 22, p. 15, and Art. 26, p. 16. The constant multiplier is $\frac{5}{27}$.

2. Here $y=p(x+z)$, $y^2=qxz$, then $\frac{y^2}{p^2}=(x+z)^2$ and $\frac{4y^2}{q}=4xz$

$$\therefore \frac{q-4p^2}{p^2q}.y^2=(x-z)^2, \text{ and } \frac{(x+z)^2}{(x-z)^2}=\frac{q}{q-4p^2}=c^2, \text{ Suppose.}$$

$$\therefore \frac{x+z}{x-z}=c, \text{ and } z=\frac{c-1}{c+1}.x, \therefore z\propto x \therefore \text{also } zx\propto x^2,$$

$$\text{but } y^2\propto xz, \therefore y^2\propto x^2 \text{ and } y\propto x.$$

3. Let $y=\frac{p}{x}$ and $x=q(z+y)$, then $p=q(yz+y^2)$,

$$\text{and } \frac{p}{q}=yz+y^2=4+4=8, \text{ when } y=2 \text{ and } z=2.$$

And since $yz=\frac{p}{q}-y^2$, when $y=4, 4z=8-16=-8$ and $z=-2$.

4. $z = 2y-x$.

5. Since $x\propto y$ and $xy\propto z^2$, then $x=py$, and $xy=qz^2$;

$$\therefore xyz=qz^3, \frac{x}{y}=p, \therefore x^2=pqz^2 \text{ and } x=p^{\frac{1}{2}}q^{\frac{1}{2}}z,$$

$$\text{also } y=\frac{x}{p}=\frac{q^{\frac{1}{2}}}{q^{\frac{1}{2}}}.z; \therefore x^3+y^3+z^3=\left\{p^{\frac{3}{2}}q^{\frac{3}{2}}+q^{\frac{3}{2}}p^{-\frac{3}{2}}+1\right\}\frac{xyz}{q}$$

$$=\left\{p^{\frac{3}{2}}q^{\frac{1}{2}}+q^{\frac{1}{2}}p^{-\frac{3}{2}}+q^{-1}\right\}xyz. \text{ Hence } x^3+y^3+z^3\propto xyz.$$

6. Since $x^2\propto yz$ and $y^2\propto xz$, let $x^2=p^2yz$, $y^2=-q^2xz$, p, q being constants.

$$\text{Then } \frac{x^2}{y^2}=\frac{p^2}{q^2}.\frac{y}{x} \text{ and } \frac{x^3}{y^3}=\frac{p^3}{q^3}, \frac{x}{y}=\frac{p}{q}; \frac{x^2}{y^2}=\frac{b^2}{q^2}, \frac{x^2+y^2}{y^2}=\frac{p^2+q^2}{q^2} \text{ and}$$

$$x^2+y^2=\frac{p^2+q^2}{q^2}.y^2=\frac{(p^2+q^2)q^2x^2}{q^2}=(p^2+q^2)qxz. \text{ Also since } \frac{y}{x}=\frac{q}{p}, y+x=\frac{p+q}{p}.x$$

$$\text{Hence } \frac{x^2+y^2}{x+y}=\frac{(p^2+q^2)qxzp}{(p+q)x}=\frac{(p^2+q^2)pq}{p+q}.z, \text{ and } x^2+y^2=\frac{(p^2+y^2)pq}{p+q}.(x+y)z.$$

7. Let $z=P+Q$, and let $P\propto x$, and $Q\propto\frac{1}{y}$, then $P=px$ and $Q=\frac{q}{y}$, $\therefore z=px+\frac{q}{y}$,

when $x=2$, $y=1$, and $z=30$, $\therefore 30=2p+q$,

$x=3$, $y=\frac{1}{3}$, and $z=15$, $15=3p+3q$, $\therefore p=25$ and $q=-20$

$\therefore z=25x-\frac{20}{y}$. And when $x=3$, and $y=4$, then $z=70$.

10. Let x, y be the two quantities, and z the third quantity,

then $x-y\propto(x^3-y^3)z$, and $x-y\propto(x^3+y^3)z$;

Let $x+y=p(x^3-y^3)z$, and $x-y=q(x^3+y^3)z$, p and q constants,

$\therefore \frac{x^3+y^3}{x^3-y^3}=\frac{q}{p}.\frac{x-y}{x+y}$; then adding and subtracting unity from the equals,

$$\frac{2x^3}{x^3-y^3}=\frac{(p+q)x+(p-q)y}{p(x+y)}, \text{ and } \frac{2y^3}{x^3-y^3}=-\frac{(p-q)x+(p+q)y}{p(x+y)}$$

$$\therefore \frac{x^3}{y^3}=-\frac{(p+q)x+(p-q)y}{(p-q)x+(p+q)y}=-\frac{x+ny}{nx+y}, \text{ if } \frac{p-q}{p+q}=n.$$

$$\text{And } z^3=-\frac{z+n}{nz+1}, \text{ if } \frac{x}{y}=z. \quad nz^4+z^3=-z-n \quad nz^2+z=-\frac{1}{z}-\frac{n}{z^2},$$

$$n\left(z^2+2+\frac{1}{z^2}\right)+\left(z+\frac{1}{z}\right)=2n, \text{ or } n\left(z+\frac{1}{z}\right)^2+\left(z+\frac{1}{z}\right)=2n.$$

$$\text{and } z+\frac{1}{z}=\frac{-1\pm(8n^2+1)^{\frac{1}{2}}}{2n}=m, \text{ suppose };$$

also z or $\frac{x}{y}=\frac{m\pm(m^2-4)^{\frac{1}{2}}}{2}$ and $x=\left\{\frac{m\pm(m^2-4)^{\frac{1}{2}}}{2}\right\}^{\frac{1}{2}}.y$

$\therefore x \propto y$, for m is constant.

XI.

1. If x, y, z denote the shares, find x in terms of y from one proportion, and z in terms of y from the other; then the sum of the shares in terms of y is equal to £1,000.

From which $y=\frac{£200\times 77}{39}$, $x=\frac{£1,000\times 11}{39}$, $z=\frac{£200\times 21}{13}$.

2. $\frac{bdc}{ac}$ horses. 3. One 72 and the other 54 years of age.

4. The gain per cent. is $\frac{100(n-m)-np}{m}$.

5. The share of gain of each partner depends both upon the amount of his capital and the time it is employed in the partnership.

ma represents the efficiency A's capital, and nb of B's.

Let x and y denote A's and B's gain respectively.

Then $\frac{ma}{nb}=\frac{x}{y}$, $\frac{ma+nb}{nb}=\frac{x+y}{y}$: and $\frac{nb}{ma}=\frac{y}{x}$, $\frac{nb+ma}{ma}=\frac{y+x}{x}$

but $x+y=p \therefore \frac{ma+nb}{ma}=\frac{p}{x}$ and $\frac{ma+nb}{nb}=\frac{p}{y}$.

or $ma+nb : ma :: p : x$, and $ma+nb : nb :: p : y$.

The method may be extended to 3, 4, or more partners.

If each partner's capital was engaged for the same time, that is, if $m=n$, then the relation will be

$a+b : a :: p : x$, and $a+b : b :: p : y$.

6. The rent was $\frac{100m}{100-p}$.

7. Let v, x, y, z denote what each creditor has to receive, then

$$v+x+y+z=21000.$$

And $\frac{v}{x}=\frac{3}{2}$, $\frac{x}{y}=\frac{4}{5}$, and $\frac{y}{z}=\frac{6}{7}$.

Then since $\frac{x}{v}=\frac{2}{3}$, $\therefore x=\frac{2v}{3}$,

$\frac{v}{x}.\frac{x}{y}=\frac{v}{y}=\frac{3}{2}.\frac{4}{5}=\frac{6}{5}$, and $y=\frac{5v}{6}$,

$\frac{v}{x}.\frac{x}{y}.\frac{y}{z}=\frac{v}{z}=\frac{3}{2}.\frac{4}{5}.\frac{6}{7}=\frac{36}{35}$, and $z=\frac{35v}{36}$

$\therefore v+x+y+z=v+\frac{2v}{3}+\frac{5v}{6}+\frac{35v}{36}=21000$, from which v can be found.

8. 929 chapters in the Old Testament, and 260 in the New.

9 The ratio of the power of exposition of Professor A to Professor B, is as 1 to 2.

10. 185 voted for, 190 against the question, and 375 members voted.

XII.

1. See note, p. 6; and Elementary Arithmetic, Sect. XI., pp. 5—7.

2. If P the principal, r the interest of one pound for one year, and M the amount in n years; then $M=P(1+nr)$ at simple interest.

If $M=2P$, then $n=\frac{1}{r}$; when $M=3P$, $n=\frac{2}{r}$.

At 5 per cent. a sum doubles itself in 20 years, and trebles itself in 40 years.

Next $M=P(1+r)^n$ at compound interest.

$$\text{when } M=2P, \text{ then } P(1+r)^n=2P, \ (1+r)^n=2, \ n\log(1+r)=\log2$$

$$\therefore\ n=\frac{\log2}{\log(1+r)}. \quad \text{Similarly, if } M=3P, \text{ then } n=\frac{\log3}{\log(1+r)}.$$

If the rate be 5 per cent., then $n=\frac{\log2}{\log1{\cdot}05}=\frac{{\cdot}3010300}{{\cdot}0211893}=14\frac{1}{5}$ years nearly, the time in which a sum of money would double itself.

Also $n=\frac{\log3}{\log1{\cdot}05}=\frac{{\cdot}6771213}{{\cdot}6211893}=22\frac{1}{2}$ years nearly, when a sum would treble itself.

3. The rate of interest is $\frac{100(B-A)}{A}$. 4. See note, p. 6.

6. The rate of interest is $6\frac{1}{4}$ per cent. per annum, and the sum is £108.

7. At simple interest the amount of one farthing for a thousand years at 5 per cent. per annum, is £20 16s. 8d.

At compound interest the amount of one farthing for 1000 years at 5 per cent. per annum, is equal to $\frac{(1{\cdot}05)^{1000}}{960}$.

The characteristic of the logarithm of this number is 18, found from tables calculated to seven places of decimals. It follows that the corresponding natural number to the logarithm will consist of 19 places of figures, the first five of which are 16117, so that the amount exceeds one quadrillion and six hundreds of thousands of trillions of pounds.

8. Here $P(1+r)^m=2P=P(1+2r)^n \ \therefore\ (1+r)^m=(1+2r)^n$, taking the logarithms of these equals, $m\log(1+r)=n\log(1+2r)$

$$\therefore\ \frac{m}{n}=\frac{\log(1+2r)}{\log(1+r)}, \text{ if interest be payable yearly.}$$

$$\frac{m}{n}=\frac{\log(1+r)}{\log(1+\frac{r}{2})}, \text{ half yearly, and } \frac{m}{n}=\frac{\log(1+\frac{r}{2})}{\log(1+\frac{r}{4})}, \text{ quarterly.}$$

10. Let x pounds be the value of one acre, then Nx is the value of N acres, the first mortgage.

When the price of land has risen n per cent, then $N_1x(1+\frac{n}{100})$ is the value of the mortgage.

$$\therefore\ Nx=N_1x(1+\frac{n}{100}), \text{ and } N-N_1=\frac{N.n}{100}.$$

Again the second mortgage is taken on N acres at the increased price, and $Nx(1+\frac{n}{100})$ is the value of the second mortgage; and this is paid off by N_2 acres at the first price of n pounds an acre.

$$\therefore\ Nx(1+\frac{n}{100})=N_2x, \text{ and } N_2-N=\frac{Nn}{100}.$$

$$\text{Hence } \frac{N-N_1}{N_2-N}=\frac{N_1}{N}, \text{ but } \frac{N_1}{N}=\frac{1}{1+\frac{n}{100}}, \ \therefore\ \frac{N-N_1}{N_2-N}=\frac{1}{1+\frac{n}{100}}, \text{ and}$$

$$N-N_1 : N_2-N :: 1 : 1+\frac{n}{100}.$$

XIII.

1. If s denote the space, and t the time, $s=16t^2$.
2. $A=157{\cdot}6$ inches.
3. The velocity of the boat in still water is 5 miles an hour.

The velocity of the stream is 2 miles an hour.

The velocity of rowing *with* the stream is $5+2=7$ miles an hour.

The velocity of rowing *against* the stream $5-2=3$ miles an hour.

$\therefore 7:10::1:1\frac{3}{7}$ hour the time of rowing 10 miles *with* the stream,

and $3:10::1:3\frac{1}{3}$ hours ,, ,, *against* the stream,

$\therefore 1\frac{3}{7}+3\frac{1}{3}=4\frac{16}{21}$ hours, the time of rowing 10 miles *with* and 10 miles *against* the stream.

Although the force of the stream accelerates the boat as much one way as it retards it the other, the *time* is not the same as if the boat had been rowed 20 miles in still water, as 10 miles *with*, and 10 miles *against* the stream would be accomplished in 4 hours.

4. Let x be the number of hours in which the distance a is rowed with the stream, then $t-x$ is the number of hours in which the same distance is rowed against the stream.

Then $x:1::a:\frac{a}{x}$ the rate per hour of rowing *with* the stream,

and $t-x:1::a:\frac{a}{t-x}$ the ,, ,, *against* the stream,

$$\therefore \frac{a}{x}:\frac{a}{t-x}::b:c, \text{ and } \frac{ac}{x}=\frac{ab}{t-x}, \therefore \frac{x}{c}=\frac{t-x}{b},$$

$\therefore x=\frac{ct}{b+c}=$ time *with* the stream, and $b-x=\frac{bt}{b+c}=$ time *against* the stream.

And $a\div\frac{ct}{b+c}=\frac{a(b+c)}{ct}$, time *with* the stream $=$ rate of rowing $+$ rate of stream.

also $a\div\frac{bt}{b+c}=\frac{a(b+c)}{bt}=$ time *against* the stream $=$ rate of rowing $-$ rate of stream.

$$\therefore 2\times\text{rate of stream}=\frac{a(b+c)}{ct}-\frac{a(b+c)}{bt}=\frac{a(b^2-c^2)}{bct}$$

and rate of stream $=\frac{a(b^2-c^2)}{2bct}$ miles an hour.

5. Here a agents produces effect e in time t, then one agent produces effect $\frac{e}{at}$ in an unit of time.

But since the units of b' are not the same as of a, a', suppose x units of the same magnitude as those of a or a' be equal to b'.

Then $a:x::b:b' \therefore x=\frac{ab'}{b}$,

and $a'+b'=a'+\frac{ab'}{b}=\frac{a'b+ab'}{b}$ units of same magnitude as of a or a'.

Then $\frac{a'b+ab'}{b}$ agents produce effect e' in time t',

$\therefore$ 1 agent produces effect $\frac{be'}{(a'b+ab')t'}$, in an unit of time.

$$\text{Hence } \frac{e}{at}=\frac{be'}{(a'b+ab')t'}, \text{ and } \frac{at}{e}=\frac{(a'b+ab')t'}{be'},$$

$$\therefore \frac{e'}{e}=\frac{(a'b+ab')t'}{abt}=\left\{\frac{a'}{a}+\frac{b'}{b}\right\}\cdot\frac{t'}{t}.$$

6. Suppose the pastures in acres, and the times in months.

Let x be the number of oxen required.

And let y denote the quantity of grass on one acre at first,

z the growth on an acre in one month.

Then $y+t_1z=$ grass on one acre in t_1 months

$y+t_2z=$,, ,, t_2 ,,

$y+t_3z=$,, ,, t_3 ,,

And $b_1(y+t_1z)=$ grass on b_1 acres eaten by a_1 oxen
$b_2(y+t_2z)=$,, b_2 ,, a_2 ,,
$b_3(y+t_3z)=$,, b_3 ,, x ,,

Since the quantity of grass eaten is proportional to the number of oxen,

$$\therefore \frac{b_1(y+t_1z)}{b_2(y+t_2z)}=\frac{a_1}{a_2}, \text{ and } z(a_2b_1t_1-a_1b_2t_2)=y(a_1b_2-a_2b_1)$$

$$\frac{b_1(y+t_1z)}{b_3(y+t_3z)}=\frac{a_1}{x}, \ z(b_1xt_1-a_1b_3t_3)=y(a_1b_3-b_1x).$$

$$\text{Whence } \frac{a_2b_1t_1-a_1b_2t_2}{b_1xt_1-a_1b_3t_3}=\frac{a_1b_2-a_2b_1}{a_1b_3-b_1x},$$

from which may be determined

$$x=\frac{t_1-t_3}{t_1-t_2}\cdot\frac{a_2b_3}{b_2}-\frac{t_2-t_3}{t_1-t_2}\cdot\frac{a_1b_3}{b_1}.$$

7. Let x denote the number of hours they travel before they meet; and let y the distance the first has travelled, then the other has travelled $a-y$ miles.

$$\text{Then } y:m-y::x:a, \text{ and } m-y:y:x:b$$

$$\frac{y}{m-y}=\frac{x}{a} \text{ and } \frac{m-y}{y}=\frac{x}{b} \ \therefore \ \frac{x^2}{ab}=1 \text{ and } x=(ab)^{\frac{1}{2}}.$$

$\therefore$ The first performed the journey in $a+(ab)^{\frac{1}{2}}$ hours, and
the second in $b+(ab)^{\frac{1}{2}}$ hours.

9. Let A and B denote the lamps a feet apart, and if the point required be x feet from A, it is $a-x$ feet from B.

Also let m, n denote the intensities of the light one foot from the lamps A, B respectively.

The intensity of light from lamp at distance x from $A : m :: \frac{1}{x^2} : \frac{1}{1^2}$

And intensity of light from lamp at distance $a-x$ from $B :: n : \frac{1}{(a-x)^2} : \frac{1}{1^2}$

$\therefore$ Intensity light of lamp at distance x from $A=\frac{m}{x^2}$

and ,, ,, ,, $(a-x)$ from $B=\frac{n}{(a-x)^2}$.

Then $\frac{m}{x^2}=\frac{2n}{(a-x)^2}$, from which the distance x can be found.

10. The earth and the planet Venus are next in conjunction after 579·858... days, or nearly 580 days.

11. Both the hour and minute hands are together at 12 o'clock. And when the hour hand points to 1 o'clock, the minute hand is at 12. It is required to find a general expression from which may be found when the minute hand is coincident with the hour hand between the hours h and $h+1$.

Suppose the minute hand points to 12 when the hour hand to h hours. Let the hour hand move over x minute divisions on the dial plate, then the minute hand in the same time moves over $5h+x$ minute divisions before it overtakes the hour hand.

But the minute hand moves 12 times as fast as the hour hand,

$$\therefore \frac{x}{5h+x}=\frac{1}{12}, \ 12x=5h+x, \ 11x=5h,$$

$\therefore$ $x=\frac{5h}{11}$, the space the hour hand has moved over,

$12x=\frac{60h}{11}$, the space the minute hand has moved over in the same time.

Here h may be 1, 2, 3, &c., to 11.

If $h=1$, $12x=\frac{60}{11}=5\frac{5}{11}$ minutes past 1 o'clock, when the hands are together between 1 and 2 o'clock.

$h=2$, $12x=\frac{120}{11}=10\frac{10}{11}$ minutes past 2, when the hands are next together, and so for the rest of the coincidences during the 12 hours.

12. The sun and moon are next in conjunction when they are in the 10° 45′ of Cancer.

XIV.

1. One part is $23\frac{1}{3}$ inches, the other $16\frac{2}{3}$.

2. Let x be the length in yards, then $\frac{1}{3}+x$, $1+x$, $220+x$, $1760+x$ form a proportion, and $x=-\frac{550}{2309}$ of one yard, or $-8\frac{1328}{2309}$ inches.

The negative sign indicates that this length is to be subtracted.

3. Let ABC be the given triangle, of which BC is the shortest side; it is required to find a point in the side AC from which a line drawn parallel to BC shall divide the triangle into two equal parts.

Suppose D the required point in AC from which the line DE is drawn meeting the side AB in E, and dividing the triangle into two equal parts.

Let $AD=x$, $DE=y$, and put $AB=c$, $AC=b$, $BC=a$.

$$\text{Then } \frac{\text{triangle } ADE}{\text{triangle } ABC}=\frac{AD^2}{AC^2}=\frac{x^2}{b^2}, \text{ (Euc. VI., 19), also } \frac{\text{triangle } ADE}{\text{triangle } ABC}=\frac{1}{2}.$$

$$\therefore \frac{x^2}{b^2}=\frac{1}{2}, \text{ and } x=\frac{b}{\sqrt{2}}=AD.$$

$\therefore$ The point D in AC is known, from which the line DE is to be drawn parallel to BC.

$$\text{Also } \frac{DE}{DA}=\frac{BC}{CA}, \text{ (Euc. VI., 2), or } \frac{y}{x}=\frac{a}{b},$$

$$\therefore y=\frac{ax}{b}=\frac{a}{b}\times\frac{b}{\sqrt{2}}=\frac{a}{\sqrt{2}}=DE, \text{ the length of the edge.}$$

4. Let x, y denote the weights of the two substances, then $x+y=a$,

$$\text{and } x:y::m:n, \text{ from which, } x=\frac{ma}{m+n}, \text{ and } y=\frac{na}{m+n}.$$

Let z be added to the former, then $x+z:y::m':n'$

$$\therefore n'x+n'z=m'y, \ \therefore z=\frac{m'y-n'x}{n'}; \text{ but } x=\frac{ma}{m+n}, \text{ and } y=\frac{na}{m+n}$$

$$\therefore z=\frac{(mn'-nn')a}{(m+n)n'}.$$

Next let z be taken from the latter.

$$\text{Then } x:y-z::m':n', \ \therefore n'x=m'y-m'z$$

$$\therefore m'z=m'y-n'x, \ \therefore z=\frac{m'y-n'x}{m'}=-\frac{(mn'-m'n)a}{(m+n)m'}.$$

5. The length is $a\left\{\frac{n}{m}\right\}^{\frac{1}{3}}$.

6. In the first vessel the water is $\frac{2}{5}$ and the wine $\frac{3}{5}$; in the second the water is $\frac{3}{10}$ and the wine $\frac{7}{10}$. If x, y denote the quantities drawn from the vessels, so that the quantity of water may be 5 and the wine 11 gallons.

$$\text{Then } \frac{2x}{5}+\frac{3y}{10}=5, \text{ and } \frac{3x}{5}+\frac{7y}{10}=11: \text{ from which } x=2,\ y=14.$$

7. Let the mass consist of x pounds of A and y pounds of B. Then $x+y=p$.
Since p pounds of A loses b pounds when weighed in water

$\therefore p : x :: b : \frac{bx}{p}$ the weight lost by x pounds of A, when weighed in water,

and $p : y :: c : \frac{cy}{p}$ „ „ y „ B „ „

$$\frac{bx}{p}+\frac{cy}{p}=a, \text{ and } bx+cy=pa, \text{ but } bx+by=bp \therefore (c-b)y=p(a-b)$$

$\therefore y=\frac{p(a-b)}{c-b}$ and $x=\frac{p(a-c)}{b-c}$, the portions of which the mass is composed.

This gives a solution of the question King Hiero proposed to Archimedes. The story is told by Vitruvius, and is briefly this. Hiero had delivered to a goldsmith a certain weight of gold to be converted into a votive crown. The king suspected that the crown he received from the smith was not of pure gold, though of the proper weight, but that it was alloyed with silver. He applied to Archimedes to ascertain, without melting the crown, whether it contained alloy. It occurred at once to him, that if a weight of pure gold equal to the weight of the crown were immersed in a vessel full of water, and the quantity of water left in the vessel measured, on the gold being taken out; by doing the same with the crown in the same vessel, he would be able to ascertain whether the bulk of the crown were greater than the bulk of an equal weight of pure gold. According to Vitruvius, as soon as he had discovered the method of solution, he leaped out of the bath and ran hastily through the streets to his own house, shouting εὕρηκα, εὕρηκα!

Suppose the crown weighed 20 ounces. If 19·64 ounces of pure gold lose one ounce when weighed in water, and 10·5 ounces of silver lose one ounce when weighed in water, it will be found that the crown was compounded of 14·78 ounces of gold, and 5·22 ounces of silver alloy.

11. If s denote the volume and x the radius of a globe, then $s \propto x^3$, and $s=px^3$ where p is some constant.

When $x=a, b$, let s become s_1, s_2 respectively, then $s_1=pa^3$, $s_2=pb^3$, and $s_1+s_2=p(a^3+b^3)$. If $s_1+s_2=s$, then $px^3=p(a^3+b^3)$, and $x=(a^3+b^3)^{\frac{1}{3}}$, the radius of the required globe.

12. Here $s \propto x^3$, and $s=px^3$ where p is constant.

When $x=3, 4, 5$, let s be denoted by s_3, s_4, s_5 respectively,

then $s_3=p.3^3$, $s_4=p.4^3$, $s_5=p.5^3$, and $s_3+s_4+s_5=p(3^3+4^3+5^3)$,

but $s=px^3$, $\therefore \frac{s}{s_3+s_4+s_5}=\frac{x^3}{3^3+4^3+5^3}$. When $x=6$, let $s=s_6$,

$$\text{then } \frac{s_6}{s_3+s_4+s_5}=\frac{6^3}{3^3+4^3+5^3}=1, \therefore s_6=s_3+s_4+s_5.$$

14. If x denote the weight at the surface of Jupiter, $x=63·5$ pounds nearly.

EDITED BY

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EUCLID'S ELEMENTS OF GEOMETRY.

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2. The School Edition, with Notes, Geometrical Exercises, &c. 12mo., pp. 418, 4*s.* 6*d.* Books I.—IV., 3*s.* Books I.—III., 2*s.* 6*d.* Books I., II., 1*s.* 6*d.* Book I., 1*s.*

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In the year 1853, the Council of Education at Calcutta were pleased to order the introduction of these Editions of Euclid's Elements into the Schools and Colleges under their control in Bengal.

In the year 1860, a Translation of the Geometrical Exercises was made into the German Language, by Hans H. Von Aller, with a Preface by Dr. Wittstein, and published at Hanover.

At the International Exhibition of 1862, in London, a Medal was awarded to R. Potts, "For the Excellence of his Works on Geometry." *Jury Awards*, Class XXIX., p. 313.

A Medal, as an award of merit, was conferred on Mr. Potts's Educational Works by the Judges at the International Exhibition, Philadelphia, 1876.

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ELEMENTARY ALGEBRA,
WITH BRIEF NOTICES OF ITS HISTORY.

SECTION X.

ARITHMETICAL, HARMONICAL, AND GEOMETRICAL PROGRESSIONS.

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1880.

ARITHMETICAL, HARMONICAL, AND GEOMETRICAL PROGRESSIONS.

Art. 1. Def. An arithmetical progression is a series of quantities which successively increase or decrease by a common difference.

The series of natural numbers, 1, 2, 3, 4, &c., form an increasing arithmetical progression whose common difference is unity. Also, the series of odd numbers, 1, 3, 5, 7, &c., and the series of even numbers, 2, 4, 6, 8, &c., form two increasing arithmetical progressions whose common difference is 2.

And generally if a be the first term, and d the common difference, then a, $a+d$, $a+2d$, $a+3d$, &c., is an increasing arithmetical progression, and a, $a-d$, $a-2d$, $a-3d$, &c., is a decreasing arithmetical progression; and the nth term of the former series is $a+(n-1)d$, and of the latter $a-(n-1)d$.

To ascertain when a series of quantities is in arithmetical progression, subtract the first from the second, and the second from the third, of any three consecutive terms of the series; if the differences are equal, the quantities are in arithmetical progression.

2. Prop. *To find an expression for the sum of any number of terms of an arithmetical progression.*

Let a denote the first term, d the common difference, n the number of terms, and s_n the sum of n terms; then

$$s_n = a+(a+d)+(a+2d)+ \; \ldots \; +\{a+(n-2)d\}+\{a+(n-1)d\},$$
$$s_n = \{a+(n-1)d\}+\{a+(n-2)d\} \; \ldots \; +(a+d)+a.$$

Taking the sum of these equals,

$$2s_n = \{2a+(n-1)d\}+\{2a+(n-1)d\}+ \; \ldots \; +\{2a+(n-1)d\}$$

or $2s_n = n\{2a+(n-1)d\}$,

and $\therefore\ s_n = \frac{n}{2}\{2a+(n-1)d\}$, an expression for the sum of n terms.

If $a+(n-1)d=l$, the last term of the series, then $s_n = \frac{n}{2}\{a+l\}$.

Hence the sum of any number of terms of an arithmetical series is equal to the sum of the first and last terms multiplied by half the number of terms.*

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* From the expression for the sum of an arithmetical progression may be deduced, by taking unity for the first term, and 0, 1, 2, 3, 4, &c., respectively for the common differences, the general terms of certain series. If the number of units in the successive terms of each series be denoted by points, these points can be arranged in the forms of all the regular polygons, and in consequence of this circumstance they have been named polygonal numbers.

In the expression $s=\frac{n}{2}\{2a+(n-1)d\}$, if unity be taken for the first terms, If any three of the four quantities contained in the expression $s_n = \frac{n}{2}\{2a+(n-1)d\}$ be given, the fourth can be found in terms of the three given quantities; also, if $a+(n-1)d=l$, any one of the five quantities can be found in terms of the other four.

3. Prop. *Given the first term and the last term of an arithmetical progression, to find the intermediate terms, or, as they are commonly called, the arithmetical means.*

Let a denote the first, and l the last term, and n the number of intermediate terms.

Then the whole series consists of $n+2$ terms.

Let x denote the common difference.

Then the last or $(n+2)$nd term is $a+(n+1)x$,

$\therefore a+(n+1)x=l$, $(n+1)x=l-a$, and $x=\frac{l-a}{n+1}$ the common difference.

---

and 0, 1, 2, 3, &c., for the successive common differences, the results will give the following general terms of the orders of polygonal numbers :—

$a=1$, $d=0$, give n, the general term of the lineal numbers.
$d=1$, ,, $\frac{1}{2}n(n+1)$,, ,, triagonal numbers.
$d=2$, ,, n^2 ,, ,, quadragonal numbers.
$d=3$, ,, $\frac{1}{2}n(3n-1)$,, ,, pentagonal numbers.
$d=4$, ,, $n(2n-1)$,, ,, hexagonal numbers.
&c. &c.
$d=r-1$,, $\frac{1}{2}n(n-1)(r-1)+n$,, r order of such numbers.

If separate units denote the series of the first order of points, the successive orders of each series of polygonal numbers will be—

1.	The units are	1,	1,	1,	1,	1,	1, &c.
2.	Lineal numbers	1,	2,	3,	4,	5,	6, &c.
3.	Triagonal	1,	3,	6,	10,	15,	21, &c.
4.	Quadragonal	1,	4,	9,	16,	25,	36, &c.
5.	Pentagonal	1,	5,	12,	22,	35,	51, &c.
6.	Hexagonal	1,	6,	15,	28,	45,	66, &c.
	&c.				&c.		
r.	rth order	1,	$(r+1)$,	$3r$,	$2(3r+1)$,	$5(2r-1)$,	$3(5r-3)$, &c.

To find an expression for the sum of n terms of the general series of polygonal numbers of the rth order.

The general term of the rth order of the polygonal numbers is $\frac{1}{2}n(n-1)(r-1)+n$.

By taking $n=1, 2, 3, 4, \ldots n$, respectively, the terms of the series is obtained, and—

The sum of n terms $=\frac{1}{2}\{1.2+2.3+3.4+\ldots+(n-1)n\}(r-1)+(1+2+3+\ldots+n)$.

First the sum of $1+2+3+4+\ldots+n=\frac{n(n+1)}{1.2}$.

Next to find the sum of $1.2+2.3+3.4+\ldots(n-1)n$;

it appears that $1.2=1.(1+1)=1^2+1$, $2.3=2(2+1)=2^2+2$, $3.4=3(3+1)=3^2+3$, &c.

$$\therefore 1.2+2.3+\ldots(n-1)n=1^2+2^2+3^2+\ldots(n-1)^2+\{1+2+3+\ldots(n-1)\}$$

$$=\frac{(n-1)n(2n-1)}{1.2.3}+\frac{n(n-1)}{1.2}=\frac{n(n-1)(n+1)}{1.2.3}$$

$$\therefore \text{sum of } n \text{ terms}=\frac{n(n-1)(n+1)(r-1)}{1.2.3}+\frac{n(n+1)}{1.2}=\frac{n(n+1)}{1.2}\cdot\left\{\frac{(n-1)(r-1)+3}{3}\right\}.$$

Then the 2nd term or 1st mean is $a+x=a+\frac{l-a}{n+1}=\frac{na+l}{n+1}$,

3rd or 2nd . $a+2x=a+\frac{2l-2a}{n+1}=\frac{(n-1)a+2l}{n+1}$,

4th or 3rd $a+3x=a+\frac{3l-3a}{n+1}=\frac{(n-2)a+3l}{n+1}$,

and so on;

and the last $a+nx=a+\frac{nl-na}{n+1}=\frac{a+nl}{n+1}$.

$\therefore \frac{na+l}{n+1}, \frac{(n-1)a+2l}{n+1}$, &c., $\frac{a+nl}{n+1}$, are the n means.

4. Prop. ***Given the first term, the common difference and the sum of an arithmetical progression, to find the number of terms.***

Assuming the notation employed, it has been shewn that the sum of an arithmetical progression of n terms is

$$s_n=\frac{n}{2}\{2a+(n-1)d\}.$$

If r be taken equal to 1, 2, 3, 4, &c., in succession, expressions will be found for the sums of n terms of the series in order of the polygonal numbers.

Thus the sum of n terms of the lineal numbers $=\frac{n(n+1)}{1.2}$.

" " trigonal numbers $=\frac{n(n+1)(n+2)}{1.2.3}$.

" " quadragonal numbers $=\frac{n(n+1)(2n+1)}{1.2.3}$.

" " pentagonal numbers $=\frac{n^2(n+1)}{1.2}$.

" " hexagonal numbers $=\frac{n(n+1)(4n-1)}{1.2.3}$.

&c.

These formulæ can be applied to find the number of balls in a pyramidal pile, whatever may be the form of the base. The piles take their names from the form of the lowest course, which in general is either an equilateral triangle, a square, or a rectangular parallelogram. In all cases each successive course from the lowest has one ball less in each of its sides than the one on which the course rests. The number of courses in the triangular pile and in the square pile is equal to the number of balls in the side of the lowest course, and each pile ends in a single ball.

It is obvious that the sum of n terms of the series of the triagonal and quadragonal numbers will denote the number of balls in a triangular and in a square pile of n courses.

If the pile be rectangular, having m balls in the length and n in the breadth of the lowest courses, its structure shews that it is composed of a square pile of n courses, and $m-n+1$ equal triangular courses lying obliquely in succession to one of the faces of the square pile, so that—

The rectangular pile is equal to the number of balls in the square pile with $(m-n+1)$ triangular courses.

Or, the number of balls $=\frac{n(n+1)(2n+1)}{1.2.3}+(m-n+1).\frac{n(n+1)}{1.2}$,

$$=\frac{n(n+1)}{1.2}.\left\{\frac{2n+1}{3}+m-n+1\right\},$$

$$=\frac{n(n+1)(3m-n+4)}{1.2.3}.$$

$$2s_n = 2na + dn^2 - dn = dn^2 + (2a-d)n,$$

or $dn^2 + 2(n-d)n = 2s_n$, a quadratic equation from which may be deduced the values of n,

$$n = -\left\{\frac{n}{d} - \frac{1}{2}\right\} \pm \left\{\left(\frac{a}{d} - \frac{1}{2}\right)^2 + \frac{2s_n}{d}\right\}^{\frac{1}{2}}.$$

Thus the two values of n obtained, may be both positive, both negative, or one positive and the other negative.

If the two values of n be positive integers, they indicate two series reckoned in the same direction.

If one value be positive and the other negative, the positive value indicates a series of terms reckoned in one direction, and the negative value indicates another series reckoned in a contrary direction. The reckoning of the second series may be considered to begin either from the last term of the first series or from the first term, but in a contrary direction.

In the general expression for the sum of the series

$$s_n = \frac{n}{2}\{2a + (a-1)d\}$$

when $n = +1$, $s = a$, the first term of the given series,
but when $n = -1$, $s = -(a-d)$;

And when $-n$ is written for $+n$ in the general expression,

$$s_n = -\frac{n}{2}\{2a + (-n-1)d\} = -\frac{n}{2}\{2(a-d) + (n-1)(-d)\},$$

$$\therefore\ -s_n = \frac{n}{2}\{2(a-d) + (n-1)(-d)\}$$

which is an expression for the sum of a decreasing arithmetical progression of n terms, of which the first term is $a-d$ and the common difference $-d$.[*]

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[*] 1. If both values of n be positive integers, there are two series which satisfy the data.

If $a = 12$, $d = -2$, $s = 72$; then $n = 12$ and 6.

The first series of 12 terms, 17, 15, 13, 11, 9, 7, 5, 3, 1, −1, −3, −5.

The second series of 6 terms, 17, 15, 13, 11, 9, 7.

And it is obvious, that when the two values of n are positive, the two series have each the same *first* terms.

2. If one value of n be a positive and the other a negative integer, the positive value furnishes an increasing series, and the negative value will indicate a number of terms of the same series reckoned in a reverse order.

If $a = 12$, $d = 4$, $s = 100$; then $n = 5$ and -10.

The first series of 5 terms is, 12, 16, 20, 24, 28.

The value $n = -10$ indicates 10 terms of the same series decreasing, and taken in the reverse order, making 28 the first term of the series. The second series of 10 terms is, 28, 24, 20, 16, 12, 8, 4, 0, −4, −8. And here it is obvious when the two values of n have different signs, the two series have, each of them, the same *last* term, and the second series may be reckoned from the last term in a contrary direction.

3. If one value of n be a positive integer, and the other a positive or negative fraction, the positive integral value of n gives the number of terms of the series.

Hence the positive value of n gives the increasing series

$$a,\ a+d,\ a+2d,\ a+3d,\ \ldots\ldots\ a+(n-1)d,$$

and the negative value of n gives the decreasing series,

$$a-d,\ a-2d,\ a-3d\ \ldots\ldots\ (a-d)-(n-1)d,$$

so that the law of continuity is preserved, and the two series constitute one series, beginning with a, the terms increasing in one direction, and decreasing in the other by the same common difference, thus :—

$$\{a-(n-1)d\}\ \ldots\ldots\ (a-2d),\ (a-d),\ a,\ (a+d),\ (a+2d)\ \ldots\ldots\ \{a+(n-1)d\}.$$

5. Def. A series of quantities is said to be in harmonical progression when the difference between the first and second of any three consecutive terms, has to the difference between the second and third, the same ratio as the first has to the third ; as if a, b, c, d, &c., be a series in harmonical progression, then $a-b : b-c :: a : c$.*

If $a=18$, $d=-5$, $s=33$; then $n=6$ and $\frac{11}{5}$,

The integral value gives the series, 18, 13, 8, 3, -2, -7.

There is no series possible for the fractional value.

4. If both values of n be positive or negative fractions, there are no series which can satisfy such conditions.

If $a=20$, $d=-8$, $s=35$; then $n=\frac{7}{2}$ and $\frac{5}{2}$.

These values of n indicate that no arithmetical progressions can exist, such as the data suppose.

* This definition is thus given by Thomas Masterson in his Third booke of Arithmeticke, which bears on its title-page the motto, "Nothing without labour, All things with reason." 1595. "The 15 definition. Those numbers are in progression harmonical, which are so rehearsed or placed one after other, that the excesse betweene the first and the second, to the excesse betweene the second and the third, hath one and the same reason, that the first hath to the third. Again, the excesse betweene the second and the third, to the excess betweene the third and the fourth, hath one and the same reason that the second hath to the fourth, and so following to the end."

The same form of definition has been extended to arithmetical and geometrical progression.

As a, b, c, d, &c., are said to be in arithmetical progression

when $a-b : b-c :: a : a$.

And a, b, c, d, &c., are said to be in geometrical progression

when $a-b : b-c :: a : b$.

First. To find the arithmetical, geometrical, and harmonical means between two quantities a and c.

Let x, y, z denote respectively the arithmetical, the geometrical, and the harmonical means between a and c.

(1) Since a, x, c are in arithmetical progression,

$$x-a : c-x :: a : a \text{ and } \therefore x-a=c-x,$$

$$\therefore 2x=a+c, \text{ and } x=\tfrac{1}{2}(a+c) \text{ the arithmetical mean.}$$

Hence the arithmetical mean between two quantities is equal to half their sum.

(2) Since a, y, c are in geometrical progression,

$$y-a : c-y :: a : y,\ \therefore y^2-ay=ac-ay,\ y^2=ac,\ \therefore y=(ac)^{\frac{1}{2}}.$$

Hence the geometrical mean between two quantities is equal to the square root of their product.

6. Prop. *To prove that the reciprocals of a series of quantities in harmonical progression are in arithmetical progression.*

Let a, b, c, d, e, &c., be a series of quantities in harmonical progression, it is required to prove that $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$, &c., are in arithmetical progression.

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(3) Since a, z, c are in harmonical progression,

$$z-a : c-z :: a : c, \therefore cz-ac=ac-az, \text{ or } (a+c)z=2ac, \therefore z=\frac{2ac}{a+c}.$$

Hence the harmonical mean between two quantities is equal to twice their product divided by their sum.

Secondly. To determine which is *the greatest* and which is *the least* of the three means between two quantities.

Here $\frac{a+c}{2}$, $(ac)^{\frac{1}{2}}$ and $\frac{2ac}{a+c}$ are the arithmetical, geometrical, and harmonical means respectively.

Suppose $a>c$, then $a-c>0$ and $a^2-2ac+c^2>0$, add $4ac$ to these unequals, and $a^2+2ac+c^2>4ac$, $\therefore a+c>2(ac)^{\frac{1}{2}}$ and $\frac{1}{2}(a+c)>(ac)^{\frac{1}{2}}$, that is, the arithmetical mean is greater than the geometrical mean.

Next, since $\frac{(a+c)}{2}>(ac)^{\frac{1}{2}}$, $\therefore \frac{2}{a+c}<\frac{1}{(ac)^{\frac{1}{2}}}$, and $\frac{2ac}{a+c}<(ac)^{\frac{1}{2}}$,

$$\therefore (ac)^{\frac{1}{2}}>\frac{2ac}{a+c}, \text{ that is,}$$

The geometrical mean is greater than the harmonical mean. But the arithmetical mean is greater than the geometrical. Hence the arithmetical mean is *the greatest*, and the harmonical is *the least* between any two given quantities.

Thirdly. To find the excess of the arithmetical mean above the geometrical, the arithmetical above the harmonical, and the geometrical above the harmonical.

(1). $\frac{a+c}{2}-(ac)^{\frac{1}{2}}=\frac{a-2(ac)^{\frac{1}{2}}+c}{2}=\frac{(a^{\frac{1}{2}}-c^{\frac{1}{2}})^2}{2}$, that is,

The arithmetical mean exceeds the geometrical by $\frac{1}{2}(a^{\frac{1}{2}}-c^{\frac{1}{2}})^2$.

(2). $\frac{a+c}{2}-\frac{2ac}{a+c}=\frac{(a+c)^2-4ac}{2(a+c)}=\frac{a^2-2ac+c^2}{2(a+c)}=\frac{(a-c)^2}{2(a+c)}$, that is

The arithmetical mean exceeds the harmonical by $\frac{(a-c)^2}{2(a+c)}$.

(3). $(ac)^{\frac{1}{2}}-\frac{2ac}{a+c}=\frac{(ac)^{\frac{1}{2}}(a+c-2(ac)^{\frac{1}{2}})}{a+c}=\frac{(ac)^{\frac{1}{2}}(a^{\frac{1}{2}}-c^{\frac{1}{2}})^2}{a+c}$, that is,

The geometrical mean exceeds the harmonical by $\frac{(ac)^{\frac{1}{2}}(a^{\frac{1}{2}}-c^{\frac{1}{2}})^2}{a+c}$.

Lastly. To find the relation of the means.

Since $(ac)^{\frac{1}{2}}\div\frac{a+c}{2}=\frac{2(ac)^{\frac{1}{2}}}{a+c}$, and $\frac{2ac}{a+c}\div(ac)^{\frac{1}{2}}=\frac{2(ac)^{\frac{1}{2}}}{a+c}$.

It follows that $\frac{a+c}{2}$, $(ac)^{\frac{1}{2}}$, and $\frac{2ac}{a+c}$ having a common ratio $\frac{2(ac)^{\frac{1}{2}}}{a+c}$, are in geometrical progression,

$$\text{or } \frac{a+c}{2} : (ac)^{\frac{1}{2}} :: (ac)^{\frac{1}{2}} : \frac{2ac}{a+c}, \text{ that is}$$

The geometrical mean between two quantities is a mean proportional between the arithmetical and harmonical means between the same quantities.

Here $a-b:b-c::a:c$ by definition; $\therefore ab-ac=ac-bc$;

$$\text{and } \frac{ab-ac}{abc}=\frac{ac-bc}{abc};\ \therefore \frac{1}{c}-\frac{1}{b}=\frac{1}{b}-\frac{1}{a};$$

or $\frac{1}{c}, \frac{1}{b}, \frac{1}{a}$ have a common difference.

Next. Since $b-c:c-d::b:d \therefore bc-bd=bd-cd$;

$$\text{and } \frac{bc-bd}{bcd}=\frac{bd-cd}{bcd};\ \therefore \frac{1}{d}-\frac{1}{c}=\frac{1}{c}-\frac{1}{b};$$

or $\frac{1}{d}, \frac{1}{c}, \frac{1}{b}$ have a common difference, and so for the rest of the terms.

Wherefore $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$, &c., are in arithmetical progression.

The converse of this proposition can be proved to be true. If a series of quantities be in arithmetical progression, their reciprocals are in harmonical progression.

7. Prop. *Given the first and last terms of a harmonical progression, to find the intermediate terms.*

Let a denote the first, l the last term, and n the number of intermediate terms; then the harmonical series will consist of $n+2$ terms. But since the reciprocals of quantities in harmonical progression are in arithmetical progression; if n arithmetical means be found between $\frac{1}{a}$ and $\frac{1}{l}$, the reciprocals of these arithmetical means will be the n harmonical means required.

Let x be the common difference of the arithmetical progression of which $\frac{1}{a}$ and $\frac{1}{l}$ are the first and last terms, then $\frac{1}{l}=\frac{1}{a}+(n+1)x$, and $x=\frac{a-b}{(n+1)al}$.

And the arithmetical means are:—

$$\frac{a+nl}{(n+1)al}, \quad \frac{2a+(n-1)l}{(n+1)al}, \quad \frac{3a+(n-2)l}{(n+1)al}, \quad \ldots \quad \frac{na+l}{(n+1)al},$$

hence the required harmonical means are

$$\frac{(n+1)al}{a+nl}, \quad \frac{(n+1)al}{2a+(n-1)bl}, \quad \frac{(n+1)al}{3a+(n-2)l}, \quad \ldots \quad \frac{(n+1)al}{na+l}.$$

8. Def. A geometrical progression is a series of quantities which *increase* or *decrease* by a common ratio.

As 1, 10, 100, 1000, &c., the successive units in the denary scale of notation form an *increasing* geometrical progression, whose first term is 1, and common ratio 10.

Also 1, $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c., form a decreasing geometrical progression whose first term is 1 and common ratio $\frac{1}{10}$.

Generally, if a be the first term and r the common ratio, then a, ar, ar^2, ar^3 is a geometrical progression, and if l denote the nth term of the series, $l=ar^{n-1}$.

A geometrical series is *convergent* when the common ratio is less than unity, but *divergent* when greater than unity.

Def. The sum of a convergent geometrical series, indefinitely continued, is the limit towards which an approximation is made by continually increasing the number of terms.

To ascertain when the terms of a series of quantities are in geometrical progression; divide the second by the first and the third by the second of any three consecutive terms of the series; if the quotients be equal, the series is in geometrical progression.

9. Prop. *To find a formula for the sum of any number of terms of a geometrical progression.*

Let a be the first term, r the common ratio, n the number of terms, and s the sum of n terms.

Then $s = a + ar + ar^2 + ar^3 + ar^4 + \ldots\ldots + ar^{n-2} + ar^{n-1}$,

and $rs = ar + ar^2 + ar^3 + ar^4 + \ldots\ldots + ar^{n-2} + ar^{n-1} + ar^n$.

By subtracting these equals,

$$\therefore (r-1)s = ar^n - a = a(r^n - 1),$$

$$\therefore s = \frac{a(r^n-1)}{r-1}, \text{ the sum } n \text{ terms of the series.}$$

It is also obvious that any three of the four quantities a, r, n, s, in this expression being given, the fourth may be determined.

10. Prop. *To find a formula for the sum of a geometrical progression continued indefinitely, the common ratio being a proper fraction.*[*]

It has been shewn that n terms of a geometrical series is expressed

$$\text{by } s = \frac{a(r^n-1)}{r-1}.$$

If $r = \frac{1}{x}$, a proper fraction, then $r^n = \frac{1}{x^n}$. As n increases, the value of x^n increases, and when n becomes indefinitely great, then $\frac{1}{x^n}$ becomes indefinitely small, and therefore less than any assignable quantity. Hence r^n or $\frac{1}{x^n}$ may be rejected without affecting the result

If s' denote the sum of the series indefinitely continued,

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[*] It may objected, that since all the terms of an infinite series are unassignable, it is therefore impossible to determine its sum. In order to obviate this, let it be considered that *a number actually infinite* is a plain contradiction to all ideas of number. For every number of which the mind can actually form any suitable idea, *is*, and *must be*, always finite, so that a number greater than any finite number can be assigned, and a greater than that last, and so on, without ever coming to an end of the increase of numbers that can be assigned. This endless progression of numbers is all that can be understood by the infinity of numbers, and to assert the number of any things is infinite, is to affirm that such a number is beyond the limit of finite numbers, and cannot be considered as having any definite relation to any finite numbers whatever.

$$s=\frac{-a}{r-1}=\frac{a}{1-r}.$$

The expression $\frac{a}{1-r}$, which is called *the sum of the series*, is more properly the limit to which the sum of the terms continually approximates, but never actually attains. The expression, however, is a true representation for the sum of the series continued indefinitely, for the series itself is reproduced by dividing a by $1-r$.*

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* The principles of arithmetical and geometrical progressions are applicable for finding the amount of annuities and their present values, both for a certain number of years, and in perpetuity.

To find an expression for the amount of an annuity, payable for a certain number of years.

Let A denote the annuity, or annual payment, r the interest of one pound for one year, and M the amount in n years.

1. If simple interest be allowed.

It has been shewn that the amount of P pounds, at simple interest, in n years, is $P(1+nr)$.

At the end of the first year A becomes due, and at the end of $(n-1)$ years amounts to $A\{1+(n-1)r\}$.

And, in like manner, at the end of the second, third, &c., years, A becomes due, and in $(n-2)$, $(n-3)$, &c., years, amounts respectively to $A\{1+(n-2)r\}$, $A\{1+(n-3)r\}$, &c.

Hence the whole amount $M=nA+rA\{1+2+3+4+\ldots(n-1)\}$

$$=nA+\frac{n(n-1)}{2}Ar.$$

2. At compound interest.

The amount of P pounds at compound interest for n years is $P(1+r)^n$.

At the end of the first year the first payment A becomes due, and in $(n-1)$ years amounts to $A(1+r)^{n-1}$.

Similarly, at the end of the second, third, &c., years, A becomes due, and in $(n-2)$, $(n-3)$, &c., years, amounts to $A(1+r)^{n-2}$, $A(1+r)^{n-3}$, &c., respectively.

Hence the whole amount $M=A\{1+(1+r)+(1+r)^2+(1+r)^3+\ldots(1+r)^{n-1}\}$

$$=\frac{A\{(1+r)^n-1\}}{(1+r)-1}=\frac{A}{r}\{(1+r)^n-1\}.$$

If the annuity be payable half-yearly or quarterly, the amounts will be respectively $\frac{A}{r}\cdot\left\{\left(1+\frac{r}{2}\right)^{2n}-1\right\}$, and $\frac{A}{r}\left\{\left((1+\frac{r}{4}\right)^{4n}-1\right\}$.

To find expressions (1) *for the present value of an annuity A, and* (2) *for the present value of the amount of an annuity A to be paid for n years.*

1. If simple interest be allowed.

Since the present values of the annuity A due at the end of the first, second, third, &c., years, are $\frac{A}{1+r}$, $\frac{A}{1+2r}$, $\frac{A}{1+3r}$, &c., respectively,

$\therefore$ the present value of the n annuities $=A\left\{\frac{1}{1+r}+\frac{1}{1+2r}+\frac{1}{1+3r}+\ldots\frac{1}{1+nr}\right\}$ which is a series in harmonical progression.

Next, if P denote the present value of the amount M of the annuities for n years, then $M=P(1+nr)$.

11. **Prop.** *Given the first term and the last term of a geometrical progression of n terms, to find the intermediate terms.*

Let a denote the first and l the last term, and n be the number of intermediate terms.

The whole series consists of $n+2$ terms.

Let x be the common ratio.

The last, or $(n+2)$th term of the series is ax^{n+1},

$$\therefore\ ax^{n+1}=l,\ x^{n+1}=\frac{l}{a},\ \text{and } x=\left\{\frac{l}{a}\right\}^{\frac{1}{n+1}}.$$

$$\text{The 2nd term is } ax=a\left\{\frac{l}{a}\right\}^{\frac{1}{n+1}}=(a^{n}l)^{\frac{1}{n+1}},$$

$$\text{3rd}\qquad ax^{2}=a\left\{\frac{l}{a}\right\}^{\frac{2}{n+1}}=(a^{n-1}l^{2})^{\frac{1}{n+1}},$$

And the amount of the n annuities $=\dfrac{nA\{(n-1)r+2\}}{2}$

$$\therefore\ P(1+nr)=\frac{nA\{(n-1)r+2\}}{2}$$

and $P=\dfrac{nA\{(n-1)r+2\}}{2(1+nr)}$, the present value of the amount of the annuity.

Hence it is obvious that the present value of the n annuities is not equal to the present value of the amount of the n annuities when calculated at simple interest.

This discrepancy arises from the fact that no consideration is allowed for the interest due and not accounted for in the several successive payments, and proves that the principle of calculating annuities by simple interest is fallacious.

The reason of the inequality may be explained.

Let P denote the present value of A due one year hence, then $P=\dfrac{A}{1+r}$ and $P+Pr=A$.

If A be left unpaid for a second year, and charged with interest, it amounts to $A(1+r)$. But the amount of P in two years is $P(1+2r)$ or $\dfrac{A(1+2r)}{1+r}$, which is different from $A(1+r)$ the amount of A, by the quantity $\dfrac{Ar^{2}}{1+r}$.

Here Pr, the interest on P for the first year, is not charged with interest for the second year; and therefore in one case, A, the annual payment was charged with interest, and in the other, only P, the present worth of it. Consequently P, the present value of A, is not the present value of the amount of A after any number of years, and the same reasoning may be applied to every payment.

If compound interest be allowed.

The present values of the n annuities due at the end of the first, second, third, &c., years, are $\dfrac{A}{1+r}$, $\dfrac{A}{(1+r)^{2}}$, $\dfrac{A}{(1+r)^{3}}\dots\dfrac{1}{(1+r)^{n}}$ respectively,

$$\therefore\ \text{the present value of all the payments}=\frac{A}{1+r}\left\{1+\frac{1}{1+r}+\frac{1}{(1+r)^{2}}+\dots\frac{1}{(1+r)^{n-1}}\right\}$$

$$=\frac{A}{r}\cdot\frac{(1+r)^{n}-1}{(1+r)^{n}}.$$

4th $$ax^3 = a\left\{\frac{l}{a}\right\}^{\frac{3}{n+1}} = (a^{n-2}l^3)^{\frac{1}{n+1}},$$

and so on,

nth $$ax^n = a\left\{\frac{l}{a}\right\}^{\frac{n}{n+1}} = (al^n)^{\frac{1}{n+1}}.$$

And $\therefore$ $(a^n l)^{\frac{1}{n+1}}, (a^{n-1}l^2)^{\frac{1}{n+1}}, \dots \dots (al^n)^{\frac{1}{n+1}}$, are the n intermediate terms, or the geometrical means, as they are usually called.

12. Prop. *To find an expression for the sum of a series of quantities in geometrical progression, but having their coefficients in arithmetical progression.*

Let a, $(a+d)x$, $(a+2d)x^2$, &c., be the series, and let s denote the sum of n terms.

Then $s = a+(a+d)x+(a+2d)x^2+ \dots\dots +\{(a+(n-1)d\}x^{n-1}$,

$$xs = ax+(a+d)x^2+ \dots\dots +\{(a+(n-2)d\}x^{n-1}+\{a+(x-1)d\}x^n,$$

$$\therefore s(1-x) = a+dx+dx^2+ \dots\dots +ndx^{n-1}-\{a+(n-1)d\}x^n$$

$$= a+\frac{dx(1-x^{n-1})}{1-x}-\{a+(n-1)d\}x^n,$$

$$\therefore s = \frac{a-\{a+(n-1)d\}x^n}{1-x}+\frac{dx(1-x^{n-1})}{(1-x)^2}.$$

Next let P denote the present value of the amount M of the annuity A for n years. If P were put out to interest for n years it should amount to the same sum as the amount of the annuities for that period.

Now the amount M of P in n years $= P(1+r)^n$.

And the amount of the annuities in n years $= \frac{A}{r}\{(1+r)^n - 1\}$

$$\therefore P(1+r)^n = \frac{A}{r}\{(1+r)^n-1\}, \text{ and } P = \frac{A\{(1+r)^n-1\}}{r(1+r)^n}.$$

Hence it appears that the present value of an annuity for n years is equivalent to the present value of the amount of the payments of the annuity when compound interest is allowed.

To deduce an expression for the present value of an annuity to continue in perpetuity.

If simple interest be allowed.

Since the present value of an annuity for n years at simple interest $= \frac{n^2Ar-2Ar+nA}{2+2nr}$.

If n become indefinitely great, the finite number 2 in the denominator, in comparison with $2nr$, an indefinitely great number, may be rejected without affecting the result, and

The present value of the annuity A in perpetuity $= \frac{n^2Ar-nAr+2nA}{2nr}$

$$= \frac{nAr-Ar+2A}{2r}$$

$= \frac{nAr}{2r}$, by rejecting $2A-Ar$, being finite, If x be less than unity, and the series be continued indefinitely, or n become indefinitely great, then x^{n-1} and x^n become less than any assignable quantity, and the limit of the sum of the series becomes $\frac{a}{1-x}+\frac{dx}{(1-x)^2}$ or $\frac{a-(a-d)x}{(1-x)^2}$.

13. Prop. *To find the sum of a series of fractions whose numerators are in arithmetical progression and denominators in geometrical progression.*

Let $a, \frac{a+d}{x}, \frac{a+2d}{x^2}$, &c., be the series of fractions, and let s denote the sum of n terms.

$=\frac{nA}{2}$, an indefinitely large sum to be paid for a definite yearly payment in perpetuity.

This conclusion further confirms the fallacy of the principle of calculating annuities by simple interest.

If compound interest be allowed.

The present value for n years $=\frac{A}{r}\cdot\frac{(1+r)^n-1}{(1+r)^n}=\frac{A}{r}\cdot\left\{1-\frac{1}{(1+r)^n}\right\}$.

When n the number of years becomes indefinitely great, then $(1+r)^n$ also becomes indefinitely great; and consequently $\frac{1}{(1+r)^n}$ becomes indefinitely small, and therefore less than any assignable number.

It may be rejected without affecting the result. And the present value of an annual payment of A in perpetuity $=\frac{A}{r}$.

To find an expression for the present value of an annuity to commence after the expiration of m years, and the payments to continue for n years, allowing compound interest.

It is obvious that the present value of such an annuity will be the difference between the present value of the annuity for $(m+n)$ years and for m years.

The present value for $(m+n)$ years $=\frac{A}{r}\left\{\frac{(1+r)^{m+n}-1}{(1+r)^{m+n}}\right\}$,

and for m years $=\frac{A}{r}\left\{\frac{(1+r)^m-1}{(1+r)^m}\right\}$.

Hence the present value for n years after the expiration of m years

$$=\frac{A}{r}\left\{\frac{(1+r)^{m+n}-1}{(1+r)^{m+n}}-\frac{(1+r)^m-1}{(1+r)^m}\right\}$$

$$=\frac{A}{r}\left\{\frac{(1+r)^n-1}{(1+r)^{m+n}}\right\}.$$

To find the present value of an annuity to commence after the expiration of m years, and to continue in perpetuity.

Here the present value for m years $=\frac{A}{r}\cdot\frac{(1+r)^m-1}{(1+r)^m}$

And the present value in perpetuity $=\frac{A}{r}$.

$\therefore$ The present value of the annuity after m years $=\frac{A}{r}\left\{1-\frac{(1+r)^m-1}{(1+r)^m}\right\}$

$$=\frac{A}{r}\cdot\frac{1}{(1+r)^m}.$$

Then $s=a+\frac{a+d}{x}+\frac{a+2d}{x^2}+\ .\ .\ .\ \frac{a+(n-1)d}{x^{n-1}}$.

$$\frac{s}{x}=\frac{a}{x}+\frac{a+d}{x}+\ .\ .\ .\ .\ \frac{a+(n-2)d}{x^{n-1}}+\frac{a+(n-1)d}{x^n},$$

$$\frac{x-1}{x}.s=a+\left\{\frac{d}{x}+\frac{d}{x^2}+\ .\ .\ .\ .\ +\frac{d}{x^{n-1}}\right\}-\frac{a+(n-1)d}{x^n}$$

$$=a+\frac{d(1-x^{n-1})}{x^{n-1}(1-x)}-\frac{a+(n-1)d}{x^n}.$$

$$\therefore\ s=\frac{a+(n-1)d}{x^{n-1}(1-x)}-\frac{ax}{1-x}-\frac{d(1-x^{n-1})}{x^{n-2}(1-x)^2}.$$

If the series be continued indefinitely, n becomes indefinitely great, and if x be greater than unity; then x^{n-1} and x^{n-2} become indefinitely great, and the limit of the sum of the series is

$$\frac{dx}{(1-x)^2}-\frac{ax(1-x)}{(1-x)^2},\ \text{or}\ \frac{ax^2-ax+dx}{(x-1)^2}.$$

14. Prop. *To find expressions,* (1) *for the sum of the squares,* (2) *for the sum of the cubes of the terms of an arithmetical progression.*

Let a_1, a_2, a_3, ... a_n denote the n terms of an arithmetical progression whose common difference is d;

and let $s_1=a_1+(a_1+d)+(a_1+2d)+\ .\ .\ .\ .\ +\{a_1+(n-1)d\}$

$$=\frac{n}{2}\left\{2a_1+(n-1)d\right\}.$$

$$s_2=a_1^2+(a_1+d)^2+(a_1+2d)^2+\ .\ .\ .\ .\ +\{a_1+(n-1)d\}^2.$$

$$s_3=a_1^3+(a_1+d)^3+(a_1+2d)^3+\ .\ .\ .\ .\ +\{a_1+(n-1)d\}^3.$$

(1) For the sum of the squares.

$$a_2^3-a_1^3=(a_1+d)^3-a_1^3=3a_1^2d+3a_1d^2+d^3,$$
$$a_3^3-a_2^3=(a_2+d)^3-a_2^3=3a_2^2d+3a_2d^2+d^3,$$
$$a_4^3-a_3^3=(a_3+d)^3-a_3^3=3a_3^2d+3a_3d^2+d^3,$$
&c., &c.,
$$a_{n+1}^3-a_n^3=(a_n+d)^3-a_n^3=3a_n^2d+3a_nd^2+d^3.$$

Taking the sums of these equals, and writing a_1+nd for a_n+d,

$$\therefore\ (a_1+nd)^3-a_1^3=3ds_2+3d^2s_1+nd^3,$$

$$\therefore\ s_2=\frac{(a_1+nd)^3-a_1^3-3d^2s_1-nd^3}{3d}.$$

(2) For the sum of the cubes.

$$a_2^4-a_1^4=(a_1+d)^4-a_1^4=4a_1^3d+6a_1^2d^2+4a_1d^3+d^4,$$
$$a_3^4-a_2^4=(a_2+d)^4-a_2^4=4a_2^3d+6a_2^2d^2+4a_2d^3+d^4,$$
&c., &c.,
$$a_{n+1}^4-a_n^4=(a_n+d)^4-a_n^4=4a_n^3d+6a_n^2d^2+4a_nd^3+d^4.$$

Taking the sums of these equals, and writing a_1+nd for a_n+d,

$$\therefore\ (a_1+nd)^4-a_1^4=4ds_3+6d^2s_2+4d^3s_1+nd^4,$$

$$\therefore\ s_3=\frac{(a_1+nd)^4-a_1^4-6d^2s_2-4d^3s_1-nd^4}{4d},$$

in which expression both s_1 and s_2 have been found.

This method may be applied to find the sums of the fourth, &c., powers of the terms of the series.*

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* These formulæ may be readily employed for finding the sums to n terms of the first, second, third, &c., powers of a series of the natural numbers, which form an arithmetical progression whose first term and common difference is unity.

(1) The sum of the first powers of the first n natural numbers,

$$s_1=1+2+3+4+\;.\;.\;.\;.\;+n=\frac{n(n+1)}{1.2}.$$

(2) The sum of the second powers, $s_2=\dfrac{(a_1+nd)^3-a_1{}^3-3d^2s_1-nd^3}{3d}$.

Here $a_1=1$, $d=1$, $n=n$, and $s_1=\dfrac{n(n+1)}{2}$,

$$\therefore\; s_2=1^2+2^2+3^2+4^2+\;.\;.\;.\;.\;+n^2=\frac{n(n+1)(2n+1)}{1\,.\,2\,.\,3}.$$

(3) The sum of the third powers, $s_3=\dfrac{(a_1+nd)^4-a_1{}^4-6d^2s_2-4d^3s_1-nd^4}{4d}$,

as before $a_1=1$, $d=1$, $n=n$, and $s_2=\dfrac{n(n+1)(2n+1)}{1\,.\,2\,.\,3}$,

$$\therefore\; s_3=1^3+2^3+3^3+4^3+\;.\;.\;.\;.\;+n^3=\frac{n^2(n^2+2n+1)}{4}=\left\{\frac{n(n+1)}{1\,.\,2}\right\}^2.$$

Hence it appears that $1^3+2^3+3^3+\;.\;.\;.\;.\;+n^3=(1+2+3+\;.\;.\;.\;+n)^2$.

In a similar manner may be found.

$$s_4=1^4+2^4+3^4+\;.\;.\;.\;.\;+n^4=\frac{n(n+1)(2n+1)(3n^2+3n-1)}{2\,.\,3\,.\,5},$$

$$s_5=1^5+2^5+3^5+\;.\;.\;.\;.\;+n^5=\frac{n^2(n+1)^2(2n^2+2n-1)}{1\,.\,3\,.\,4},$$

$$s_6=1^6+2^6+3^6+4^6+\;.\;.\;.\;.\;+n^6=\frac{n(n+1)(2n+1)(3n^4+6n^3-3n+1)}{2\,.\,3\,.\,7}.$$

The general doctrine of series, and especially of infinite series, is perhaps one of the most useful and difficult subjects of mathematical science. Only the most elementary properties of Arithmetical, Harmonical, and Geometrical series have been exhibited in these pages. Archimedes was the first who determined the limit of the sum of an infinite series of squares, whose roots are in arithmetical progression, and the common difference equal to the first term. In the Lilavati are given rules for finding the sum of the squares and the sum of the cubes of the natural numbers. Wallis, in his "Arithmetica Infinitorum," 1655, first shewed the quotient of a divided by $1-r$ produced the infinite series $a+ar+ar^2+$ &c., which became the germ of many important improvements. Demoivre was led to the discovery of recurring series by his inquiries in the calculation of probabilities, which constitute a part of his treatise entitled, "Miscellanea Analytica de Seriebus, &c., 1730." Newton, Brounker, the Bernoullis, Mercator, James Gregory, and others in later times, have largely extended the knowledge of properties of series, the investigations of which are dependent on principles and processes in the more advanced parts of the science.

EXERCISES.

I.

Find the sums of the following arithmetical series :—

1. $1+2+3+4+\;.\;.\;.\;.$ to n terms.
2. $1+3+5+7+\;.\;.\;.\;.$ to n terms.
3. $2+4+6+8+\;.\;.\;.\;.$ to n terms.
4. $2+8+14+\;.\;.\;.\;.$ to 12 terms.
5. $3+10+17+\;.\;.\;.\;.$ to 40 terms.
6. $1+9+17+\;.\;.\;.\;.$ to 100 terms.
7. $1+5+9+\;.\;.\;.\;.$ to 1,000 terms.
8. $0+3+6+\;.\;.\;.\;.$ to 51 terms.
9. $16+14+12+\;.\;.\;.\;.$ to 25 terms.
10. $6+2-2-\;.\;.\;.\;.$ to 31 terms.
11. $-9-7-5-\;.\;.\;.\;.$ to 20 terms.
12. $-5-3-1-\;.\;.\;.\;.$ to 72 terms.
13. $1-3+5-7+\;.\;.\;.\;.$ to 45, and to $2n$ terms.
14. $1-8+15-22+\;.\;.\;.\;.$ to 50, and to $2n+1$ terms.

II.

Find the sums of the arithmetical progressions :—

1. $1+\frac{3}{2}+2+\frac{5}{2}+\;.\;.\;.\;.$ to 12 terms.
2. $5+4\frac{1}{12}+3\frac{1}{6}+\;.\;.\;.\;.$ to 20 terms.
3. $3+\frac{9}{2}+6+\;.\;.\;.\;.$ to 17 terms.
4. $\frac{5}{7}+\frac{2}{9}+\;.\;.\;.\;.$ to 50, and to n terms.
5. $\frac{1}{2}+\frac{1}{3}+\frac{1}{6}+\;.\;.\;.\;.$ to 100, and to $2n$ terms.
6. $\frac{1}{3}+\frac{2}{3}+1+\;.\;.\;.\;.$ to 51 terms.
7. $\frac{1}{6}+\frac{1}{3}+\frac{1}{2}+\;.\;.\;.\;.$ to 100 terms.
8. $a-2a+3a-4a+\;.\;.\;.\;.$ to $2n$ terms.
9. $(a+n)^2+(a^2+n^2)+(a-n)^2+\;.\;.\;.\;.$ to $4n$ terms.
10. $\frac{n-1}{n}+\frac{n-2}{n}+\frac{n-3}{n}+\;.\;.\;.\;.$ to n and to $2n$ terms.
11. $1+\frac{n-3}{n-1}+\frac{n-5}{n-1}+\;.\;.\;.\;.$ to $3n$ terms.
12. $\frac{a+b}{a-b}+\frac{3a-2b}{a+b}+\;.\;.\;.\;.$ to $(2n+1)$ terms.
13. $\cdot1+\cdot2+\cdot3+\cdot4+\;.\;.\;.\;.$ to 100 terms.
14. $\frac{a}{n^2}+\frac{2a}{n^2}+\frac{3a}{n^2}+\;.\;.\;.\;.$ to n terms.

III.

1. Insert three arithmetical means between ·012 and ·132.
2. Insert nine arithmetical means between 9 and 109.

3. Find ten arithmetical means between a and b.

4. Find a series of arithmetical means between 1 and 21, such that their sum has to the sum of the two greatest of them, the ratio of 11 to 4.

5. The sum of n arithmetical means between 1 and 19 is to the sum of the first $n-2$ of them as 5 is to 3; find the means.

6. The sum of any number of arithmetical means between a and b is to the same number between $\frac{1}{a}$ and $\frac{1}{b}$, as ab is to unity.

7. Shew that the difference between the sum of an arithmetical series and the sum of the arithmetical means between every two consecutive terms, is equal to half the sum of the first and last terms of the arithmetical series.

8. If n arithmetical means be inserted between a and b; shew that the sum of the whole series is $\frac{1}{2}n(a+b)$, and $n=\frac{a-b+m}{a-b-m}$, where m is the difference between the first and last mean.

IV.

1. In every arithmetical progression, shew that the sum of any two terms equidistant from the first and last terms is equal to the sum of the first and last terms; and the sum of the first and third of three equidistant terms, is equal to twice the second.

2. The successive second differences of the squares of the natural numbers form an arithmetical progression whose common difference is 2.

3. Write down the arithmetical series whose 7th and 10th terms are respectively 15 and 21.

4. Shew that a certain number of terms of an arithmetical progression may be found of which the algebraic sum is equal to zero, provided that twice the first term be divisible by the common difference, and the series ascending or descending as the first term is negative or positive.

5. The sum of an even number of terms of any arithmetical progression, whose common difference is equal to the least term, is four times the sum of half that number of terms diminished by half the last term.

6. In an arithmetical series, if the common difference be equal to twice the first term; prove that the quotient of the sum divided by the first term is a perfect square.

7. If s denote the sum of n terms of the series of the natural numbers beginning with unity; shew that $8s+1$ is a square number.

8. Having given the sum, the first term, and common difference of an arithmetical progression, find the number of terms, and shew generally, that if one of the roots be a negative integer, the solution may be adapted to another series.

9. The first two terms of an arithmetical progression being together equal to 18, and the three next to 12; how many terms beginning with the first must be taken to make 28? Explain the reason of the double solution.

10. The sum of x terms of the series 3, 4, 5, &c., is 25, and y terms of the series 5, 4, 3, &c., is 14; find the values of x and y, and explain their meaning.

11. The squares of all the odd numbers are found among the terms of the arithmetical series 1, 9, 17, 25, 33, 41, &c. Prove this, and shew at what intervals they occur. Apply the result to discover how many of the first 100 terms of the series are not square numbers.

12. How many terms of the series 1, 3, 5, 7, &c., are equal to the number 123454321?

V.

1. Find the first term, common difference, and number of terms of the arithmetical series, the mth and nth terms of which are a and b, and its last term $a+b$.

2. Given the $(m+n)$th and the $(m-n)$th terms of an arithmetical series; find the mth and nth terms; and conversely.

3. The nth term of an arithmetical series is $\frac{1}{6}(3n-1)$, and the sum of n terms is $\frac{n}{12}(3n+1)$; find the series.

4. The sum of n terms of an arithmetical progression is $n(a-b)^2+n^2ab$; determine the series, and write the nth term.

5. Prove that 1, 3, 5, &c., is the only arithmetical series whose first term is 1, in which the first half of any even number of terms is one-third of the second half.

6. Of the two series 2, 5, 8, 11, &c., and 3, 7, 11, &c., each extending to 100 terms; how many terms of the two series will be identical?

7. If the sum of m terms of an arithmetical progression be equal to the sum of the next n terms and also equal to the sum of the next p terms; prove that $\frac{(m+n)(p-n)}{np}=\frac{(n+p)(n-m)}{mn}$.

8. If a, b be the first and nth terms of an arithmetical series, a', b' the pth terms from the extremes, and d the common difference; shew that $a'b'-ab=(p-1)(n-p)d^2$.

9. If s be the sum and d the common difference of an arithmetical series of n terms; shew that the difference of the squares of the first and nth terms is $\frac{2(n-1)ds}{n}$.

10. If the series of the natural numbers be arranged in the groups 1, 2+3, 4+5+6, &c.; find the sum of the nth group. Also if the odd numbers be arranged in a similar manner, 1, 3+5, 7+9+11, &c.; find the sum of the nth group.

11. If there be a series of terms forming an arithmetical progression whose common difference is d, and if sets of p consecutive terms be taken, the first of each set being the same as the last term of the preceding set, then the series of the sets will form an arithmetical progression whose common difference is $p(p-1)d$.

12. If a series of terms in arithmetical progression be divided into groups of n terms in each group, and the terms of each group be added together; the results shall form an arithmetical progression whose common difference is n^2 times the original common difference.

VI.

1. If a, b, c be in arithmetical progression; shew that

$$(a-c)^2 = 2\{(a-b)^2+(b-c)^2\}.$$

2. If $\frac{1}{b+c}$, $\frac{1}{a+c}$, $\frac{1}{a+b}$ be in arithmetical progression; shew that $a^2+c^2=2b^2$.

3. If $\frac{a}{b-c}$, $\frac{b}{c-a}$, $\frac{c}{a-b}$, be in arithmetical progression, prove that $\frac{a^3+c^3-2b^3}{a^2+c^2-2b^2}=\frac{a+b+c}{2}$.

4. If $\frac{a}{b+c-a}$, $\frac{b}{c+a-b}$, $\frac{c}{a+b-c}$ be in arithmetical progression, prove that $\frac{a-c}{b-c}=\frac{2b}{a+b-c}$.

5. If a, b, c be the mth, nth, and pth terms respectively of an arithmetical progression; shew that $(m-n)c+(p-m)b+(n-p)a=0$.

6. If a, b, c, d be in arithmetical progression,

then shall $4(ab+bc+cd+da)=a^2+b^2+c^2+d^2+6(ac+bd)$,

and $3(ac+bd)-3(bc+ad)=a^2+d^2+2(b^2+c^2)-2(ad+bc+cd)$.

VII.

1. If s_1, s_2, s_3 be the sums of n terms of three arithmetical progressions of which the first term of each is unity, and the common differences 1, 2, 3, respectively, then shall $s_3+s_1=2s_2$.

2. If s_1, s_2, s_3 be the sums of n, $2n$, $3n$ terms respectively of an arithmetical series; shew that $s_3=3(s_2-s_1)$.

3. If s_n represent the sum of n terms of an arithmetical series; shew that $s_{n+4}-2s_{n+3}+s_{n+2}=b$, the common difference.

4. There are n arithmetical series, the first term of each of which is a and common difference b; shew that if there be taken n terms of the first series, $(n-1)$ of the second, &c., to the nth inclusive, the sum of the results will be $\frac{n(n+1)}{1.2}.\{a+\frac{1}{3}(n-1)b\}$.

5. If there be n arithmetical series, the first terms of which are

1, 2, 3 .... and their common differences 3, 5, 7 ....; shew that the sum of n terms of all the series is $\frac{1}{2}\{2n^3+(n-1)^2\}$.

6. There are m arithmetical series, each beginning with 1, and having 1, 2, 3, &c., as their common differences; shew that the sum of their nth terms is $\frac{1}{2}\{(n-1)m^2+(n+1)m\}$.

7. If s_1, s_2, $s_3 \ldots s_p$ denote the sums of p arithmetical progressions continued to n terms; the first terms being 1, 2, 3, &c., and the common differences 1, 3, 5, &c.; shew that $s_1+s_2+s_3+\ldots s_p=\frac{1}{2}(np+1)np$.

8. If s_1, s_2, $s_3 \ldots s_p$ denote the sums of p arithmetical series each to n terms, the first terms are the first p even numbers, and the common differences are the first p odd numbers respectively; shew that

$$s_1+s_2+s_3+\ldots+s_p=np+\frac{n(n+1)}{1.2}.p^2.$$

9. If s_1, s_2, s_3, $s_4 \ldots s_{2n}$ denote the sums of n terms of $2n$ arithmetical progressions whose first terms are the same, and whose common differences are respectively b, $2b$, $3b$, ... $2nb$; shew that

$$s_2+s_4+s_6+\ldots+s_{2n}-\{s_1+s_3+s_5+\ldots+s_{2n-1}\}=\frac{n^2(n-1).b}{2}.$$

VIII.

1. How many strokes does a clock strike in twelve hours?

2. The three digits of a certain number are in arithmetical progression; if the number be divided by the sum of the digits in the units and tens places, the quotient is 107; but if 396 be subtracted from the number its digits will be inverted. What is the number?

3. The interior angles of a rectilinear figure are in arithmetical progression; the least angle is 120°, and the common difference is 5°; find the number of sides, and construct the figure.

4. A debt can be discharged by paying 2s. 6d. the first week, 5s. the second, and so on, in one year; what is the last payment and the amount of the debt?

5. If 100 stones be placed in a straight line in succession, one yard from each other, the first being one yard from a basket; what distance will a person go over who gathers them up singly, returning with each to the basket?

6. *A* sets off and travels 5 miles the first hour, 6 the second, 7 the third, and so on; *B* sets off 3 hours later, and travels 15 miles an hour; find when *B* overtakes *A*, and when *A* overtakes *B*.

7. Suppose two persons, *A* and *B*, go round the world at the equator. *A* goes eastward, one mile the first day, two the second, and so on, in arithmetical progression. *B* goes westward, at an uniform rate of 20 miles a day. Find where they will meet, and interpret the negative result, supposing the circuit of the earth to be 23,661 miles.

8. It was observed that from the 8th to the 19th of June of a certain year the thermometer rose half a degree daily, and that the

arithmetical mean of these twelve different positions of the thermometer was $18\frac{3}{4}$ degrees. At what degree did it stand on the 8th day of June?

IX.

1. Insert three harmonical means between a and b.

2. Insert four harmonical means between 6 and 26.

3. Insert six harmonical means between 1 and 20.

4. Continue the harmonical progression .... 3, 4, 6, ... both ways; how far can it be continued?

5. If $b-a$ be the harmonical mean between $c-a$ and $d-a$; then shall $d-c$ be the harmonical mean between $a-c$ and $b-c$.

6. Shew that the difference between the first and last of n harmonical means between a and b, bears to the difference between a and b, a less ratio than $n-1$ to $n+1$.

7. In any harmonical progression, the product of the first two terms is to the product of any two adjacent terms, as the difference between the first two is to the difference between the other two.

8. Given the mth and nth terms of a harmonical series; determine the first and second terms.

9. If m and n be two arithmetical means between a and b, and p and q, two harmonical means; then shall $mq = np$.

10. If $b+c$, $a+c$, $a+b$ be in harmonical progression; shew that a^2, b^2, c^2 are in arithmetical progression; and conversely.

11. If a, b, c be in arithmetical progression, then ab, ac, and bc shall be in harmonical progression.

12. If $\frac{a+b}{1-ab}$, b, $\frac{b+c}{1-bc}$ be in arithmetical progression; then shall a, $\frac{1}{b}$, c be in harmonical progression.

X.

If a, b, c be in harmonical progression, prove that the following quantities are respectively in harmonical progression:—

1. $a,\ a-c,\ a-b$. 2. $c,\ c-a,\ c-b$. 3. $\frac{a}{b+c},\ \frac{b}{c+a},\ \frac{c}{a+b}$.

4. $\frac{ab}{a+b},\ \frac{ac}{a+c},\ \frac{bc}{b+c}$. 5. $\frac{a}{b+c-a},\ \frac{b}{a+c-b},\ \frac{c}{a+b-c}$.

6. $\frac{1}{a}+\frac{1}{b+c},\ \frac{1}{b}+\frac{1}{a+c},\ \frac{1}{c}+\frac{1}{a+b}$.

XI.

If a, b, c, d be in harmonical progression, verify the truth of the following equivalents:—

1. $\frac{1}{b-a}+\frac{1}{b-c}=\frac{1}{a}+\frac{1}{c}$. 2. $\frac{1}{a-b}+\frac{1}{b-c}+\frac{4}{c-a}=\frac{1}{c}-\frac{1}{a}$.

3. $\frac{a(a-b)}{2a-b}+\frac{c(c-b)}{2c-b}=0$. 4. $a^3+c^3>2b^3$, and generally $a^n+c^n>2b^n$.

5. $\frac{b+a}{b-a}+\frac{b+c}{b-c}=2$.

6. $3(b-a)(d-c)=(c-b)(d-a)$.

XII.

1. If a, b, c are positive quantities in harmonical progression, and $a^2+b^2+c^2=1$; shew that b is less than the reciprocal of the square root of 3.

2. If $\frac{a}{b}=\frac{2a^2+bc}{2b^2+ac}$; then is c a harmonical mean between a and b.

3. Shew that $(x^2+y^2)(x^2+xy+y^2)$, $x^4+x^2y^2+y^4$, $(x^2+y^2)(x^2-xy+y^2)$ are in harmonical progression.

4. If a, b, c be the mth, nth, and pth terms respectively of a harmonical progression, then $(m-n)ab+(p-m)ac+(n-r)bc=0$.

5. If $\frac{a-x}{px}=\frac{a-y}{qy}=\frac{a-z}{rz}$, and p, q, r be in arithmetical progression, then shall x, y, z be in harmonical progression.

6. If a, b, c be in arithmetical progression, and a, b, d in harmonical progression; shew that $\frac{c}{d}=1-\frac{2(a-b)^2}{ab}$.

7. If the harmonical means between each pair of the three quantities a, b, c be in arithmetical progression, then b^2, a^2, c^2 shall be in harmonical progression. But if the harmonical means be in harmonical progression, then a, b, c shall be in harmonical progression.

8. If n arithmetical, and the same number of harmonical means be inserted between two quantities a and b; and a series of n terms be formed by dividing each arithmetical mean by the corresponding harmonical mean; the sum of the series will be equal to

$$n\left\{1+\frac{n+2}{n+1}\cdot\frac{(a-b)^2}{6ab}\right\}.$$

XIII.

Find the sums of the following geometrical series:—

1. $1+2+4+8+$ .... to 10 terms and to n terms.

2. $1-2+4-8+$ .... to 30 terms and to n terms.

3. $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+$ .... to 20 terms, and ad infin.

4. $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+$ .... to 20 terms, and ad infin.

5. $9-6+4$ .... to 20 terms, and ad infin.

6. $100+40+16+$ .... to 50 terms.

7. $\frac{5}{4}+1+\frac{4}{5}+$ . . . . ad infin.

8. $7-\frac{49}{8}+\frac{343}{64}-$ &c. . . . . 14 terms.

9. $\frac{5}{4}+\frac{4}{5}+\frac{16}{25}+$ . . . . ad infin.

10. $1+\frac{1}{4}+\frac{1}{16}+$ . . . . to n terms.

11. $2+3+\frac{9}{2}+$ . . . . to 20 terms, and to $2n$ terms.

12. $\frac{3}{4}-\frac{4}{3}+\frac{64}{27}-$ . . . . to $3n$ terms.

13. $1+\frac{1}{2}+\frac{1}{2.2^3}+\frac{1}{2.2^6}+$ . . . . ad infin.

14. $1-\frac{1}{2}+\frac{1}{2.3}-\frac{1}{2.3^2}+$ . . . . ad infin.

15. $3\frac{3}{8}+2\frac{1}{4}+1\frac{1}{2}+$ . . . . to 30 terms.

16. $1+\frac{2}{3}+2+\frac{4}{9}+4+\frac{8}{27}+$, &c., to n terms.

17. $\frac{3}{5}+\frac{6}{25}+\frac{9}{125}+$, &c., ad infin.

18. $\frac{1}{5}-\frac{2}{15}+\frac{4}{45}-$, &c., ad infin.

19. $a^{n-1}+a^{n-2}b+a^{n-3}b^2+$ . . . . to n terms.

20. $\left(\frac{3}{5}\right)^{\frac{1}{2}}-6^{\frac{1}{2}}+2.15^{\frac{1}{2}}-$ . . . . to 8 terms.

21. $\frac{1}{\sqrt{2}(1+\sqrt{2})}+\frac{1}{(1+\sqrt{2})(2+\sqrt{2})}+\frac{1}{(2+\sqrt{2})(3+\sqrt{2})}+$ . . . . ad infin.

XIV.

1. In every geometrical progression of a finite number of terms, the product of any two terms equidistant from the first and last terms is equal to the product of the first and last terms.

2. Shew that every term of the series 1, 2, 4, 8, &c., is greater by unity than the sum of the terms which precede it.

3. In every geometrical progression consisting of an odd number of terms; the sum of the squares of the terms is equal to the sum of all the terms multiplied by the excess of the odd terms above the even.

4. In every geometrical series, the sums of every successive n terms are in geometrical progression; find the sum of m such series, and

shew that the result is the same as that for the sum of mn terms of the first series.

5. In any geometrical series, if the $(p+q)$th term be equal to m, and the $(p-q)$th term to n; the pth term is equal to $(mn)^{\frac{1}{2}}$, and the qth term to $m\left(\frac{n}{m}\right)^{\frac{1}{2q}}$; prove also the converse.

6. Explain what is meant by a limit, and prove that the sum of the series $a+ar+ar^2+ \dots ar^{n-1}$, approaches a finite limit when n increases indefinitely, and find that limit.

7. If a, b, c, d, &c., be n quantities in geometrical progression; shew that $\frac{1}{a^2-b^2}$, $\frac{1}{b^2-c^2}$, $\frac{1}{c^2-d^2}$, &c., are also in geometrical progression, and find their sum.

8. Which is the greater sum, $\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+$, &c., ad infin., or $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+$ &c., ad infin.

9. The first term of an infinite geometrical series is 2^n, whose common ratio is $\frac{1}{2}$; shew that the sum of all the terms after the nth term is 2.

10. The first term of a geometrical series continued *ad infinitum* is 1; and any term is equal to the sum of all the succeeding terms. Required the series.

11. If a, b, c be the pth, qth, and rth terms of a geometrical progression; shew that $a^{q-r}.b^{r-p}.c^{p-q}=1$.

12. If P be the product and S the sum of the terms of a geometrical series of n terms, and S^1 the sum of their reciprocals; shew that

$$P^2=\left\{\frac{S}{S^1}\right\}^n.$$

13. If s, the sum of $2n$ terms of a geometrical series whose first term is a and common ratio r, be equal to k times the $(n+1)$th term; shew that $k=s\{a^2+(r-1)sa\}^{-\frac{1}{2}}$.

14. If a, b, c, d be in geometrical progression; then shall

$$(a\pm b)^2,\ (b\pm c)^2,\ (c\pm d)^2,$$

be also in geometrical progression.

XV.

1. If s_1, s_2, s_3 be the sums of three geometrical series, having their first terms in geometrical progression, the number of terms and the common ratio being the same in each; prove that $s_1.s_3=s_2^2$.

2. If s_1, s_2 denote the sums to be terms of two geometrical series, having a for their first terms, the same in each series, and the common ratio of one the reciprocal of that of the other, and if l be the last term of the first series; then $as_1=ls_2$.

3. If s_r denote the sum of a geometrical series, of which a is the first term, r the common ratio, and n the number of terms;

$$\text{then } s_r^{2n+1} - s_{-r}^{2n+1} = 2rs^n_{r^2}.$$

4. If P be the sum of a series formed by taking the first and every pth term of an infinite geometrical series whose first term is 1, and the common ratio a proper fraction; Q the sum of the series formed by taking the first and every qth term; prove that $P^q(Q-1)^p = Q^p(P-1)^q$.

5. If s be the sum of n terms of the series $x+x^3+x^5+\dots$, and s' be the sum of the reciprocals of the terms; then $\frac{s-s'}{s+s'} = \frac{x^{2n}-1}{x^{2n}+1}$.

6. If $s_1, s_2, s_3 \dots s_n$ denote the sums of n geometrical series whose first terms are $a, 2a, 3a, \dots na$ respectively, n the number of terms and r the common ratio; then $s_1+s_2+s_3+\dots+s_n = \frac{r^n-1}{r-1}\cdot\frac{(n+1)na}{2}$.

7. If s_1 represent the sum of an infinite geometrical series whose first term is a and common ratio r, s_2 the sum of the squares of the terms, s_3 the sum of the cubes, &c.; shew that $\frac{1}{s_1}+\frac{1}{s_2}+\frac{1}{s_3}+$, &c., indefinitely continued $= \frac{1}{a-1} - \frac{a-r}{r}$.

8. $s_1, s_3, s_5 \dots s_{2n-1}$ be sums of n geometrical progressions each to n terms, and whose first terms are respectively $a, 3a, 5a \dots (2x-1)a$, and r the common ratio; prove that $s_1+s_3+s_5+\dots s_{2n-1} = \frac{(r^n-1)n^2a}{r-1}$.

9. If s_1 denote the sum of an infinite geometrical series, of which a is the first term and r the common ratio; s_2 the sum of the squares of each term, s_3 of the cubes, and so on;

$$\text{Then shall } s_1 - s_2 + s_3 - \text{ \&c.}, = \frac{a}{1-a} \cdot \frac{1-r}{r+a}.$$

10. Between each pair of the $(n+1)$ quantities x, y; $x, 2y$; $x, 4y$; &c., are inserted n geometrical means, and if m_1, m_2, m_3, &c., be the nth means respectively of each pair; prove that

$$\frac{m_1}{m_2}+\frac{m_2}{m_3}+\frac{m_3}{m_4}+ \dots = \left\{\frac{n^{n+1}}{2^n}\right\}^{\frac{1}{n+1}}.$$

XVI.

1. When unity is added to the greatest of three numbers in arithmetical progression, they are in geometrical progression; shew that the least term is equal to the square of the common difference.

2. If a, b, c, d be four numbers in geometrical progression; find what number added to each of them, the squares of the sums shall be in arithmetical progression.

3. If the arithmetical mean between a and b be double of the geometrical mean; then $a : b :: 2+\sqrt{3} : 2-\sqrt{3}$.

4. If a, b, c be in geometrical progression, and x, y be the arithmetical means between a, b and b, c respectively;

$$\text{prove } \frac{2}{b}=\frac{1}{x}+\frac{1}{y}, \text{ and } 2=\frac{a}{x}+\frac{c}{y}.$$

5. If the pth, qth, rth, and sth terms of an arithmetical progression be also in geometrical progression; prove that $p-q$, $q-r$, $r-s$ are in geometrical progression.

6. If three quantities are in an increasing arithmetical progression; shew that the second has a greater ratio to the first than the third has to the second.

7. Given the sums of the even and the odd terms of a geometrical progression of $2n$ terms; required the rth arithmetical mean between the pth and qth terms.

8. Determine four numbers in arithmetical progression, which being increased by 2, 4, 8, and 15 respectively; the sums shall be in geometrical progression.

9. Prove that the arithmetical mean of any number of positive quantities is greater than the geometrical mean.

10. If t_n, t'_n, denote the nth terms of two geometrical progressions whose first and second terms are respectively a, $a+b$, and a, $a-b$; then shall $t_n \times t'_n$ be the nth term of a geometrical progression whose first and second terms are a^2, a^2-b^2.

11. If a, b, c be three numbers, and x, y, z other three, each in geometrical progression; and if $\frac{a}{x}$, $\frac{b}{y}$, $\frac{c}{z}$ be in arithmetical progression; then $a+mx$, $b+my$, $c+mz$ shall also be in geometrical progression, m being any integral number.

XVII.

1. If a, b, c be in geometrical progression, and $a+x$, $b+x$, $c+x$ be in harmonical progression; find the value of x.

2. If between each two successive terms of the series a, ar, ar^2, &c., consisting of $n+1$ terms, a harmonical mean be found; find the sum of the n means.

3. If the geometrical mean between two quantities x and y be to the harmonical mean as 1 to n; shew that x is to y as $1+\sqrt{1-n^2}$ to $1-\sqrt{1-n^2}$.

4. If between two numbers there be inserted two arithmetical means a_1, a_2; and two harmonical means b_1, b_2; and if a harmonical mean be inserted between a_1 and a_2; and an arithmetical mean between b_1 and b_2; then the geometrical mean between these is equal to the geometrical mean between the original numbers.

5. If the arithmetical means between the pairs of quantities a, x; b, y; c, z be in arithmetical progression, the geometrical means in

geometrical progression, and the harmonical means in harmonical progression; shew that $by\left\{\frac{1}{a}+\frac{1}{c}-\frac{z}{b}\right\}=2b-a-c$, or $by=ac$.

6. Three quantities a, b, c are (1) in arithmetical progression, (2) in geometrical progression, (3) in harmonical progression; find the value of $\frac{a(b-c)}{a-b}$ in each case.

7. If a, b, c be in arithmetical, x, y, z in harmonical, and ax, by, cz in geometrical progression; then shall $\frac{x}{y}+\frac{y}{x}=\frac{a}{c}+\frac{c}{a}$.

8. If $\frac{x}{a}$, $\frac{y}{b}$, $\frac{z}{c}$ be in arithmetical progression, ax, by, cz in geometrical progression, and $\frac{a^2}{yz}$, $\frac{b^2}{xz}$, $\frac{c^2}{xy}$ in harmonical progression; prove that $b^4(c^2-a^2)^2+4a^2c^2(a^2-bc)(c^2-ab)=0$.

9. If y be the harmonical mean between x and z, and x and z be respectively the arithmetical and geometrical means between a and b; prove that $y=\frac{2(a+b)}{\left\{\left(\frac{a}{b}\right)^{\frac{1}{4}}+\left(\frac{b}{a}\right)^{\frac{1}{4}}\right\}^2}$.

10. If $a^x=b^y=c^z$, and if a, b, c be in geometrical progression, then shall x, y, z be in harmonical progression.

XVIII.

1. Prove that a geometrical mean between two quantities is a mean proportional between an arithmetical and a harmonical mean between the same two quantities, and shew of these three mean terms which is the greatest.

2. If to three quantities in arithmetical progression a fourth proportional be found, the three last terms of this proportion will be in harmonical progression.

3. Prove $a_2a_3-a_1a_4$ is positive, zero, or negative, according as a_1, a_2, a_3, a_4 are in arithmetical, geometrical, or harmonical progression.

4. If a, b, c are in arithmetical progression, and b, c, d are in harmonical progression; shew that a, b, c, d are in geometrical progression.

5. If a, b, c be in arithmetical progression, b, c, d in geometrical progression, and c, d, e in harmonical progression; prove that a, c, e are in geometrical progression.

6. If a, b, c be three quantities such that a is the arithmetical mean between b and c, c the harmonical mean between a and b; prove that b is the geometrical mean between a and c.

7. There are three numbers in harmonical progression. If unity be subtracted from the first, the progression becomes geometrical;

but if 4 be subtracted from the third, it becomes arithmetical; what are the numbers?

8. If a be the difference between the arithmetical and geometrical means between two quantities; and b the difference between the arithmetical and harmonical means; find the values of the two quantities.

9. If three numbers be in arithmetical progression, the products of the first and second, first and third, second and third, are in harmonical progression.

10. If a, b, c be in geometrical progression; then shall $a+b$, $2b$, $b+c$ be in harmonical progression.

11. If a, b, c be in harmonical progression; then $a-\frac{1}{2}b$, $\frac{1}{2}b$, $c-\frac{1}{2}b$ are in geometrical progression.

12. If x and y be the arithmetical and harmonical mean respectively between a and b; then $\frac{1}{x}$ and $\frac{1}{y}$ will be respectively the harmonical and arithmetical mean between $\frac{1}{a}$ and $\frac{1}{b}$.

13. The difference of the arithmetical and geometrical means between two numbers is 13, and the difference between the geometrical and harmonical means is 12; what are the two numbers?

XIX.

1. Find the amount of an annuity of £140 payable quarterly for three years, compound interest being allowed at 5 per cent.

2. If £600 be lent without interest, and to be repaid by a bond for £800 in 3 years; what interest did lender in fact take if compound interest be allowed?

3. What annuity can be purchased for £1,000 to continue 20 years, allowing (1) 3 per cent., (2) 5 per cent., at compound interest?

4. A person has a capital of £2,000, which produces him interest at 5 per cent.; if he spend every year £180, find in how many years he will be ruined, having given

log. 2 = ·30103, log. 3 = ·47712, log. 7 = ·84510.

5. A person borrows £300 to be paid back by equal instalments in 30 years, and pays interest each year at 4 per cent. on the portion not paid back; find the whole amount of interest he pays.

6. A person places £10,000 in the funds at 5 per cent.; the first year he spends the whole interest; the second, third, &c., years he spends twice, three times, &c., the same interest; how long will his property last?

7. A sum of £1,000 is lent to be repaid with interest at 4 per cent. per annum by annual instalments, beginning with £40 at the end of the first year, and increasing 30 per cent. each year on the last preceding instalment; how soon will the debt be paid off?

log. 2 = ·3010300, log. 3 = ·47712.

8. If a national debt of a thousand millions of pounds has a tenth part of its existing amount paid off every year; in how many years will it be reduced to less than half of its original amount, supposing money bear interest at 3 per cent.?

9. If an estate be sold for £5,000, which produces a clear rental of £200 a year; at what rate was the interest of money reckoned?

10. Freehold estates are usually valued at so many years' purchase, that is, at so many years' rent; find how many years' purchase must be given for an estate, (1) when money bears interest at 3¼ per cent., (2) at 4 per cent., (3) at 5 per cent.

11. If a person purchase the reversion of an estate after 20 years for £500; what rent ought it to produce that he may make 6 per cent. of his money?

12. Find what sum would be required to purchase a government annuity of £20, to continue 15 years, when the price of the 3 per cent. consols is such as to yield an interest of £3 5s. per cent., both the interest and the annuity being payable half-yearly?

log. 1·01625 = ·0070006, log. 6·16571 = ·789983.

XX.

1. A person owes P pounds, on which he pays interest yearly. The principal is to be repaid by fixed annual payments in n years; what annual payment must be made at the same rate of interest so as to be just able to pay the interest annually and the principal when it becomes due?

2. If there be n annuities of 1, 2, 3 .... n pounds respectively left unpaid for n years; shew that their amount at simple interest will be $\frac{n^2(n+1)}{2}+\frac{(n^2-1)n^2r}{4}$; r being the interest of one pound for one year.

3. If an annuity be payable for $2n$ years, r the interest of one pound for one year, P and Q the present values of the annuity for the first and last halves of the time respectively; shew that $P=Q(1+r)^n$.

4. A person left a pounds to purchase three annuities, each of the same yearly value, but to continue for m, n, p years respectively; determine what portions of the sum must be apportioned for the three several annuities, compound interest being allowed.

5. If a debt of a pounds increasing at compound interest be discharged in n years by annual payments of $\frac{a}{m}$ pounds,

then $(1+r)^n(1-mr)=1$.

6. The present value of an annuity, to continue for a term of years at a given rate at compound interest, is equal to m times the present value of the same annuity to be paid only during the latter half of the same term; find when the annuity will cease.

7. Find the present worth of A, $2A$, $3A$, . . . nA pounds due at the end of 1, 2, 3 . . . n years respectively from the present time, and find the value when n is indefinitely great, compound interest being allowed.

8. If P_1 represent the sum of the present values of n annuities, to commence at the expiration of 2, 4, 6, &c., years respectively, and to continue in perpetuity; and P_2 the sum of n others, to commence at the expiration of 1, 3, 5, &c., years, and to continue in perpetuity; shew that $P_2 = (1+r)P_1$, where r is the interest of one pound for one year, compound interest being allowed.

9. Shew that the sum of the present values of n separate annuities is equal to the present value of the amount of the annuities when calculated at compound interest, but is not equal when simple interest is allowed.

10. If P be the present value of an annuity, to continue for p years, and $P+Q$ for $2p$ years; shew that the annuity is equal to

$$\frac{P^2}{P-Q}\left\{\left(\frac{P}{Q}\right)^{\frac{1}{p}}-1\right\}.$$

11. A person spends a certain part of his income every year, and adds the remainder to his principal; shew that his expenditure increases in geometrical progression, and find his comparative income at the end of the nth year.

12. If a man spends p pounds a year more than his income, of which the deficit is taken from his capital, which was originally P pounds, bearing interest payable yearly; find in what time he will become penniless.

13. A freehold estate of A pounds a year is left for the equal benefit of two charitable institutions; how many years must one of them hold it before the other resumes it to possess it in perpetuity, supposing compound interest be allowed?

XXI.

1. There are two numbers in the proportion of $\frac{1}{2}$ to $\frac{2}{3}$; which being increased respectively by 6 and 5, are in the proportion of $\frac{2}{5}$ to $\frac{1}{2}$. Required the numbers.

2. Two numbers are in the ratio of 4 to 5; if one be increased and the other diminished by 10, the ratio of the resulting numbers is inverted; find the numbers.

3. There are two numbers such that if unity be added to the first and 5 to the second, the results are in proportion as 4 to 5; but if 19 be subtracted from the first and 15 from the second, they shall be as 19 to 24; find the two numbers.

4. Three numbers are in continued proportion, the sum of the least and greatest is 51, and the sum of the two greater ones is 60; required the numbers.

5. There are three numbers in geometrical progression, whose sum

is 14; and the sum of the first and second is to the sum of the second and third as 1 to 2; required the numbers.

6. There are four numbers proportional, of which the third is the product of the first and second, while the fourth is the sum of the second and third, and the product of the first and third is equal to the sum of the second and fourth; required the numbers.

7. Given the sum of three quantities in geometrical progression, and the sum of their reciprocals, to find the quantities themselves.

8. Divide 111 into three parts so that the products of each pair may be as 4 : 5 : 6.

XXII.

1. The sum of four numbers in arithmetical progression is 56, and the sum of their squares is 864; what are the numbers?

2. The sum of seven numbers in arithmetical progression is 28, and the sum of their cubes is 784; find the numbers.

3. There are four numbers in arithmetical progression, the sum of the squares of the first and second is 34, and the sum of the squares of the third and fourth is 130; find the four numbers.

4. The sum of three numbers in harmonical progression is 11, and the sum of their squares is 49; what are the numbers?

5. If the product of three numbers in harmonical progression be 576; what are the numbers if their sum be 26?

6. The sum of three numbers in harmonical progression is 191, and the product of the first and third is 4032; find the numbers.

7. From each of three numbers in harmonical progression; what number must be subtracted that the three differences may be in geometrical progression?

8. Four numbers are in harmonical progression, the first and fourth are 6 and 10 respectively; find the equation which connects the second and third.

9. Determine the arithmetical progression when the number of terms is 11, the sum of the terms 220, and the sum of their cubes 147400.

XXIII.

Find the sums of the following series :—

1. $1^2+3^2+5^2+ \dots\dots +n^2$. 2. $1^3+3^3+5^3+ \dots\dots +n^3$.
3. $1^4+3^4+5^4+ \dots\dots +n^4$. 4. $1^2-3^2+5^2-7^2+ \dots\dots \pm n^2$.
5. $1^3-3^3+5^3-7^3+ \dots\dots \pm n^3$. 6. $1^4-3^4+5^4-7^4+ \dots\dots \pm n^4$.
7. $1.2^2+2.3^2+3.4^2+ \dots\dots +n(n+1)^2$.
8. $1.3^2+3.5^2+5.7^2+ \dots\dots +(2n-1).(2n+1)^2$.
9. $1.2^3+2.3^3+3.4^3+ \dots\dots n.(n+1)^3$.

XXIV.

Verify the truth of the sums to *n* terms of each of the following series :—

1. $2+6+14+30+62+\ldots\ldots+(2^n-2)=2^{n+2}-2(n+2)$.
2. $4+10+28+82+\ldots\ldots+(3^n+1)=\frac{1}{2}\{3^{n+1}+2n-3\}$.
3. $3+6+11+20+\ldots\ldots+(2^n+n)=\frac{1}{2}\{2^{n+2}+n^2+n-4\}$.
4. $1+3+7+15+\ldots\ldots+(2^n-1)=2^n-(n+1)$.
5. $1+5+13+29+\ldots\ldots+(2^{n+1}-3)=2^{n+2}-3n-4$.
6. $1+6+23+76+237+\ldots\ldots+\{3^n-(n+1)\}$
$=\frac{1}{2}\{3^{n+1}-(n+1)(n-2)+1\}$.
7. $1+11+111+1111+\ldots\ldots+\frac{1}{9}(10^n-1)=\frac{10}{81}\{10^n-1\}-\frac{n}{9}$.
8. $7+77+777+7777+\ldots\ldots+\frac{7}{9}(10^n-1)=\frac{70}{81}\{10^n-1\}-\frac{7n}{9}$.

XXV.

1. $1+\frac{3}{2}+\frac{7}{4}+\frac{15}{8}+\frac{31}{16}\ldots\ldots$ to n terms.
2. $\frac{3}{10}+\frac{7}{10^2}+\frac{11}{10^3}+\frac{15}{10^4}\ldots\ldots$ to n terms.
3. $1+\frac{2}{2}+\frac{3}{2^2}+\frac{4}{2^3}+\ldots\ldots$ ad infin.
4. $\frac{4}{10}+\frac{5}{20}+\frac{6}{40}+\frac{7}{80}+\ldots\ldots$ ad infin.
5. $\frac{1}{4}+\frac{3}{4^2}+\frac{5}{4^3}+\frac{7}{4^4}+\ldots\ldots$ to n terms.
6. $1+2x+3x^2+4x^3+\ldots\ldots$ to n terms.
7. $1+3x+5x^2+7x^3+\ldots\ldots$ to n terms.
8. $1+4x+9x^2+16x^3+\ldots\ldots$ ad infin.
9. $1+8x+27x^2+64x^5+\ldots\ldots$ ad infin.
10. $2+6x+12x^2+20x^3+\ldots\ldots$ ad infin.
11. $1+16x+81x^2+256x^3+\ldots\ldots$ ad infin.
12. $1+32x+243x^2+1024x^3+\ldots\ldots$ ad infin.
13. $1+\frac{2b}{a}+\frac{3b^2}{a^2}+\frac{4b^3}{a^3}+\ldots\ldots$ ad infin.
14. $(x+x^{-1})+2(x^2+x^{-2})+3(x^3+x^{-3})+\ldots\ldots$ to n terms.

XXVI.

1. Find the sum of the squares and the sum of the cubes of the first n natural numbers without assuming any general formulæ.

2. If n be an even number, shew that

$$n+2(n-1)+3(n-2)+\ldots\ldots+\frac{n}{2}\left(\frac{n}{2}+1\right)=\frac{1}{2}\cdot\frac{n(n+1)(n+2)}{1.2.3}.$$

3. Having given the expression for the sum of the squares of the first n natural numbers; shew from the expanded form of

$$(1+2+3+\ldots+n)^2,$$

that the sum of the products of the first n numbers taken two and two together is $\frac{1}{12}n(n^2-1)(3n+2)$.

4. Between each two consecutive terms of a series, consisting of the squares of a series of numbers in arithmetical progression, a geometrical mean is inserted; shew that each term in the original

series is an arithmetical mean between the inserted terms adjacent to it, and find the sum of n terms of the series.

5. What is the general form of triangonal, what of pentagonal numbers? Shew that there is one and only one case in which the same number can be both triagonal and pentagonal.

6. Write the first five and the nth of the series of triagonal numbers, and find an expression for the sum; and shew that the sum of any two consecutive triagonal numbers is a square number.

7. If any triagonal number be multiplied by 8 and the product increased by unity; the result is a square number.

8. Shew that the sum of the series $1^3+3^3+5^3+ \ldots n^3$ is an hexagonal number.

XXVII.

1. Find an expression for the number of balls in a pyramidal pile, (1) when the base of the pile is an equilateral triangle having n balls in each side, (2) when the base is a square with n balls in each side, (3) when the base is a rectangular parallelogram having m balls in its length and n in its breadth.

2. What is the difference of the number of balls in a square pile and in a triangular pile, each consisting of 20 courses?

3. Apply the formula to find the number of balls in a rectangular pile which contains 16 in length and 7 in breadth; and verify the correctness by finding the sum of the successive courses of the pile.

4. What is the number of balls in a rectangular pile of which the length of the lowest course is double the breadth?

5. There is a pile of iron shot, each six inches in diameter, on a square base. Had there been one shot less in the side of the square, the weight of the pile would have been less than before in the ratio of 55 to 91; find the number of shot in the base of the pile; and it being given that a ball of iron one foot in radius weighs 2048 pounds, and that the volumes of spheres vary as the cubes of their radii; find the weight of the shot in the pile.

XXVIII.

1. If $a_1, a_2, a_3, a_4 \ldots . a_n$ be a series of quantities, such that $a^3_2=a_1a^2_3$, $a^3_3=a_2a^2_4$, $a^3_4=a_3a^2_5$, &c.; determine a_4 in terms of a_1 and a_2.

2. In the series $a_1+a_2+a_3+ \ldots . +a_n+a_{n+1}+ \ldots .$ the $(n+1)$th term is derived from the preceding terms by the formula

$$a_{n+1}=\frac{n(n+1)}{2n-an}; \text{ shew that } a_4=\frac{4(3-2a_1)}{4-3a_1}.$$

3. If n become $1+\frac{1}{n}$, where n is a positive integer; prove that the sum of the terms $1+2n+3n^2+$, &c., indefinitely continued, is equal to n^2.

4. If any three consecutive terms be taken of the series

$$1^{\frac{1}{2}}+2^{\frac{1}{2}}+3^{\frac{1}{2}}+4^{\frac{1}{2}}+ \ldots .$$

the sum of the first and third is always less than double of the second term.

5. Verify the correctness of the reversion of the series.

(1) If $x = y - \frac{y^2}{2} + \frac{y^3}{3} - \frac{y^4}{4}$, &c.;

$$\text{then } y = x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4}, \text{ \&c.}$$

(2) If $x = y + 2y^2 + 3y^3 + 4y^4 + \&c.$:

$$\text{then } y = x - 2x^2 + 5x^3 - 14x^4 + \&c.$$

6. Shew that $(1+x)(1+x^2)(1+x^4) + \ldots\ldots$ to n factors $= \frac{1-x^{2^n}}{1-x}$.

7. Find the sum of $(x-1) + 3(x-2) + 5(x-3) + \ldots\ldots$ to n terms.

8. Determine the limits of the ratios of the following series when the number of terms is indefinitely continued:—

(1) $\frac{1+2x+3x^2+4x^3+\ldots\ldots}{1-2x+3x^2-4x^3+\ldots\ldots}$. (2) $\frac{1-3x^2+5x^4-7x^6+\ldots\ldots}{x-2x^3+3x^5-4x^7+\ldots\ldots}$.

9. If $s = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \ldots\ldots$ ad infin.

and $s' = 1 - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \ldots\ldots$ ad infin.;

shew that $s : s' :: 2^{n-1} : (2^{n-1} - 1)$.

10. Given $x^2 + (x+1)^2 + (x+2)^2 + \ldots\ldots$ to 8 terms $= 380$; find x.

11. If $\frac{x^3+1^3}{x+1} + \frac{x^3+2^3}{x+2} + \frac{x^3+3^3}{x+3} + \ldots\ldots$ to x terms $= \frac{x(2x+1)^2}{9}$; find x.

XXIX.

1. Shew that double the discount of any sum of money is a harmonical mean between the sum itself and the interest upon it for the same time.

2. Compare the sides of a right angled triangle, when the squares described on the three sides are in harmonical progression.

3. On the perpendicular of an equilateral triangle another equilateral triangle is described, and on the perpendicular of this, another, and so on continually; find the sum of the areas of all the triangles.

4. The sum of £100 is to be divided among four persons so that the shares shall be in geometrical progression, and the difference between the extremes shall be to the difference between the means as 37 to 12.

5. If s_1, s_2, s_3 denote the sums of n terms of each of three arithmetical series whose first terms are unity, and their common differences in harmonical progression; prove that $n = \frac{2s_1.s_3 - s_1.s_2 - s_2.s_3}{s_1 + s_3 - 2s_2}$.

6. If a mixture be made of two substances, first in equal volumes and next in equal weights, the specific gravities of the mixtures may be found, and they will be in the first case an arithmetical mean, in the second a harmonical mean between the specific gravities of the simple substances.

RESULTS, HINTS, ETC., FOR THE EXERCISES ON ARITHMETICAL, HARMONICAL, AND GEOMETRICAL PROGRESSION.

I.

1. $\frac{n(n+1)}{1.2}$. 2. n^2. 3. $n(n+1)$. 4. 420. 5. 5580.

6. 39700. 7. 1999000. 8. 3725. 9. −200.

10. −1674. 11. 200. 12. 4752.

13. $1-3+5-7+\ldots$ to 45 terms $=(1+5+9+\ldots$ to 23 terms$)$
$-(3+7+11+\ldots$ to 22 terms$)$.
$1-3+5-7+9-\ldots$ to $2n$ terms $=(1+5+9+$ to n terms$)$
$-(3+7+11+\ldots$ to n terms$)$.

14. $1-8+15-22+\ldots$ to 50 terms $=(1+15+29+\ldots$ to 25 terms$)$
$-(8+22+36+\ldots$ to 25 terms$)$.
$1-8+15-22+\ldots$ to $(2n+1)$ terms $=(1+15+29+\ldots$ to $(n+1)$ terms$)$
$-(8+22+36+\ldots$ to n terms$)$.

II.

1. 45. 2. $-248\frac{1}{3}$. 3. $280\frac{1}{2}$. 4. $-605\frac{60}{63}$ and $\frac{n(121-31n)}{126}$.

5. −2425 and $\frac{n(3-n)}{4}$. 6. 442. 7. $841\frac{1}{3}$. 8. $-na$.

9. $4n\{a^2-(4n-3)an+n^2\}$. 10. $\frac{1}{2}(n-1)$ and −1. 11. $\frac{-6n^2}{n-1}$.

12. $(2n+1).\left\{\frac{a+b}{a-b}+\frac{n(2a-3b)}{a+b}\right\}$. 13. 505. 14. $\frac{(n+1)a}{2n}$.

III.

1. Since there are 3 means, there are 5 terms in the series.

Let x=com. diff. Then $\cdot012+x$, $\cdot012+2x$, $\cdot012+3x$, denote the means,
and $\cdot012+4x$=the fifth term $=\cdot132$, $\therefore$ $4x=\cdot12$ and $x=\cdot03$.
Hence $\cdot012+x=\cdot012+\cdot03=\cdot042$, $\cdot012+2x=\cdot012+\cdot068=\cdot072$,
$\cdot012+3x=\cdot012+\cdot09=\cdot102$

$\therefore$ ·042, ·072, ·102, are the three arithmetical means between ·012 and ·132.

2. The nine means are 19, 29, 39, 49, 59, 69, 79, 89, 99.

3. The means are $\frac{10a+b}{11}$, $\frac{9a+2b}{11}$, $\frac{8a+3b}{11}$, $\frac{7a+4b}{11}$, $\frac{6a+5b}{11}$, $\frac{5a+6b}{11}$, $\frac{4a+7b}{11}$, $\frac{3a+8b}{11}$, $\frac{2a+9b}{11}$, $\frac{a+10b}{11}$.

4. The number of means is 9, and the common difference is 2.

5. The number of means is 8, and the common difference is 2.

IV.

1. Let a denote the first term, d the common difference, and n the number of terms of the series. Also let T_n denote the nth term.

The pth term reckoned from the last term of the series will be the $(n-p+1)$th term reckoned from the first.

$$\text{Then } T_p=a+(p-1)d,\ T_{n-p+1}=a+(n-p)d,\ T_n=a+\overline{n-1}d$$
$$\therefore T_p+T_{n-p+1}=2a+(n-1)d, \text{ and } T_1+T_n=2a+(n-1)d.$$
$$\text{Also } T_p+T_{p+2}=a+(p-1)d+a+(p+1)d=2(a+pd)=2T_{p+1}.$$

3. Let x the first term, y the common difference; then $x+6y=15$, and $x+9y=21$, from which the first term and common difference can be found.

7. $8s+1=(2n+1)^2$. 8. See Art. 4, pp. 3—5.

10. $x=5$ and -10 : $y=7$ and 4. See Art. 4, p. 4.

12. Let x be the number of terms of the series, then $x^2=123454321$: find x.

V.

1. Let x denote the first term, y the common difference, and z the number of terms;

then $x+(m-1)y=a$, $x+(n-1)y=b$, $x+(z-1)y=a+b$: find x, y, z.

2. Here $T_{m+n}=a+(m+n-1)d$, $T_{m-n}=a+(m-n-1)d$,

$$\therefore T_{m+n}+T_{m-n}=2a+2(m-1)d=2\{a+(m-1)d\}=2T_m,$$
$$\text{Hence } T_m=\tfrac{1}{2}\{T_{m+n}+T_{m-n}\},$$

or the mth term is equal to half the sum of the $(m+n)$th and $(m-n)$th terms.

3. Let x denote the first term, y the common difference,

$$\text{then } x+(n-1)y=\frac{1}{6}(3n-1) \text{ and } \frac{n}{2}\{2x+(n-1)y\}=\frac{n}{12}(3n+1).$$
$$\text{Whence } x=\tfrac{1}{3} \text{ and } y=\tfrac{1}{2}.$$

4. Let x denote the first term and y the common difference, then $S_n=\frac{n}{2}\{2x+(n-1)y\}$, which is an identical expression with $n(a-b)^2+n^2ab$, so that

$$n\left(x-\frac{y}{2}\right)+\frac{n^2y}{2}=n(a-b)^2+n^2ab. \quad \therefore \frac{y}{2}=ab, \text{ and } x-\frac{y}{2}=(a-b)^2$$
$$\therefore y=2ab \text{ and } x=a^2-ab+b^2.$$

6. Let T_m and T_n denote the general terms of the respective series, then $T_m=2+(m-1)3=3m-1$, $T_n=3+(m-1)4=4n-1$, $\therefore T_n-T_m=4n-3m$. now $4n-3m=o$, when identical terms occur in the two series, and $\therefore 4n=3m$,

$$\therefore \frac{n}{m}=\frac{3}{4}=\frac{6}{8}=\frac{9}{12}=\frac{12}{16}=\frac{15}{20}= \text{\&c.}$$

Hence the 4th, 8th, 12th, 16th, &c., terms in the first series are respectively identical to the 3rd, 6th, 9th, 12th, &c., terms in the second.

8. $a'b'-ab=\{a+(p-1)d\}\cdot\{a+(n-p)d\}-a\{a+(n-1)d\}=(p-1)(n-p)d^2$.

9. Here $T_n{}^2+T_1{}^n=\{a+(n-1)d\}^2-a^2=2(n-1)ad+(n-1)^2d^2$

$$=(n-1)d\{2a+(n-1)d\}\times\frac{n}{2}\times\frac{2}{n}$$
$$=\frac{2(n-1)d}{n}\cdot\frac{n}{2}\cdot\{2a+(n-1)d\}=\frac{2(n-1)ds}{n}$$

10. In the first group, the common difference being unity, the first term of the nth group is $\frac{1}{2}(n^2-n+2)$, and the sum is $\frac{1}{2}(n^2+n)$.

In the second group, the common difference is 2, and the first term of the nth group is n^2-n+1, and the sum is n^3.

11. Let a be the first term, and s_1, s_2, s_3, &c., be the sums of the sets of the p terms;

$$\text{then } s_1=pa+\frac{(p^2-p)d}{2},\ s_2=pa+\frac{(3p^2-3p)d}{2},\ s_3=pa+\frac{(5p^2-5p)d}{2}, \text{ \&c.}$$
$$\text{and } s_2-s_1=p(p-1)d,\ s_3-s_2=p(p-1)d, \text{ \&c.}$$

12. Let a denote the first term, d the common difference, and s_1, s_2, s_3, &c., the sums of the successive groups of n terms each.

$$\text{Then } s_1=na+\frac{(n^2-n)d}{2},\ \ s_2=na+\frac{(3n^2-n)d}{2},\ \ s_3=na+\frac{(5n^2-n)d}{2},\ \text{\&c.}$$

$$\therefore s_2-s_1=n^2d,\ \ s_3-s_2=n^2d,\ \text{\&c.}$$

VI.

1. Here $a-b=b-c$, and $a+c=2b$, also $a-c=2b-2c=2(b-c)$

$\therefore (a-c)^2=4(b-c)^2=2(b-c)^2+2(b-c)^2=2(b-c)^2+2(a-b)^2=2\{(b-)^2+(a-b)^2\}$.

2. Here $\frac{1}{b+c}+\frac{1}{a+b}=\frac{2}{a+c}$, then multiplying these equals by $(a+b)(b+c)(a+c)$ there results $(b+c)(a+c)+(a+b)(a+c)=2(a+b)(b+c)$, whence $a^2+c^2=2b^2$.

5. Note. The expression $(m-n)c+(p-m)b+(m-p)a=o$ can be arranged in the form $(c-b)m+(a-c)n+(b-a)p=o$, which indicates also an arithmetical progression in which m, n, p are respectively the ath, bth, and cth terms; and that the two series have their common differences, one the reciprocal of the other.

6. Here $a-b=b-c=c-d$; $::\ a+c=2b,\ b+d=2c,\ a+d=b+c$

$$\text{and } (a+c)^2+(b+d)^2=4b^2+4c^2.$$

$$\text{or } a^2+b^2+c^2+d^2+2ac+2bd=4b^2+4c^2$$

$$\text{adding } 4ac+4bd \text{ to these equals}$$

$$\therefore a^2+b^2+c^2+d^2+6(ac+bd)=4(b^2+c^2+ac+bd):$$

$$\text{but since } b+d=2c,\ \therefore b^2+bd=2bc$$

$$\text{and } a+c=2b\ \therefore ac+c^2=2bc$$

$$\therefore b^2+bd+ac+c^2=4bc$$

$$\text{also } 4bc=(b+d)(a+c)=ab+ad+bc+cd.$$

$$\text{Hence } \therefore a^2+b^2+c^2+d^2+6(ac+bd)=4(ab+ad+bc+cd).$$

VII.

1. Here $s_1+s_3=\frac{n}{2}(n+1)+\frac{n}{2}(3n-1)=2n^2=2s_2$.

2. If a denote the first term, and d the common difference,

$$\text{then } s_1=\frac{n}{2}\{2a+(n-1)d\},\ \ s_2=\frac{2n}{2}\{2a+(2n-1)d\},\ \ s_3=\frac{3n}{2}\{2a+(3n-1)d\}$$

$$\text{and } s_2-s_1=\frac{n}{2}\{2a+(3n-1)d\}=\frac{1}{3}\cdot\frac{3n}{2}\{2a+(3n-1)d\}=\frac{1}{3}\cdot s_3$$

$$\therefore 3(s_2-s_1)=s_3.$$

3. If a denote the first term,

$$\text{then } s_{n+4}+s_{n+2}=\frac{n+4}{2}\{2a+(n+3)b\}+\frac{n+2}{2}\{2a+(n+1)b\}$$

$$=2(n+3)a+(n^2+5n+7)b$$

$$\text{and } 2s_{n+3}=\frac{2n+3}{2}\{2a+(n+2)b\}=(n+3)a+(n^2+5n+6)b,\ \therefore s_{n+4}+s_{n+2}-2s_{n+3}=b.$$

4. Here $\frac{n}{2}\{2a+(n-1)b\}=na+\frac{n(n-1)b}{2}$,

$$\frac{n-1}{2}\{2a+(n-2)b\}=(n-1)a+\frac{(n-1)(n-2)b}{2},$$

$$\frac{n-2}{2}\{2a+(n-3)b\}=(n-2)a+\frac{(n-2)(n-3)b}{2},$$

$$\frac{n-3}{2}\{2a+(n-4)b\}=(n-3)a+\frac{(n-3)(n-4)b}{2},$$

Then the sum of $n+(n-1)+(n-2)+\ldots$ to n terms $=\frac{1}{2}(n+1)n$.

Next to find the sum of $n(n-1)+(n-1)(n-2)+(n-2)(n-3)+$ &c., to $(n-1)$ terms, or of $1.2+2.3+3.4+\ldots(n-2)(n-1)+(n-1)n$, reversing the order.

$$\text{Since } \left.\begin{array}{r} 1.2.3+2.3.4+3.4.5+\ldots\quad (n-2)(n-1)n+(n-1)n(n+1) \\ -1.2.3-2.3.4-\ldots-(n-2)(n-1)n-(n-1)n(n+1) \end{array}\right\}=o.$$

$$\text{or } 1.2.3+2.3.(4-1)+3.4(5-2)+\ldots+(n-1)n\{(n+1)-(n-2)\}=(n-1)n(n+1)$$

$$\text{or } 1.2.3+2.3.3.+3.4.3+\ldots+(n-1)n.3=(n-1)n(n+1),$$

$$\therefore 1.2+2.3+3.4+\ldots+(n-1)n=\tfrac{1}{3}(n-1)n(n+1).$$

Hence the sum of the results $=\frac{1}{2}(n+1)na+\frac{1}{3}(n-1)n(n+1)b=\frac{1}{2}n(n+1)\{a+\frac{1}{3}(n-1)b\}$.

9. $s_2+s_4+s_6+\ldots+s_{2n}=\dfrac{n}{2}\{2na+(n-1)n(n+1)b\}$,

$$\text{and } s_1+s_3+s_5+\ldots s_{2n-1}=\frac{n}{2}\{2na+\overline{n-1}.n^2b\}.$$

VIII.

1. 78 strokes. 2. $x=6$, $y=4$, $z=2$, the number is 642.

3. By aid of Euc. I. 32, cor. 1, the equation $\dfrac{x}{2}(235+5x)+360=180x$.

from which $x=6$ and -1, and the figure has six sides.

4. Last payment £6 10*s*. Amount of debt £338.

5. 5 miles 1300 yards.

6. Suppose *A* travels x hours, then $\dfrac{x(x+9)}{2}=15(x-3)$, whence $x=15$ and 6 hours, also *B* travels 12 and 3 hours. Explain these results.

7. Let x be the number of days, then $\dfrac{x(x+1)}{2}+20x=23661$. Find values of x.

8. 16°.

IX.

1. The three harmonical means are $\dfrac{4ab}{a+3b}$, $\dfrac{4ab}{2a+2b}$, and $\dfrac{4ab}{3a+b}$.

2. The four means are $7\frac{1}{11}$, $8\frac{2}{3}$, $11\frac{1}{7}$, and $15\frac{3}{5}$.

3. The six means are $1\frac{19}{121}$, $1\frac{19}{51}$, $1\frac{57}{83}$, $2\frac{3}{16}$, $3\frac{1}{9}$, $5\frac{5}{13}$.

4. Shew that x, 3, 4 are in harmonical progression, also that 4, 6, y are in harmonical progression.

6. Shew that the ratio of $\left\{\dfrac{(n+1)ab}{a+nb}-\dfrac{(n+1)ab}{na+b}\right\}$ to $(a-b)$ is less than the ratio of $n-1$ to $n+1$.

7. See Art. 7, p. 7, and shew that the ratio of $\dfrac{a^2(n+1)l}{a+nl}$ to $\dfrac{a(n+1)l^2}{na+l}$ is equal to the ratio of $\dfrac{(n+1)al}{a+nl}-a$ to $l-\dfrac{(n+1)al}{na+l}$.

8. Let a, b denote the mth and nth terms respectively of a harmonical progression; then $\dfrac{1}{a}$, $\dfrac{1}{b}$ are the mth and nth terms of the corresponding arithmetical progression.

Let x denote the first term, and y the common difference,

$$\text{then } \frac{1}{a}=x+(m-1)y \text{ and } \frac{1}{b}=x+(n-1)y\text{: find } x \text{ and } y.$$

9. Let x denote the common difference when a, m, n, b are in arithmetical progression; then $a+x=m$ and $a+2x=n$; and when a, p, q, b, are in harmonical progression, $\dfrac{1}{a+x}=p$, and $\dfrac{1}{a+2x}=y$.

$$\text{Then } \frac{a+x}{a+2x}=\frac{m}{n}, \text{ and } \frac{a+x}{a+2x}=\frac{q}{p}: \therefore \frac{m}{n}=\frac{q}{p}, \text{ and } mp=nq.$$

10. From $a+c=\frac{2(a+b)(b+c)}{a+c+2b}$ may be deduced $a^2-b^2=b^2-c^2$.

11. Since a, b, c are in arithmetical progression, $a+c=2b$, multiply these equals by abc, then $ac(ab+bc)=2ab.bc$, and $\therefore ac=\frac{2ab.bc}{ab+bc}$, or ac is a harmonical mean between ab and bc.

12. Here $\frac{a+b}{1-ab}+\frac{b+c}{1-bc}=2b$, whence by reduction there arises

$$(a+c)(1+b^2)=2abc(1+b^2), \therefore \frac{1}{b}=\frac{2ac}{a+c}, \text{ i.e. } a, \frac{1}{b}, c \text{ are in } H.P.$$

X.

1. Here $b=\frac{2ac}{a+c}$, and $ab+bc=2ac$, $\therefore ab+bc-2ac=o$, add $2a^2-2ab$ to each side, $2a^2-2ac-ab+bc=2a^2-2ab=2a(a-b)$ or $(2a-b)(a-c)=2a(a-b)$,

$\therefore a-c=\frac{2a(a-b)}{(a-b)+(a-c)}$, or $a-c$ is the harmonical mean between a and $a-b$.

3. Since a, b, c, and $\frac{a}{b+c}$, $\frac{b}{c+a}$, $\frac{c}{a+b}$, are respectively in harmonical progression,

$\therefore \frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, and $\frac{b+c}{a}$, $\frac{c+a}{b}$, $\frac{a+b}{c}$, are respectively in arithmetical progression.

$$\frac{1}{a}+\frac{1}{c}=\frac{2}{b}, \text{ and } \frac{2(a+c)}{b}=\frac{b+c}{a}+\frac{a+b}{c}.$$

Deduce the former from the latter, and then reverse the process.

$$\text{Here } \frac{2(a+c)}{b}=\frac{b+c}{a}+\frac{a+b}{c}=\frac{bc+c^2+a^2+ab}{ac},$$
$$\text{and } 2a^2c+2ac^2=b^2c+bc^2+a^2b+ab^2,$$
$$\text{whence } c(a-b)\{c+a+b\}=a(b-c)\{a+b+c\},$$
$$\therefore ac-bc=ab-ac, \text{ and } ab+bc=2ac \therefore \frac{1}{a}+\frac{1}{c}=\frac{2}{b}.$$

4. Here $\frac{2(a+c)}{ac}=\frac{a+b}{ab}+\frac{b+c}{bc}$, or $\frac{1}{a}+\frac{1}{c}+\frac{1}{a}+\frac{1}{c}=\frac{1}{b}+\frac{1}{a}+\frac{1}{c}+\frac{1}{b}$, and $\frac{1}{a}+\frac{1}{c}=\frac{2}{b}$. Reverse the process.

5. From $\frac{b+c-a}{a}+\frac{a+b-c}{c}=\frac{2(a+c-b)}{b}$, may be deduced $b^2c+bc^2+a^2b+ab^2=2a^2c+2ac^2$, from which may be found $b(a+c)=2ac$, then reverse the process, as in Ex. 3.

XI.

1. Here $b=\frac{2ac}{a+c}$, then $b-a=\frac{ac-a^2}{a+c}$, and $b-c=\frac{ac-c^2}{a+c}$,

$$\therefore \frac{1}{b-a}+\frac{1}{b-c}=\frac{a+c}{a(c-a)}+\frac{a+c}{c(a-c)}=\frac{a+c}{c+a}.\left\{\frac{1}{a}-\frac{1}{c}\right\}=\frac{a+c}{c-a}.\frac{c-a}{ac}=\frac{a+c}{ac}=\frac{1}{c}+\frac{1}{a}.$$

3. Here $b=\frac{2ac}{a+c}$, $\frac{b}{a}=\frac{2c}{a+c}$, $\frac{a-b}{a}=\frac{a-c}{a+c}$, and $a-b=\frac{a(a-c)}{a+c}$.

$$\text{Next } b=\frac{2ac}{a+c}, \frac{b}{c}=\frac{2a}{a+c}, \frac{c-b}{c}=\frac{c-a}{a+c}, \text{ and } c-b=\frac{c(c-a)}{a+c}.$$

$$\text{Again } 2a-b=2a-\frac{2ac}{a+c}=\frac{2a^2}{a+c}, \text{ and } \frac{2a-b}{a}=\frac{2a}{a+c}, \text{ and } \frac{2c-b}{c}=\frac{2c}{a+c}.$$

Hence $\frac{a(a-b)}{2a-b}+\frac{2(c-b)}{2c-b}=\frac{a+c}{2}+\frac{c-a}{2}=o.$

5. Here $b=\frac{2a}{a+c}$, $b+a=\frac{3ac+a^2}{a+c}$, $b-a=\frac{ac-a^2}{a+c}$, $\therefore \frac{b+a}{b-a}=\frac{3c+a}{c-a}$.

Next $b+c=\frac{3ac+c^2}{a+c}$, and $b-c=\frac{ac-c^2}{a+c}$,

$\therefore \frac{b+c}{b-c}=\frac{3a+c}{ac-c^2}$. Then $\frac{b+a}{b-a}+\frac{b+c}{b-c}=\frac{3c+a}{c-a}+\frac{3a+c}{a-c}=2.$

XII.

1. This follows at once from XI. Example 4. $a^2+c^2>2b^2$ $\therefore a^2+b^2+c^2>3b^2$; but $a^2+b^2+c^2=1$, $\therefore 1>3b^2$, and $3b^2<1$, $\therefore b<\frac{1}{\sqrt{3}}$.

2. By clearing the equation from fractions, and transposing one term from each side, the common factor may be removed, and the result is $2ab=ac+bc$.

4. $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ are the *m*th, *n*th, *p*th terms respectively of an arithmetical progression. Let x be the first term, y the common difference, then $x+(m-1)y=\frac{1}{a}$, $x+(n-1)y=\frac{1}{b}$, $x+(p-1)y=\frac{1}{c}$. Eliminate x, y.

5. First $\frac{a-x}{px}=\frac{a-y}{qy}$, $\therefore aqy-qxy=apx-pxy$, and $a(qy-px)=xy(q-p)$.

Next $\frac{a-y}{qy}=\frac{a-z}{rz}$, $\therefore arz-ryz=aqy-qyz$, and $a(rz-qy)=yz(r-q)$.

Hence $\frac{qy-px}{rz-qy}=\frac{x(q-p)}{z(r-q)}$, but $q-p=r-q$, because p, q, r are in arithmetical progression. $\therefore \frac{qy-px}{rz-qy}=\frac{x}{z}$, and $qyz-pxz=rxz-qxy$.

$\therefore q(yz+xy)=(p+r)xz$, and $\frac{yz+xy}{xz}=\frac{p+r}{q}=2$, $\therefore$ $y=\frac{2xz}{x+z}$,

$\therefore$ x, y, z, are in harmonical progression.

6. $a+c=2b$, and $\frac{1}{a}+\frac{1}{d}=\frac{2}{b}$, then $c=2b-a$, and $\frac{1}{d}=\frac{2}{b}-\frac{1}{a}=\frac{2a-b}{ab}$.

$\therefore \frac{c}{d}=\frac{(2a-b)(2b-a)}{ab}=\frac{ab-2(a-b)^2}{ab}=1-\frac{2(a-b)^2}{ab}$.

7. The harmonical means between a, b; a, c; b, c; are $\frac{2ab}{a+b}$, $\frac{2ac}{a+c}$, $\frac{2bc}{b+c}$;

and these are in arithmetical progression, $\therefore$ $\frac{2ab}{a+b}+\frac{2bc}{b+c}=\frac{4ac}{a+c}$.

From this equality by reduction, $(a^2+c^2)b^2=2ac^2$.

8. See Art. 3, pp. 2, 3; and Art. 6, pp. 6, 7.

XIII.

1. $2^{10}-1$, and 2^n-1. 2. $-\frac{1}{3}(2^{30}-1)$, and $-\frac{1}{3}\{(-2)_n-1\}$. 3. $2^{-19}(2^{20}-1)$, and 2. 4. $\frac{1}{3}\cdot\frac{2^{20}-1}{2^{19}}$, and $\frac{2}{3}$. 5. $\frac{1}{5}\left\{\frac{3^{20}-2^{20}}{3^{17}}\right\}$, and $5\frac{2}{3}$. 6. $\frac{4}{3}\cdot\frac{5^{50}-2^{50}}{5^{47}}$.
7. $6\frac{1}{4}$. 8. $\frac{7}{15}\cdot\frac{8^{14}-7^{14}}{8^{13}}$. 9. $3\frac{17}{36}$. 10. $\frac{1}{3}\cdot\frac{4^n-1}{4^{n-1}}$. 11. $\frac{3^{20}-2^{20}}{2^{18}}$, and $\frac{3^{2n}-2^{2n}}{2^{2n-2}}$. 12. $\frac{3}{100}\cdot\frac{16^{3n}+9^{3n}}{9^{3n-1}}$. 13. $1\frac{4}{7}$. 14. $\frac{5}{8}$. 15. $\frac{1}{8}\cdot\frac{3^{30}-2^{30}}{3^{24}}$. 16. This

series obviously can be separated into the following two series :—

$1+2+4+\ldots\ldots$ to $\frac{n}{2}$ terms if n be even, or $\frac{n+1}{2}$ terms if n be odd,

and $\frac{2}{3}+\frac{4}{9}+\frac{8}{27}+\ldots\ldots$ to $\frac{n}{2}$ terms if n be even, or $\frac{n-1}{2}$ terms if n be odd.

17. 1. 18. $\frac{3}{25}$. 19. $\frac{a_n-b_n}{a-b}$. 20. $\frac{(10^4-1)3^{\frac{1}{4}}}{5^{\frac{1}{4}}(10^{\frac{1}{4}}-1)}$. 21. $\frac{1}{2}$.

XIV.

3. Let a denote the first term, r the common ratio.

Then $a+ar+ar^2+\ldots$ to $(2n+1)$ terms $=\frac{a(r^{2n+1}-1)}{r-1}$,

and $a^2+a^2r^2+a^2r^4+\ldots$ to $(2n+1)$ terms $=\frac{a^2(r^{4n+2}-1)}{r^2-1}$.

Also $a+ar^2+ar^4+\ldots$ to $(n+1)$ terms $=\frac{a(r^{2n+2}-1)}{r^2-1}$,

$ar+ar^3+ar^5+\ldots$ to n terms $=\frac{a(r^{2n+1}-r)}{r^2-1}$.

Excess of odd above even terms $=\frac{a(r^{2n+2}-1)}{r^2-1}-\frac{a(r^{2n+1}-r)}{r^2-1}=\frac{a(r^{2n+1}+1)}{r+1}$.

Hence $\frac{a(r^{2n+1}-1)}{r-1}\times\frac{a(r^{2n+1}+1)}{r+1}=\frac{a^2(r^{4n+2}-1)}{r^2-1}$.

5. Let x denote the first term of the series and y the common ratio, then $xy^{p+q-1}=m$, and $xy^{p-q-1}=n$; by taking the product of these equals, $x^2y^{2p-2}=mn$, and $xy^{p-1}=(mn)^{\frac{1}{2}}$ the pth term.

Next take the quotient, $\frac{xy^{p+q-1}}{xy^{p-q-1}}=\frac{m}{n}$, or $y^{2q}=\frac{m}{n}$, $\therefore y=\left(\frac{m}{n}\right)^{\frac{1}{2q}}$.

And since $xy^{p+q-1}=xy^{q-1}.y^p=m$, $\therefore xy^{q-1}=\frac{m}{y^p}=m\left(\frac{n}{m}\right)^{\frac{1}{2q}}$, the qth term.

6. See Art. 10, p. 8.

7. It may be shewn that the common ratio of every two successive terms of the series is $\frac{a}{c}$, the first term being $\frac{1}{a^2-b^2}$. The sum of n terms will be found to be equal to $\frac{1}{a^2-b^2}\cdot\frac{a^4-c^4}{a-c}\cdot\frac{1}{c^{n-1}}$.

8. The limit of each series is the same.

9. The nth term of the series is $2^n\left(\frac{1}{2}\right)^{n-1}=2$, and the series after the nth term is $1+\frac{1}{2}+\frac{1}{4}+\ldots\ldots$ the limit of whose sum is $\frac{1}{1-\frac{1}{2}}=2$.

10. Let x denote the common ratio, the first term being unity. Then the nth term is x^{n-1}, and according to the problem,

$x^{n-1}=x^n+x^{n+1}+x^{n+2}+$ ad infin. $=\frac{x^n}{1-x}$; $\therefore 1=\frac{x}{1-x}$, $1-x=n$;

$\therefore 2x=1$ and $x=\frac{1}{2}$, the common ratio. The series is $1+\frac{1}{2}+\frac{1}{4}+$ &c.

11. Let x be the first term and y the common ratio, then $xy^{p-1}=a$, $xy^{q-1}=b$ $xy^{r-1}=c$, and $y^{p-q}=\frac{a}{b}$, $y^{q-r}=\frac{b}{c}$; $\therefore y=\left(\frac{a}{b}\right)^{\frac{1}{p-q}}$, and $y=\left(\frac{b}{c}\right)^{\frac{1}{q-r}}$.

Hence $\left(\frac{a}{b}\right)^{\frac{1}{p-q}}=\left(\frac{b}{c}\right)^{\frac{1}{q-r}}$, and $\left(\frac{a}{b}\right)^{q-r}=\left(\frac{b}{c}\right)^{p-q}$; $\therefore\ a^{q-r}.c^{p-q}=b^{p-r}$,

and $a^{q-r}.b^{r-p}.c^{p-q}=1$.

12. Let a denote the first term, r the common ratio.

Then $s=\frac{a(r^n-1)}{r-1}$, $s'=\frac{1}{ar^{n-1}}\cdot\frac{r_n-1}{r-1}$, $\therefore\ \frac{s}{s'}=a^2r^{n-1}$, and $\left\{\frac{s}{s'}\right\}^n=a^{2n}.r^{n(n-1)}$.

$P=a.ar.ar^2+\ .\ .\ .\ ar^{n-1}=a^n.r^{1+2+\ldots+(n-1)}=a^n.r^{\frac{n(n-1)}{2}}$ $\therefore\ P^2=a^{2n}.r^{n(n-1)}$.

Hence $P^2=\left\{\frac{s}{s'}\right\}^n$.

13. $s=\frac{a(r^{2n}-1)}{r-1}$, and $s=k.r^n$. Eliminate r^n.

14. Suppose r the common ratio, then a, ar, ar^2, ar^3 are respectively equal to a, b, c, d.

And $(a\pm b)^2=a^2(1\pm r)^2$, $(a\pm c)^2=a^2r^2(1\pm r)^2$, $(c\pm d)^2=a^2r^4(1\pm r)^2$.

Hence $(a\pm b)^2$, $(b\pm c)^2$, $(c\pm d)^2$ have a common ratio $r^2(1\pm r)^2$.

XV.

1. Let a, ab, ab^2 be the respective first terms of the three series, r the common ratio, and n the number of terms.

Then $s_1=\frac{a(r^n-1)}{r-1}$, $s_2=\frac{ab(r^n-1}{r-1}$, $s_3=\frac{ab^2(r^n-1)}{r-1}$; $\therefore\ \frac{s_1}{s_2}=\frac{1}{b}$,

$\frac{s_2}{s_3}=\frac{1}{b}$; $\therefore\ \frac{s_1}{s_2}=\frac{s^2}{s^3}$ and $s_1s_3=s_2{}^2$.

2. Let r denote the common ratio in the first series, and n the number of terms in each series.

Then $s_1=\frac{a(r^n-1)}{r-1}$, $s_2=\frac{a}{r^{n-1}}$, $\frac{r^n-1}{r-1}$, $ar^{n-1}=l$. Whence $as_1=ls_2$.

3. $s_r{}^{2n+1}-s_{-r}{}^{2n+1}=\frac{a(r^{2n+1}-1)}{r-1}-\frac{a\{(-r)^{2n+1}-1\}}{-r-1}=a.\left\{\frac{r^{2n+1}-1}{r-1}-\frac{r^{2n+1}+1}{r+1}\right\}$

$=\frac{a(2r^{2n+1}-2r)}{r^2-1}=\frac{2ar\{(r^2)^n-1\}}{r^2-1}=2rs_{r^2}^{\ n}$.

4. Instead of the pth and qth terms as printed in the question, read the $(p+1)$th and $(q+1)$th terms.

The first series is $1+r^p+r^{2p}+\ .\ .\ .\ .$ ad infin. $=\frac{1}{1-r^p}=P$.

The second series $1+r^q+r^{2q}+\ .\ .\ .\ .$ ad infin. $=\frac{1}{1-r^q}=Q$.

$$\therefore\ 1=P-Pr^p,\ Pr^p=P-1,\ r^p=\frac{P-1}{P},\ \therefore\ r=\left\{\frac{P-1}{P}\right\}^{\frac{1}{p}}$$

$$1=Q-Qr^q,\ Qr^q=Q-1,\ r^q=\frac{Q-1}{Q},\ \therefore\ r=\left\{\frac{Q-1}{Q}\right\}^{\frac{1}{q}}.$$

Hence $\left\{\frac{P-1}{P}\right\}^{\frac{1}{p}}=\left\{\frac{Q-1}{Q}\right\}^{\frac{1}{q}}$, and $\frac{(P-1)^q}{P^q}=\frac{(Q-1)^p}{Q^p}$,

$$\therefore\ (P-1)^q.Q^p=(Q-1)^p.P^q.$$

6. $s_1=\frac{a(r^n-1)}{r-1}$, $s_2=\frac{2a(r^n-1)}{r-1}$, $s_3=\frac{3a(r^n-1)}{r-1}$, &c.

$\therefore\ s_1+s_2+s_3+\ .\ .\ .\ s_n=\frac{a(r^n-1)}{r-1}\left\{1+2+3+\ .\ .\ .\ n\right\}=\frac{a(r_n-1)}{r-1}\cdot\frac{n(n+1)}{1.2}$.

10. See Art. 11, p. 10.

XVI.

1. If a be the least term, d the common difference, then a, $a+d$, $a+2d$ are in arithmetical progression, and a, $a+d$, $a+2d+1$ are in geometrical progression.

2. Let x be added to each term, then the sums are $a+x$, $b+x$, $c+x$, $d+x$, and the squares of these quantities are in arithmetical progression, or

$(a+x)^2+(d+x)^2=(b+c)^2+(c+x)^2$, the value of x is the number required.

3. Here $\frac{1}{2}(a+b)=2(ab)^{\frac{1}{2}}$, $\therefore a+b=4(ab)^{\frac{1}{2}}$, and $a^2+2ab+b^2=16ab$, subtracting $4ab$ from these equals, $a^2-2ab+b^2=12ab$, $\therefore a-b=2\sqrt{3}(ab)^{\frac{1}{2}}$, and $a+b=4(ab)^{\frac{1}{2}}$. Hence $a:b::2+\sqrt{3}:2-\sqrt{3}$.

4. Here $ac=b^2$, $x=\frac{1}{2}(a+b)$, $y=\frac{1}{2}(b+c)$.

$$\text{Then } \frac{1}{x}+\frac{1}{y}=\frac{2}{a+b}+\frac{2}{b+c}=\frac{2}{a^{\frac{1}{2}}+c^{\frac{1}{2}}}\left\{\frac{1}{c^{\frac{1}{2}}}+\frac{1}{a^{\frac{1}{2}}}\right\}=\frac{2(a^{\frac{1}{2}}+c^{\frac{1}{2}})}{(a^{\frac{1}{2}}+c^{\frac{1}{2}})(ac)^{\frac{1}{2}}}=\frac{2}{(ac)^{\frac{1}{2}}}=\frac{2}{b}.$$

5. Let x denote the first term, and y the common difference, then $x+(p-1)y$, $x+(q-1)y$, $x+(r-1)y$, $x+(s-1)y$, are in arithmetical progression, and these are also in geometrical progression. Shew that $(p-q)(r-s)=(q-r)^2$.

6. If a, $a+d$, $a+2d$ be the terms of the progression, shew that $\frac{a+d}{a}$ is greater than $\frac{a+2d}{a+d}$.

8. Let v, x, y, z be in arithmetical progression, then

$$2x=v+y,\ 2y=x+2, \text{ and } v+z=x+y,$$

and $v+2$, $x+4$, $y+8$, $z+15$ are in geometrical progression,

$\therefore (v+2)(z+15)=(x+4)(y+8)$; find the values of v, x, y, z.

11. If m be added to $\frac{a}{x}$, $\frac{b}{y}$, $\frac{c}{z}$ respectively,

they become $\frac{a+mx}{x}$, $\frac{b+my}{y}$, $\frac{c+mz}{z}$.

Make the product of the first and third equal to the square of the second.

Then $\frac{(a+mx)(c+mz)}{xz}=\frac{(b+my)^2}{y^2}$, but $xz=y^2$, because x, y, z are in geometrical progression, $\therefore (a+mx)(c+mz)=(b+my)^2$, or

$$ac+(cx+az)m+m^2xz=b^2+2bmy+m^2y^2;$$

but $ac=b^2$, because a, b, c are in geometrical progression, and $m^2xz=m^2y^2$.

Hence $m(cx+az)=2mby$, and $\frac{(cx+az)m}{xz}=\frac{2bym}{y^2}$, or $\frac{c}{z}+\frac{a}{x}=\frac{2b}{y}$, or $\frac{c}{z}$, $\frac{b}{y}$, $\frac{a}{x}$, are in arithmetical progression,

$\therefore a+mx$, $b+my$, $c+mz$, are in geometrical progression.

XVII.

1. $x=\frac{(a+c)b-2ac}{a+c-2b}$. 2. See Art. 7, p. 7.

4. Let p, q denote the two numbers, then the two arithmetical means are

$$a_1=\tfrac{1}{3}(2p+q), \text{ and } a_2=\tfrac{1}{3}(p+2q).$$

And the harmonical mean between a_1 and a_2, $\frac{2a_1a_2}{a_1+a_2}=\frac{2(2p+q)(p+2q)}{9(p+q)}$.

Next, the two harmonical means are $b_1=\frac{3pq}{p+2q}$, $b_2=\frac{3pq}{2p+q}$.

And the arithmetical mean between b_1 and b_2 is $\frac{b_1+b_2}{2}=\frac{9pq(p+q)}{2(p+2q)(2p+q)}$,

$\therefore$ geometrical mean between $\frac{2a_1a_2}{a_1+a_2}$ and $\frac{b_1+b_2}{2}=(pq)^{\frac{1}{2}}$.

5. The arithmetical means $\frac{1}{2}(a+x)$, $\frac{1}{2}(b+y)$, $\frac{1}{2}(c+z)$ are in arithmetical progression. The geometrical means $(ax)^{\frac{1}{2}}$, $(by)^{\frac{1}{2}}$, $(cz)^{\frac{1}{2}}$, in geometrical progression, and the harmonical means $\frac{2ax}{a+x}$, $\frac{2by}{b+y}$, $\frac{2cz}{c+z}$, in harmonical progression. Form the equations and eliminate x.

8. Here $\frac{2y}{b}=\frac{x}{a}+\frac{z}{c}$, $b^2y^2=acxz$, and $\frac{b^2}{xz}=\frac{2a^2c^2}{a^2xy+c^2yz}$.

Eliminate x, y, z, and the result will give the equation required.

9. Here $y=\frac{2xz}{x+z}$, $2x=a+b$, and $z^2=ab$.

$$\text{Then } y=\frac{2xz}{x+z}=\frac{2(a+b)(ab)^{\frac{1}{2}}}{a+2(ab)^{\frac{1}{2}}+b}=\frac{2(a+b)}{\left\{\frac{a^{\frac{1}{4}}}{b^{\frac{1}{4}}}+\frac{b^{\frac{1}{4}}}{a^{\frac{1}{4}}}\right\}^2}.$$

10. Since $a^x=b^y$, $\therefore\ a=b^{\frac{y}{x}}$, also since $c^z=b^y$, $\therefore\ c=b^{\frac{y}{z}}$, $\therefore\ ac=b^{\frac{y}{x}+\frac{y}{z}}$, but $ac=b^2$, $\therefore\ b^{\frac{y}{x}+\frac{y}{z}}=b^2$, $\therefore\ \frac{y}{x}+\frac{y}{z}=2$, $\therefore\ y=\frac{2xz}{x+z}$, that is y is the harmonical mean between x and z.

XVIII.

1. See Art. 5, note, pp. 5, 6.

5. Here $b=\frac{a+c}{2}$, $c^2=bd$, $d=\frac{2ce}{c+e}$. From the first and third equality,

$$bd=\frac{ce(a+c)}{c+e},\ \text{and}\ c^2=bd,\ \therefore\ c^2=\frac{ce(a+c)}{c+e},\ \text{whence}\ ae=c^2.$$

6. Since $a=\frac{b+c}{2}$, and $c=\frac{2ab}{a+b}$; $\therefore\ ac=b(2a-c)$ and $b=2a-c$, whence $ac=b^2$.

7. Let x, y, z denote the three numbers in harmonical progression, then $y=\frac{2xz}{x+y}$, $(x-1)z=y^2$, and $2y=x+z-4$: from these equations find the values of x, y, z.

8. Let x, y denote the two quantities; then $\frac{x+y}{z}-(xy)^{\frac{1}{2}}=a$, and

$$\frac{x+y}{2}-\frac{2xy}{x+y}=b.\ \text{Find } x \text{ and } y \text{ from these equations.}$$

9. Let a, b, c, be in arithmetical progression, then $b=\frac{1}{2}(a+c)$; shew from this that $ac=\frac{2ab.bc}{ab+bc}$, or that ac is the harmonical mean between ab and bc. Reverse the operation.

13. Let x and y denote the two numbers, then $\frac{1}{2}(x+y)-(xy)^{\frac{1}{2}}=13$,

and $(xy)^{\frac{1}{2}}-\frac{2xy}{x+y}=12$. From the first equation $(x^{\frac{1}{2}}-y^{\frac{1}{2}})^2=26$,

and from the second $\frac{(xy)^{\frac{1}{2}}(x^{\frac{1}{2}}-y^{\frac{1}{2}})^2}{x+y}=12$. Whence $\frac{(xy)^{\frac{1}{2}}}{x+y}=\frac{12}{26}=\frac{6}{13}$,

and $\frac{2(xy)^{\frac{1}{2}}}{x+y}=\frac{12}{13}$: then $\frac{(x^{\frac{1}{2}}+y^{\frac{1}{2}})^2}{x+y}=\frac{25}{13}$, and $\frac{(x^{\frac{1}{2}}-y^{\frac{1}{2}})^2}{x+y}=\frac{1}{13}$, $\therefore\ \frac{(x^{\frac{1}{2}}+y^{\frac{1}{2}})^2}{(x^{\frac{1}{2}}-y^{\frac{1}{2}})^2}=25$,

$\therefore\ \frac{x^{\frac{1}{2}}+y^{\frac{1}{2}}}{x^{\frac{1}{2}}-y^{\frac{1}{2}}}=25$, $y^{\frac{1}{2}}=\frac{2x^{\frac{1}{2}}}{3}$, and $y=\frac{4x}{9}$, $\therefore\ \frac{x+y}{2}=\frac{13x}{18}$, $(xy)^{\frac{1}{2}}=\frac{2x}{3}$.

Substitute these values in the first equation, $x=234$, and $y=104$.

XIX.

1. £450 2*s.* 2¾*d.*

2. A little more than 10 per cent.

3. At 3 per cent., £67 2*s*. 9*d*. nearly; at 5 per cent., £80 4*s*. 6*d*. nearly.

4. If n be the number of years, then $n=\frac{2(\log 3-\log 2)}{\log 7+\log 3-\log 2-1}$, which can be determined from the logarithms given.

5. Each annual payment is £17 9*s*. 3*d*., and the whole sum paid for interest is £220 9*s*. 3*d*. The $\log 1{\cdot}04={\cdot}0170333$ and $\log 3{\cdot}243389={\cdot}5109990$ will facilitate the calculation.

6. Six years and scarcely five months.

8. It will be reduced to one half in 4¾ years nearly, and in 5 years to less than one half of the amount.

9. The interest was reckoned at 4 per cent. per annum.

10. $30\frac{10}{13}$ years' purchase, when money is at 3¼ per cent.; 25 years' when at 4 per cent., and 20 years' when at five per cent.

11. The estate is an annuity in perpetuity after the expiration of 20 years, for which the sum of £500 is paid down. The annual income, reckoning money at 6 per cent., is only £96 4*s*. 3¼*d*.

XX.

1. See Art. 10, note, p. 12, on the present value of an annuity.

3. See p. 12, note on the present value of an annuity to commence after m years, and to continue for n years.

4. Let x, y, z be the three portions, then $x+y+z=a$, and if p pounds be the value of each annual payment,

$$\text{then } x=\frac{p}{r}\cdot\frac{(1+r)^m-1}{(1+r)^m},\quad y=\frac{p}{r}\cdot\frac{(1+r)^n-1}{(1+r)^n},\quad z=\frac{p}{r}\cdot\frac{(1+r)^p-1}{(1+r)^p}.$$

$$\text{Then } \frac{y}{x}=\frac{1-(1+r)^{-n}}{1-(1+r)^{-m}},\quad \frac{z}{x}=\frac{1-(1+r)^{-p}}{1-(1+r)^{-m}}.\quad \text{But } x+y+z=a.$$

$$\text{Whence } x=\frac{\{1-(1+r)^{-m}\}a}{3-\{(1+r)^{-m}+(1+r)^{-n}+(1+r)\}^{-p}}.$$

Similarly for y and z.

5. The amount of the debt a is $a(1+r)^n$; and the amount of the annual payments of $\frac{a}{m}$ is $\frac{a}{m}\left\{\frac{(1+r)^n-1}{r}\right\}$.

7. Let P_1, P_2, P_3 . . . P_n, denote the respective present values of the n sums.

$$\text{Then } P_1=\frac{A}{2}\left\{1-\frac{1}{1+r}\right\},\ P_2=\frac{2A}{2}\left\{1-\frac{1}{(1+r)^n}\right\},\ P_3=\frac{3A}{r}\left\{1-\frac{1}{(1+r)^3}\right\},$$

$$\text{and } P_n=\frac{nA}{r}\left\{1-\frac{1}{(1+r)^n}\right\}.$$

$$\therefore P_1+P_2+\ldots+P_n=\frac{A}{r}\{1+2+3+\ldots+n\}$$
$$-\frac{A}{r(1+r)}\left\{1+\frac{2}{(1+r)}+\frac{3}{(1+r)^2}+\ldots+\frac{n}{(1+r)^n}\right\}$$
$$=\left\{\frac{n(n+1)}{1.2}-\frac{(1+r)^n-nr-1}{(1+r)^{n-1}r^2}\right\}\cdot\frac{A}{r}.$$

8. Here $P_1=\frac{A}{r}\left\{\frac{1}{(1+r)^2}+\frac{1}{(1+r)^4}+\ldots+\frac{1}{(1+r)^{2n}}\right\}$

$$=\frac{A}{r(1+r)^2}\left\{1+\frac{1}{(1+r)^2}+\frac{1}{(1+r)^4}+\ldots+\frac{1}{(1+r)^{2n-2}}\right\},$$

$$\text{and } P_2=\frac{A}{r}\left\{\frac{1}{1+r}+\frac{1}{(1+r)^3}+\ldots+\frac{1}{(1+r)^{2n-1}}\right\}$$

$$=\frac{A}{r(1+r)}\left\{1+\frac{1}{(1+r)^2}+\frac{1}{(1+r)^4}+\ldots+\frac{1}{(1+r)^{2n-2}}\right\}.$$

Hence $\frac{P_1}{P_2}=\frac{A}{r(1+r)^2}\times\frac{r(1+r)}{A}=\frac{1}{(1+r)}$, $\therefore\ P_2=P_1(1+r)$.

9. See notes pp. 11, 12.

10. Here $P=\frac{A}{r}\left\{1-\frac{1}{(1+r)^p}\right\}$, and $P+Q=\frac{A}{r}\left\{1-\frac{1}{(1+r)^{2p}}\right\}$. Dividing the latter by the former equality, $\therefore\ \frac{P+Q}{P}=1+\frac{1}{(1+r)^p}$, whence $\frac{Q}{P}=\frac{1}{(1+r)^p}$,

$$\therefore\ (1+r)^p=\frac{P}{Q}, \text{ and } 1+r=\left(\frac{P}{Q}\right)^{\frac{1}{p}},\ r=\left(\frac{P}{Q}\right)^{\frac{1}{p}}-1.$$

But $P=\frac{A}{r}\left\{1-\frac{1}{(1+r)^p}\right\}=\frac{A}{r}.\left\{1-\frac{Q}{P}\right\}$ $\therefore\ A=P\times\frac{Pr}{P-Q}=\frac{P^2}{P-Q}\left\{\left(\frac{P}{Q}\right)^{\frac{1}{p}}-1\right\}$.

13. If one of the institutions enjòy the benefaction for m years, the other is to enjoy it in perpetuity after the expiration of m years. If A denote the annual income, and r the interest of one pound for one year. Note, p. 12.

Then the present value for m years$=\frac{A}{r}.\left\{\frac{(1+r)^m-1}{(1+r)^m}\right\}$.

And the present value in perpetuity after m years$=\frac{A}{r}.\frac{1}{(1+r)^m}$.

$$\therefore\ \frac{A}{r}.\left\{\frac{(1+r)^m-1}{(1+r)^m}\right\}=\frac{A}{r}.\frac{1}{(1+r)^m}, \text{ and } (1+r)^m-1=1,\ \therefore\ (1+r)^m=2,$$

$$\text{and } m=\frac{\log 2}{\log(1+r)}.$$

If money bear interest at 5 per cent., the first institution must retain the estate for 14¼ years nearly, before the other enters on the perpetuity.

This method may be extended to calculate how long 2, 3, 4, &c., institutions should in succession hold the estate before a 3rd, 4th, 5th, &c., respectively enter upon it to hold it in perpetuity.

If a second institution is to hold it for n years after the first has held it m years, before the third enter upon the perpetuity.

The first, as before, holds for m years, and present value$=\frac{A}{r}\cdot\left\{\frac{(1+r)^m-1}{(1+r)^m}\right\}$.

The second holds for n years after m, and present value$=\frac{A}{r}\cdot\left\{\frac{(1+r)^n-1}{(1+r)^{m+n}}\right\}$.

And the third holds in perpetuity after $(m+n)$ years, present worth$=\frac{A}{r}\cdot\frac{1}{(1+r)^{m+n}}$.

$$\text{Then } \frac{A}{r}.\frac{(1+r)^n-1}{(1+r)^m}=\frac{A}{r}.\frac{(1+r)^n-1}{(1+r)^{m+n}},\ \frac{A}{r}.\frac{(1+r)^n-1}{(1+r)^{m+n}}=\frac{A}{r}.\frac{1}{(1+r)^{m+n}}.$$

From these equations $(1+r)^n=2$, and $(1+r)^m=\frac{3}{2}$, and $n=\frac{\log 2}{\log(1+r)}$, $m=\frac{\log 3-\log 2}{\log(1+r)}$

In the same manner, the calculation may be made for 4 or more institutions.

XXI.

1. Let x and y denote the numbers, then $\frac{x}{y}=\frac{3}{4}$, and $\frac{x+6}{y+5}=\frac{4}{5}$; find x and y.

2. Let x and y denote the numbers, then $\frac{x}{y}=\frac{4}{5}$, and $\frac{x+10}{y-10}=\frac{5}{4}$.

4. Let x, y, z be the three numbers, then $y^2=xz$, $x+z=51$, $y+z=60$, from which may be found $y=22\frac{1}{2}$ and 12, $z=37\frac{1}{2}$ and 48, $x=13\frac{1}{2}$ and 3.

6. Let v, x, y, z be the numbers, then $vz=xy$, $y=vx$, $z=x+y$, $vy=x+z$. Find the values of v, x, y, z.

7. Let x, y, z denote the numbers, a the sum of the numbers, b the sum of their

reciprocals, then $y^2=xz$, $x+y+z=a$, $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=b$. Find x, y, z.

8. Let x, y, z denote the parts, then $x+y+z=11$, and $\frac{xy}{4}=\frac{xz}{5}=\frac{yz}{6}$.

XXII.

1. Let $x-2y$, $x-y$, $x+y$, $x+2y$ denote the four numbers,

$$\text{then } (x-2y)+(x-y)+(x+y)+(x+2y)=56,$$
$$\text{and } (x-2y)^2+(x-y)^2+(x+y)^2+(x+2y)^2=864.$$

4. Let x, y, z denote the numbers, then $y=\frac{2xz}{x+z}$, $x+y+z=11$, $x^2+y^2+z^2=49$.

5. Let x, y, z be the numbers, $y=\frac{2xz}{x+z}$, $xyz=576$, $x+y+z=26$.

7. Let a, b, c be the three numbers, then $b=\frac{2ac}{a+c}$, and let x be the number subtracted from each, then $(b-x)^2=(a-x)(c-x)$, from which equation $x=\frac{b^2-ac}{2b-(a+c)}$, but $a+c=\frac{2ac}{b}$, $\therefore$ $x=\frac{b(b^2-ac)}{2(b^2-ac)}=\frac{b}{2}$, the number required.

8. $13x=11y$, and $13x+11y=180$.

9. Let a denote the first term, d the common difference, then $220=\frac{11}{2}\{2a+10d\}$ and $a+5d=20$. For the sum of the cubes of the eleven terms, see Art. 14, p. 13.
The series is 5, 8, 11, 14, 17, 20, 23, 26, 29, 32.

XXIII.

1. n even, $\frac{n(n^2-1)}{2.3}$. 2. n even, $\frac{n^2(n^2-3}{2.4}$. 3. n even, $\frac{n(n^2-1)(3n^2-7)}{2.3.5}$.

4. The series $1^2-3^2+5^2-7^2+9^2 \ldots +n^2$, may be divided into the two series $1^2+5^2+9^2+$, &c., and $-(3^2+7^2+11^2+$, &c.). and the sums of each series can be found by means of the formula for the sum of the squares of a series of numbers in arithmetical progression. Art. 14, p. 13. 5 and 6 may be found by a similar process.

7. Since $2^2.1=2^2(2-1)=2^3-2^2$, $3^2.2=3^2(3-1)=3^3-3^2$,
$4^2.3=4^2(4-1)=4^3-4^2$, &c., $\therefore$ $1.2^2+2.3^2+3.4^2+\ldots\ldots+n(n+1)^2$
$=\{2^3+3^3+4^3+\ldots\ldots+(n+1)^3\}-\{2^2+3^2+4^2+\ldots\ldots+(n+1)^2\}$.

XXIV.

1. The general term of this series suggests how the terms are to be divided. Here $2^2-2=4-2=2$, $2^3-2=8-2=6$, $2^4-2=16-2=14$, $2^5-2=32-2=30$, &c.,

and $2+6+14+30+\ldots\ldots+(2^n-2)=(2^2+2^3+2^4+2^5+\ldots\ldots 2^{n+1}$
$-(2+2+2+\ldots\ldots \text{to } n \text{ terms})=2^2(2^n-1)-2n=2^{n+2}-2(n+2)$.

NOTE. The series numbered 2, 3, 4, 5, 6 may be verified in a similar manner.

7. Here $11=\frac{99}{9}=\frac{10^2-1}{10-1}$, $111=\frac{999}{9}=\frac{10^3-1}{10-1}$, $1111=\frac{10^4-1}{10-1}$, &c.

8. Here $7=7\times1=\frac{7\times9}{9}=\frac{7(10-1)}{10-1}$, $77=\frac{7\times99}{9}=\frac{7(10^2-1)}{10-1}$, &c

XXV.

1. The first five series come under the general form in Art. 13, p. 12. Each sum, however, may be found separately.

$$\text{Let } s=1+\tfrac{3}{2}+\tfrac{7}{4}+\tfrac{15}{8}+\tfrac{31}{16}+\ \ldots\ +\frac{2^{n-1}-1}{2^{n-1}}.$$

$$\frac{s}{2}=\tfrac{1}{2}+\tfrac{3}{4}+\tfrac{7}{8}+\tfrac{15}{16}+\ \ldots\ +\frac{2^{n-1}-1}{2^{n-1}}+\frac{2^{n-1}-1}{2^{n}}.$$

$$s(1-\tfrac{1}{2})=1+\tfrac{2}{2}+\tfrac{4}{4}+\tfrac{8}{8}+\tfrac{16}{16}+\ \ldots\ +\frac{2^{n-1}}{2^{n-1}}-\frac{2^{n-1}-1}{2^{n}}.$$

$$=n-\frac{2^{n-1}-1}{2^{n}}=\frac{(2n-1)2^{n-1}+1}{2^{n}},$$

$$\therefore\ s=\frac{(2n-1)2^{n-1}+1}{2^{n-1}}.$$

2. $\dfrac{4.10^{n}-36n+5}{81.10^{n}}+\dfrac{1}{3}$. 3. 4. 4. $1\dfrac{86}{95}$. 5. $\dfrac{2.4^{n}-6n-5}{3^{2}.4^{n}}+\dfrac{1}{3}$.

6. Let $s=1+2x+3x^{2}++4x^{3}+\ \ldots.\ nx^{n-1}$,

$$sx=x+2x^{2}+3x^{3}+\ \ldots.\ (n-1)x^{n-1}+nx^{n}$$

$$\therefore\ (1-x)s=1+x+x^{2}+x^{3}+\ \ldots.\ +x^{n-1}-nx^{n}$$

$$=\frac{x^{n}-1}{x-1}-nx^{n}=\frac{1-(n+1)x^{n}+nx^{n+1}}{1-x};$$

$$\therefore\ s=\frac{1-(n+1)x^{n}+nx^{n+1}}{(1-x)^{2}}.$$

7. $\dfrac{1+x-2x^{n-1}+(1-2n)(1-x)x^{n}}{(1-x)^{2}}$. 8. $\dfrac{1+x}{(1-x)^{3}}$. 9. $\dfrac{1+4x+x^{2}}{(1-x)^{4}}$

10. $\dfrac{2}{(1-x)^{3}}$. 11. $\dfrac{1+11x+11x^{2}+x^{3}}{(1-x)^{5}}$. 12. $\dfrac{1+26x+66x^{2}+26x^{3}+x^{4}}{(1-x)^{6}}$.

13. $\dfrac{a^{2}}{(a-b)^{2}}$.

14. $(x+x^{-1})+2(x^{2}+x^{-2})+3(x^{3}+x^{-3})+\ \ldots.$ to n terms

$$=(x+2x^{2}+3x^{3}+\ \ldots.\ nx^{n})+(x^{-1}+2x^{-2}+3x^{-3}+\ \ldots.\ nx^{-n}).$$

XXVI.

1. Suppose the possibility of expressing the sum of the squares and the sum of the cubes of the first n natural numbers in a series of the powers of n, and first for the sum of the squares.

Let $1^{2}+2^{2}+3^{2}+\ \ldots.\ +n^{2}=An^{3}+Bn^{2}+Cn+D$,

then $1^{2}+2^{2}+3^{2}+\ \ldots.\ +n^{2}+(n+1)^{2}=A(n+1)^{3}+B(n+1)^{2}+C(n+1)+D$.

By subtracting the upper from the lower equality,

$$\therefore\ n^{2}+2n+1=A\{(n+1)^{3}-n^{3}\}+B\{(n+1)^{2}-n^{2}\}+C\{(n+1)-n\},$$

$$=3An^{2}+(3A+2B)n+(A+B+C).$$

Since this expression is an identical equation, it follows that the corresponding terms are equal,

$$\therefore\ 3A=1,\qquad \therefore\ A=\tfrac{1}{3},$$

$$3A+2B=2,\qquad 2B=2-3A=2-1=1,\ \therefore\ B=\tfrac{1}{2},$$

$$A+B+C=1,\qquad C=1-B-A=1-\tfrac{1}{3}-\tfrac{1}{2}=\tfrac{1}{6}.$$

$$\therefore\ 1^{2}+2^{2}+3^{2}+4^{2}+\ \ldots.\ +n^{2}=\frac{n^{3}}{3}+\frac{n^{2}}{2}+\frac{n}{6}=\frac{n(n+1)(2n+1)}{1.2.3}.$$

Next for the sum of the cubes.

Let $1^{3}+2^{3}+3^{3}+\ \ldots.\ +n^{3}=An^{4}+Bn^{3}+Cn^{2}+Dn+E$,

Then $1^{3}+2^{3}+3^{3}+\ \ldots.\ +n^{3}+(n+1)^{3}=A(n+1)^{4}+B(n+1)^{3}+C(n+1)^{2}$
$+D(n+1)+E$.

Subtracting the upper line from the lower,

$$\therefore\ n^{3}+3n^{2}+3n+1$$

$$=A\{(n+1)^{4}-n^{4})+B(n+1)^{3}-n^{3}\}+C\{(n+1)^{2}-n^{2}\}+D\{(n+1)-n\}$$

$$=4An^{3}+(6A+3B)n^{2}+(4A+3B+2C)n+(A+B+C+D).$$

By equating corresponding terms of this identity,

$$\begin{aligned} &\therefore 4A=1, && \therefore A=\tfrac{1}{4}, \\ &6A+3B=3, && 3B=3-\tfrac{3}{2}=\tfrac{3}{2}, \therefore B=\tfrac{1}{2}, \\ &4A+3B+2C=3, && 2C=3-1-\tfrac{3}{2}=\tfrac{1}{2}, \therefore C=\tfrac{1}{4}, \\ &A+B+C+D=1, && D=1-\tfrac{1}{4}-\tfrac{1}{2}-\tfrac{1}{4}=0. \end{aligned}$$

$$\therefore 1^3+2^3+3^3+\ldots\ldots+n^3=\frac{n^4}{4}+\frac{n^3}{2}+\frac{n^2}{4}=\frac{n^2(n+1)^2}{1^2.2^2}=\left\{\frac{n(n+1)}{1.2}\right\}^2.$$

NOTE. If higher powers of n than the third and fourth had been assumed in the expressions for the sum of the squares, and of the cubes, respectively, of the natural numbers; it would have been found that all the coefficients of these powers would be zero.

XXVII.

1. The number of balls in a pyramidal pile of n courses having an equilateral triangle for its base, is equal to the sum of the first n of the series of triangular numbers.

And the number of balls in a pyramidal pile of n courses, having a square for its base, is equal to the sum of the first n of the series of quadrangular numbers. See note, p. 3.

As the rectangular base of this pile has m balls in its length, and n in its breadth, the number of balls in the ridge of the pile will be $m-n+1$, also the lowest course of the pile is composed of $n(m+n)$ or $mn+n^2$ balls.

5. Let x denote the number of balls in a side of the lowest course, then $\frac{x(x+1)(2x+1)}{1.2.3}$ is the number of balls in the pile, and if w denote the weight of each ball, then $\frac{x(x+1)(2x+1)w}{1.2.3}$ denotes the weight of the pile.

But if the side of the lowest course had contained $x-1$ balls, then $\frac{(x-1)x(2x-1)}{1.2.3}$ would have been the number of the pile, and $\frac{(x-1)x(2x-1)w}{1.2.3}$ its weight. And the ratio of their weights is $\frac{(x-1)(2x-1)}{(x+1)(2x+1)}=\frac{55}{91}$, from which x, the number of balls in the side of the lowest course, may be found.

To find the weight of the shot in the pile.

Here 2048lb. = weight of an iron ball 12 inches in radius,
$w$lb. = „ „ 6 „ „

$$\frac{w}{2048}=\frac{6^3}{12^3}=\frac{1}{2^3}=\frac{1}{8}, \therefore x=\frac{2048}{8}=256\text{lb}.$$

And the weight of the pile $=\frac{x(x+1)(2x+1)\times 256}{1.2.3}$ pounds, when the value of x is substituted.

XXVIII.

1. $a_4=\left\{\frac{a_7^2}{a_1^3}\right\}^{\frac{1}{2}}$.

2. In the given formula, take n equal to 1, 2, 3 respectively, and eliminate a_2 and a_3 from the three equations.

3. Here $1+2n+3n^2+\ldots$ ad infin. $=\frac{1}{(1-n)^2}=n^2$, when n becomes $1+\frac{1}{n}$.

4. This may be shewn to be true in general for any three terms whatever.

Here $(n-1)^{\frac{1}{2}}+(n+1)^{\frac{1}{2}}<2n^{\frac{1}{2}}$, $(n+1)^{\frac{1}{2}}<2n^{\frac{1}{2}}-(n-1)$,

$$n+1 < 4n-4(n^2-n)^{\frac{1}{2}}+n-1,\ 4(n^2-n)^{\frac{1}{2}} < 4n-2,\ 2(n^2-n)^{\frac{1}{2}} < 2n-1,$$
$$4n^2-4n < 4n^2-4n+1.$$

5. For the series (1) let $y=An+Bn^2+Cn^3+Dn^4+$, &c., in which A, B, C are unknown,

$$\text{then} -\frac{y^2}{2}= -\tfrac{1}{2}\{A^2n^2+2ABn^3.+(2AC+B^2)n^4+\ \text{&c.}\}$$
$$+\frac{y^3}{3}= \qquad \tfrac{1}{3}\{A^3n^3+3A^2Bn^4+\ \text{&c.}\}$$
$$-\frac{y^4}{4}= \qquad -\tfrac{1}{4}\{A^4n^4+\ \text{&c.}\}$$

$$\text{Then } x=y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\ \text{&c.}$$

$$=Ax+\left(B-\frac{A^2}{2}\right)x^2+\left(C-AB+\frac{A^3}{3}\right)x^3+\left\{D-AC-\frac{B^2}{2}+A^2B-\frac{A^4}{4}\right\}x^4+\ \text{&c.},$$

which is an indentity, and the corresponding terms involving the same powers of x are equal.

$\therefore\ A=1,$

$B-\frac{A^2}{2}=0,\ \therefore\ B=\frac{A^2}{2}=\frac{1}{1.2}.$

$C-AB+\frac{A^3}{3}=0,\ C=AB-\frac{A^3}{3}=\frac{1}{2}-\frac{1}{3}=\frac{1}{1.2.3}.$

$D-AC-\frac{B^2}{2}+A^2B-\frac{A^4}{4}=0,\ D=AC+\frac{B^2}{2}-A^2B+\frac{A^4}{4}=\frac{1}{6}+\frac{1}{8}\ \ \frac{1}{2}+\frac{1}{4}=\frac{1}{1.2.3.4}$, &c.

$\therefore\ y=x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\frac{x^4}{1.2.3.4}+$ &c.

8. The former ratio is $\left\{\frac{1+x}{1-x}\right\}^2$, and the latter $\frac{1-x^2}{x}$.

9. Here $s=1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\ \ldots$ ad infin.

$$\therefore\ \frac{2s}{2^n}=\frac{2}{2^n}+\frac{2}{4^n}+\frac{2}{6^n}+\ \ldots\ \text{ad infin.}$$

$$\therefore\ s\left\{1-\frac{1}{2^{n-1}}\right\}=1-\frac{1}{2^n}+\frac{1}{3^n}-\frac{1}{4^n}+\frac{1}{5^n}-\ \ldots\ \text{ad infin.}\ =s'$$

$$\therefore\ \frac{s}{s'}=\frac{2^{n-1}}{2^{n-1}-1}$$

10. $x=3$ and -10. 11. $x=\frac{1}{4}$.

XXIX.

1. See Ratio, Proportion, and Variation, Art. 5, note, pp. 7, 8.

2. Let a, b, the sides, c, the hypotenuse of the right-angled triangle, having a^2, b^2, c^2 in harmonical progression.

Then $\frac{1}{a^2}-\frac{1}{b^2}=\frac{1}{b^2}-\frac{1}{c^2}$, and $a^2+b^2=c^2$. Whence may be found

$$c=\pm a(1\pm\sqrt{2})^{\frac{1}{2}} \text{ and } b=\pm 2^{\frac{1}{4}}a.$$

3. The sum is $\frac{3a^2\sqrt{3}}{16}.\left\{1+\frac{3}{4}+\left(\frac{3}{4}\right)^2+\ldots\ \text{ad infin.}\right\}$.

4. Let x, xy, xy^2, xy^3 denote the four shares, then $x+xy+xy^2+xy^3=100$, and $xy^3-x:xy^2-xy::37:12$, determine x and y.

5. Let a, b, c denote the common differences in harmonical progression,

then $b=\frac{2ac}{a+c}$: $s_1=\frac{n}{2}\{2+(n-1)a\}$, $s_2=\frac{n}{2}\{2+(n-1)b\}$, $s_3=\frac{n}{2}\{2+(n-1)c\}$.

$$\text{And } 2s_1s_3 - s_1s_2 - s_2s_3 = \frac{n^2(n-1)}{2}\{a-2b+c\} + \frac{(n-1)^2n^2}{4}\{2ac-b(a+c)\},$$

$$\text{but } 2ac-b(a+c)=o, \therefore 2s_1s_3 - s_1s_2 - s_2s_3 = \frac{n^2(n-1)}{2}\{a-2b+c\},$$

$$\text{also } s_1+s_3-2s_2 = \frac{n(n-1)}{2}\{a-2b+c\} \therefore \frac{2s_1s_3 - s_1s_2 - s_2s_3}{s_1+s_3-2s_2} = n.$$

6. Let v, v' be the volumes, w, w' the weights, and s, s' the specific gravities of the two substances. Then since the weight is equal to the product of the volume and specific gravity, supposing the specific gravity to mean the weight of an unit of volume; then $w=sv$, and $w'=s'v'$. If s'' be the specific gravity of the mixture, then $sv+s'v'=w+w'=(v+v')s''$, and $v'(s''-s')=v(s-s'')$. If the volumes be equal or $v=v'$, then $s''-s'=s-s''$, and $2s''=s+s'$, $\therefore s''=\frac{1}{2}(s+s')$.

Next let the weights be equal, or $w=w'$, then $\frac{v}{v'}=\frac{s'}{s}$ and $\frac{v}{v'}=\frac{s-s''}{s''-s'}$, $\therefore \frac{s-s''}{s''-s'}=\frac{s'}{s}$,

$$\text{whence } s''=\frac{2ss'}{s+s'}.$$

ELEMENTARY ALGEBRA,
WITH BRIEF NOTICES OF ITS HISTORY.

SECTION XI.

PERMUTATIONS AND COMBINATIONS,
BINOMIAL AND POLYNOMIAL THEOREMS,
AND PROBABILITIES.

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VARIATIONS, PERMUTATIONS, AND COMBINATIONS.

Art. 1. The different sets that can be formed of any number of things without regard to the order in which they are placed, are called their combinations, as the combinations of three things, a, b, c, taken two at a time, are ab, ac, bc.

The different orders in which any number of things less than the whole can be arranged are called their variations, as the variations of three things a, b, c, taken two at a time, are ab, ba, ac, ca, cb, bc.

The different arrangements which can be made of the whole of any number of things are called their permutations, as the permutations of three things, a, b, c, are abc, cba, acb, bca, bac, cab.

The number of the combinations of n different things taken r at a time is denoted by ${}^nC_r$, the number of variations by ${}^nV_r$, and the number of permutations of n different things by ${}^nP$.

If the combinations of n things be taken r at a time, and $n-r$ at a time, the latter are named the *complementary combinations* of the former.

2. *To find expressions for the numbers of variations of n different things, when they are taken* 1, 2, 3, 4 . . . n *at a time.*

Let a_1, a_2, a_3, a_4, ... a_n denote the n different things.

First. The number of these n things when taken one by one is obviously n, and ${}^nV_1 = n$.

Secondly. To find the number of the variations when the things are taken two at a time.

Of the n things a_1 may be taken and placed successively before each of the remaining $n-1$ things, a_2, a_3, a_4 . . . . a_n, and thus form $n-1$ variations taken two at a time, in each of which a_1 stands first; and there can be no more variations in which a_1 stands first.

Similarly, a_2 may be taken and placed successively before a_1, a_3, a_4 . . . . a_n, and thus form $n-1$ variations, in each of which a_2 stands first.

And in the same may be done with a_3, a_4 . . . a_n successively; there being $n-1$ variations corresponding to each letter in the first place.

Since therefore there are n different sets, and $n-1$ variations in each set, $\therefore {}^nV_2 = n(n-1)$.

Thirdly. To find the number of variations when the n things are taken three at a time.

Omit a_1, and the number of variations formed of the remaining $n-1$ things taken two at a time will be $(n-1)(n-2)$.

Then placing a_1 before each of these, there will $(n-1)(n-2)$ variations taken three at a time, in each of which a_1 stands first; and there can be no more variations in which a_1 stands first; otherwise there would be more than $(n-1)(n-2)$ variations of the remaining $n-1$ things taken two at a time.

Similarly, omitting a_2, there will be $(n-1)(n-2)$ variations of the remaining $n-1$ things taken two at a time. Then placing a_2 before each of these, there will be $(n-1)(n-2)$ variations, in each of which a_2 stands first.

And the same may be said of $a_3, a_4, \ldots . a_n$ successively.

Hence, since there are n sets and $(n-1)(n-2)$ variations in each set, $\therefore {}^nV_3 = n(n-1)(n-2)$.

By a similar process of reasoning, ${}^nV_4 = n(n-1)(n-2)(n-3)$.

And generally, if r things be taken at a time, and it be assumed that

${}^nV_r = n(n-1)(n-2)\ldots.(n-r+1)$, it is required to shew that

${}^nV_{r+1} = n(n-1)(n-2)\ldots.(n-r+1)(n-r)$;

that is, if the law be true for the number of variations of n things taken r at a time, it is also true when they are taken $r+1$ at a time.

For, omit a_1, then all the variations of the remaining $n-1$ things taken r at a time, are $(n-1)(n-2)(n-3)\ldots.(n-r)$.

If a_1 be placed before each of these variations, there will be formed $(n-1)(n-2)(n-3)\ldots(n-r)$ variations of n things taken $r+1$ at a time, in each of which a_1 stands first.

And the same may be said of $a_2, a_3, a_4 \ldots . a_n$ successively. Since there are n sets and $(n-1)(n-2)(n-3)\ldots.(n-r)$ variations in each set, therefore the number of variations of n things $r+1$ at a time, or ${}^nV_{n+1} = n(n-1)(n-2)\ldots.(n-r)$.

Hence, if the law be true for the variations of n things taken r at a time, it is also true for the variations of n things taken $r+1$ at a time.

But it has been proved true for n things taken two at a time, and three at a time. Therefore it is generally true of n things taken r at a time,*

$$\text{or, } {}^nV_r = n(n-1)(n-2)(n-3)\ldots.(n-r+1).$$

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* *To find the number of variations of* n *different things taken* r *and* r *together, when each thing is contained* 1, 2, 3 .... r *times in each variation.*

First let a_1 be placed before each of the n things $a_1, a_2 \ldots a_n$, and thus form n variations; and in the same manner each of the rest of the n things. Then since there are n sets and n variations in each set, there will be $n \times n$ or n^2 variations of the n things taken two and two, including the variations a_1a_1, a_2a_2, &c.

Next, let a_1 be placed before each of the n^2 variations, and form n^2 variations taken 3 and 3 together, in each of which a_1 stands first, and the same may be done with each of the remaining things; therefore there will be $n \times n^2$, or n^3 variations when three things are taken 3 and 3 together, and similarly when the things are

It is obvious that

$${}^{n}V_{r+1} = n(n-1)(n-2)\dots(n-r+1)(n-r) = (n-r).{}^{n}V_{r}$$

And if r be taken equal to n in the expression for the number of variations of n things taken r at a time, the result then becomes an expression for the number of permutations of n different things; that is,

$${}^{n}P = n(n-1)(n-2)(n-3)\dots\dots 3.2.1.$$

3. *To find an expression for the number of permutations of n things when r of them are identical.*[*]

If the r things which are identical be supposed to be different, they admit of $r(r-1)(r-2)\dots\dots 3.2.1$ permutations, but when they become identical, this number is reduced to one.

Hence the permutations of n things when all are different are $r(r-1)(r-2)\dots 3.2.1$ times as great as when r of them are identical. But $n(n-1)(n-2)\dots\dots 3.2.1$ denotes the number of permutations of n things when all are different.

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taken 4 and 4 together. And generally, when the things are taken r and r together, the whole number of variations will be n^r.

To find the number of variations of n different things when they are taken 1, 2, 3....r at a time, with repetitions.

Instead of taking away a_1, let a_1 be annexed to each of the n things, and the number of things becomes $n+1$, and when a_1 is placed before each of these $n+1$ things, there will be formed $n+1$ variations taken 2 and 2 (with repetitions of a_1), in each of which a_1 stands first.

And each of the remaining $n-1$ things can stand first in each set of $n+1$ variations, with repetitions of the a_2, a_3, &c. Then since there are n sets of the variations, and $n+1$ variations in each set, the whole number of variations of n things with repetitions when taken 2 and 2 together is $n(n+1)$.

Next, when a_2 is repeated as well as a_1, the number of the variations with repetitions of the n things is $(n+1)(n+2)$, and when a_1 is placed before each of these $(n+1)(n+2)$ variations, taken two at a time, there will be $(n+1)(n+2)$ variations of the n things, taken 3 and 3 together with repetitions, in each of which a_1 stands first. And the same may be done with the rest of the n things; therefore on the whole there will be $n(n+1)(n+2)$ variations, with repetitions of the n things taken 3 and 3 at a time. And, generally, when the variations with repetitions are taken r and r together, the whole number of variations will be

$$n(n+1)(n+2)\dots(n+r-1).$$

[*] The truth of this proposition may be exhibited in the following form:—

Of the n different things let the last r of them be supposed to become alike, then these r like things admit of only one permutation.

If a_1 be taken from the rest, it may occupy the first, second, third, &c.... and $(r+1)$th place in the order of the r like things, and thus form $r+1$ different permutations.

Next, if a_2 be taken, it may occupy the first, second, third, ... and $(r+2)$nd place in these permutations, and thus form on the whole $(r+1)(r+2)$ permutations.

Similarly, if a_3 be taken and employed in the same manner, there will result $(r+1)(r+2)(r+3)$ permutations.

And generally, when the rest of the $n-r$ things have been taken, there will be on the whole, $(r+1)(r+2)\dots\{r+(n-r)-1\}\{r+(n-r)\}$,

or $n(n-1)+\dots(r+2)(r+1)$ permutations.

Hence the number of permutations of n things when r of them are identical is equal to $\frac{n(n-1)(n-2)\ldots(r+1)r(r-1)\ldots.3.2.1}{1.2.3\ldots.r.}$

$$=n(n-1)(n-2)\ldots.(r+1).$$

If in addition, there were also s other things identical, but different from the r identical things. It may in the same way be shewn that the number of permutations is equal to

$$\frac{n(n-1)(n-2)\ldots.3.2.1}{1.2.3\ldots.r.1.2.3\ldots.s}.$$

Also if t things besides the above named were identical, the number of permutations would be

$$\frac{n(n-1)(n-2)\ldots.3.2.1}{1.2.3\ldots.r.1.2.3\ldots.s.1.2.3\ldots.t}.$$

And similarly for any other sets of identical quantities.

4. *To find an expression for the number of the combinations of n different things when taken 2, 3, 4 . . . . n at a time.*

Let $a_1, a_2, a_3 \ldots. a_n$ denote the n different things, and ${}^nC_2$ the number of combinations of the n things taken 2 at a time. Since every combination of two things admits of 2.1 permutations, therefore $2.1.{}^nC_2$ is equal to the number of the variations of n things taken 2 at a time.

But $n(n-1)$ is also equal to the number of the variations of n things taken 2 at a time. Therefore $2.1.{}^nC_2=n(n-1)$, and ${}^nC_2=\frac{n(n-1)}{1.2}$.

Also in the same manner, if ${}^nC_3$ denote the number of combinations of n things taken 3 at a time.

Since every combination of 3 things admits of 3.2.1 permutations; then $3.2.1.{}^nC_3$ denotes the number of variations of n things taken 3 at a time; but $n(n-1)(n-2)$ has been shewn to be the number of the same variations; therefore $3.2.1.{}^nC_3=n(n-1)(n-2)$, and ${}^nC_3=\frac{n(n-1)(n-2)}{1.2.3}$.

And generally,

Since every combination of r things admits of

$$r(r-1)(r-2)\ldots.3.2.1 \text{ permutations.}$$

Therefore $r.(r-1)(r-2)\ldots.3.2.1$ times the number of combinations of n things is equal to the corresponding number of variations of the same things; that is

$$r(r-1)(r-2)\ldots.3.2.1.{}^nC_r={}^nV_r=n(n-1)(n-2)\ldots.(n-r+1),$$

$$\text{and consequently } {}^nC_r=\frac{n(n-1)(n-2)\ldots..(n-r+1)}{1.2.3\ldots\ldots r}.$$

$$\text{And } {}^nC_{r+1}=\frac{n(n-1)(n-2)\ldots.(n-r+1)(n-r)}{1.2.3\ldots r.(r+1)}=\frac{n-r}{r+1}.{}^nC_r.$$

Or, the number of combinations of n things taken $r+1$ at a time is $\frac{n-r}{r+1}$ times the number of n things taken r at a time.[*]

5. *The number of the combinations of n things when taken r at a time is equal to the number of combinations of the same n things when they are taken $n-r$ at a time.*[†]

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[*] *To find the number of combinations that can be formed out of two sets, one set consisting of m like things, and the other set of n like things, the latter being different from the former, when they are combined in every possible way.*

Let one set consist of m things each equal to a, and the other set of n things each equal to b.

Then the combinations made out of the first set of m like things are a, aa, aaa, &c., or $a, a^2, a^3 \dots a^m$, are m in number. And the combinations made out of the other set of n like things, b, bb, bbb, &c., or $b, b^2, b^3 \dots b^n$, are n in number.

The number of all the possible combinations that can be formed out of these different sets of like things taken singly, 2 and 2, &c., will be $(m+1)(n+1)-1$.

If a third set of p things each equal to c, were taken with each of these combinations, it may be shown that the total number of all the possible combinations are $(m+1)(n+1)(p+1)-1$. And it is obvious that of these combinations the number which contains c is $(m+1)(n+1)p$.

[†] In illustration may be exhibited the series of combinations of 7 and of 8 things when taken 1, 2, 3, &c., at a time.

$${}^7C_1=\frac{7}{1}=7.$$

$${}^7C_2=\frac{7.6}{1.2}=21.$$

$${}^7C_3=\frac{7.6.5}{1.2.3}=35.$$

$${}^7C_4=\frac{7.6.5.4}{1.2.3.4}=35.$$

$${}^7C_5=\frac{7.6.5.4.3}{1.2.3.4.5}=21.$$

$${}^7C_6=\frac{7.6.5.4.3.2}{1.2.3.4.5.6}=7.$$

$${}^7C_7=\frac{7.6.5.4.3.2.1}{1.2.3.4.5.6.7}=1.$$

$${}^8C_1=\frac{8}{1}=8.$$

$${}^8C_2=\frac{8.7}{1.2}=28.$$

$${}^8C_3=\frac{8.7.6}{1.2.3}=56.$$

$${}^8C_4=\frac{8.7.6.5}{1.2.3.4}=70.$$

$${}^8C_5=\frac{8.7.6.5.4}{1.2.3.4.5}=56.$$

$${}^8C_6=\frac{8.7.6.5.4.3}{1.2.3.4.5.6}=28.$$

$${}^8C_7=\frac{8.7.6.5.4.3.2}{1.2.3.4.5.6.7}=8.$$

$${}^8C_8=\frac{8.7.6.5.4.3.2.1}{1.2.3.4.5.6.7.8}=1.$$

It may here be observed that the whole number of combinations of 8 things is greater than twice the whole number of combinations of 7 things by one.

From these illustrations it appears that when the given number of things is 7, an odd number, the maximum number of combinations is 35, and when the given number is 8, an even number, the maximum number of combinations is 70.

These examples also exhibit the truth of the proposition respecting complementary combinations.

When the number of things combined is an odd number, as 7, the number of combinations is the same when the things are taken 1 and 6, 2 and 5, 3 and 4 together, respectively; and the number is the greatest possible when the things are combined 3 and 4 at a time.

When the number of things is even, the number of combinations is the same

$$\text{For } {}^nC_r = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{1.2.3\ldots\ldots r}$$

$$\text{and } {}^nC_{n-r} = \frac{n(n-1)(n-2)\ldots\ldots\{n-(n-r)+1\}}{1.2.3\ldots\ldots(n-r)}$$

$$= \frac{n(n-1)(n-2)\ldots\ldots(r+1)}{1.2.3\ldots\ldots(n-r)}.$$

Dividing the former equation by the latter,

$$\frac{{}^nC_r}{{}^nC_{n-r}} = \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{1.2.3\ldots\ldots r} \div \frac{n(n-1)(n-2)\ldots\ldots(r+1)}{1.2.3\ldots\ldots(n-r)}$$

$$= \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{1.2.3\ldots\ldots r} \times \frac{1\,.\,2\,.\,3\ldots\ldots(n-r)}{n.(n-1)(n-2)\ldots\ldots(r+1)}$$

$$= \frac{n(n-1)(n-2)\ldots\ldots(n-r+1)(n-r)\ldots\ldots3.2.1}{1.2.3\ldots\ldots r(r+1)\ldots\ldots(n-1)n} = 1.$$

Hence ${}^nC_r = {}^nC_{n-r}$.

6. *To find when the number of the combinations of n things is the greatest possible.*

Let n be the given number of things, and x of them be supposed to be taken at a time; then ${}^nC_x = \dfrac{n(n-1)(n-2)\ldots(n-x+1)}{1.2.3\ldots\ldots\ldots\ldots x}$.

Since each successive factor of the numerator of this expression is diminished by unity, while each successive factor of the denominator is increased by unity; the value of ${}^nC_x$ will continually increase until the last factor in the numerator *be not less than* the corresponding factor of the denominator.

First. Let the xth factor of the numerator be *equal to* the corresponding factor of the denominator, then $n-x+1=x$, and $2x=n+1$, therefore $x=\dfrac{n+1}{2}$. But as x is necessarily a whole number, it follows that $\dfrac{n+1}{2}$ is a whole number, and consequently n must be an *odd* number.

Therefore, when n is an odd number, the number of the combinations of n things will be the greatest possible when they are taken $\dfrac{n+1}{2}$ at a time.

Secondly. If the xth factor of the numerator be the last which exceeds the corresponding factor in the denominator.

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when taken 1 and 7, 2 and 6, 3 and 5 together, respectively; and when they are taken 4 at a time, the number of combinations is a maximum.

When the number of things taken to form a combination is greater than half the whole number of things, some labour in practice may be saved by using the complementary combinations. For instance, the number of the combinations of 20 things taken 13 at a time is the same as the number of the combinations of the same things taken 7 at a time.

Since each factor of the numerator is one less, and each factor of the denominator is one more than each preceding factor, it follows that the last factor of the numerator which exceeds the corresponding factor of the denominator can exceed it by one only; then $n-x+1=x+1$, and $2x=n$, therefore $x=\frac{n}{2}$, and consequently n must be an even number.

Then when n is even, the number of combinations of n things will be the greatest possible when they are taken $\frac{n}{2}$ at a time.

Since ${}^nC_x={}^nC_{n-x}$; and $x=\frac{n+1}{2}$, when n is *odd*, $\therefore\ n-x=\frac{n-1}{2}$; and the number of the combinations of n things when n is odd, is the greatest possible when they are taken $\frac{n+1}{2}$ and $\frac{n-1}{2}$ at a time.

Also $x=\frac{n}{2}$, when n is *even*, $\therefore\ n-x=\frac{n}{2}$; and the number of combinations of n things when n is *even* is greatest when they are taken $\frac{n}{2}$ at a time.

7. *The sum of the number of the combinations of n things taken r and r—1 at a time, is equal to the number of combinations of n+1 things taken r at a time.*

$$\text{Here } {}^nC_r=\frac{n(n-1)(n-2)\,\ldots\ldots\,(n-r+1)}{1.2.3\,\ldots\ldots\,r},$$

$$\text{and } {}^nC_{r-1}=\frac{n(n-1)(n-2)\,\ldots\ldots\,(n-r+2)}{1.2.3\,\ldots\ldots\,(r-1)},$$

$$\begin{aligned}\therefore\ {}^nC_r+{}^nC_{r-1}&=\frac{n(n-1)(n-2)\,\ldots\ldots\,(n-r+1)}{1.2.3\,\ldots\ldots\,r}\\&\qquad+\frac{n(n-1)(n-2)\,\ldots\ldots\,(n-r+2)}{1.2.3\,\ldots\ldots\,(r-1)},\\&=\frac{n(n-1)(n-2)\,\ldots\ldots\,(n-r+2)}{1.2.3\,\ldots\ldots\,(r-1)}.\left\{\frac{n-r+1}{r}+1\right\},\\&=\frac{n(n-1)(n-2)\,\ldots\ldots\,(n-r+2)(n+1)}{n(n-1)(n-2)\,\ldots\ldots\,(n-1)r},\\&=\frac{(n+1)n(n-1)\,\ldots\ldots\,\{(n+1)-r+1\}}{1.2.3\,\ldots\ldots\,(n-1)r},\\&={}^{n+1}C_r.\end{aligned}$$

8. *To find an expression for the number of combinations that can be formed by taking of the combinations of m things r at a time, with the combinations of n things s at a time.*

$$\text{Here } {}^mC_r=\frac{m(m-1)(m-2)\,\ldots\ldots\,(m-r+1)}{1.2.3\,\ldots\ldots\,r},$$

$${}^nC_s=\frac{n(n-1)(n-2)\,\ldots\ldots\,(n-s+1)}{1.2.3\,\ldots\ldots\,s}.$$

Now when each combination of the former set of m things is taken with each combination of the latter set of n things, the combinations which result will each consist of $r+s$ things, and the total number will be $\frac{m(m-1)\ldots(m-r+1)}{1.2\ldots r}\times\frac{n(n-1)(n-2)\ldots(n-s+1)}{1.2.3\ldots s}$.

This process can be extended to the combinations of three or more sets of things, each set forming separate and independent combinations.*

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* There are certain series of numbers which have been named *Figurate Numbers*, and may be supposed to originate from a series of units. Each series of these numbers is formed from the foregoing series, by adding 1, 2, 3, &c., of its terms to form the corresponding terms of the succeeding series, as will be obvious in the following orders of the figurate numbers, with their general terms.

Series of units	1,	1,	1,	1,	1,	1,	1,	1,	1,	1, &c.
F. N. of										
First order	1,	2,	3,	4,	5,	6,	7,	8,	9,	10, ... n.
Second order	1,	3,	6,	10,	15,	21,	28,	36,	45,	55, ... $\frac{n(n+1)}{1.2}$.
Third order	1,	4,	10,	20,	35,	56,	84,	120,	165,	220, ... $\frac{n(n+1)(n+2)}{1.2.3}$.
Fourth order	1,	5,	15,	35,	70,	126,	210,	330,	495,	715, ... $\frac{n(n+1)(n+2)(n+3)}{1.2.3.4}$.

&c., &c.

And the general term of the series of figurative numbers of the rth order will be

$$\frac{n(n+1)(n+2)\ldots\ldots(n+r-1)}{1.2.3\ldots\ldots r}.$$

It may here be remarked that the successive orders of the series of figurate numbers arise from a series of units; but the different series of polygonal numbers are produced from the sums of arithmetical progressions, which have the same first terms, but not the same common difference. Hence it cannot be affirmed that any series of figurate numbers (with one exception) is identical to any series of polygonal numbers.

EXERCISES.

I.

1. Explain the meanings of the words combinations, permutations, and variations, and give examples.

2. Of the letters a, b, c, d, e, write down the variations, when taken two and two, and their combinations when taken three and three together.

3. Required the variations of every four of the six characters a, a, a, b, b, c.

4. Of what number of things is the number of permutations 120?

5. How many different numbers can be formed by the permutations of the figures of the number 48555? Write them out fully. (*Lilavati.*)

6. How many numbers can be formed out of any five of the figures 122333444455555?

7. How many numbers, integral and decimal, can be formed with the digits 012345, taking three to form each number?

8. How many fractions can be formed with the nine digits, taking one only for the numerator and denominator of each fraction? Shew how many are identical in value, and how many pairs of which one is the reciprocal of the other.

9. Find the number of variations of five things, a, b, c, d, e, with repetitions, when they are taken 2 at a time, and when 3 at a time; write also all the variations at length.

II.

1. If the number of variations of n things taken 3 at a time, be 12 times the number of variations of $\frac{n}{2}$ things taken 3 at a time; what is the number of permutations of the same n things?

2. The number of combinations of $n+1$ things taken $n-1$ at a time is 36; find the number of permutations of n things.

3. If the number of combinations of n things taken 5 together be proportional to the number taken 3 together, as 3 is to 5; find n.

4. The number of combinations of a certain number of things taken four at a time bears to the number taken two at time the ratio of 15 to 2; what is the number of things?

5. The number of variations of n things taken r at a time is 272, and the number of combinations 136; find n and r.

6. If the ratio of the numbers of combinations of n things taken r at a time be to the number taken $r-1$ at a time as 9 to 4, and their variations be as 9 to 1; find the values of n and r.

7. If the variations of n letters taken three together be six times as

great as the number of combinations of the same letters taken four at a time; find the value of n.

8. If the number of the combinations of $2n$ things taken $n-1$ at a time be to the number of $(2n-1)$ things taken n at a time as 396 to 231; find the value of n.

III.

1. Express in words the meaning of the equation $^nC_2 = \frac{2}{15}\,^nC_4$, and find the value of n?

2. There are n things of which p are alike, and the remainder $n-p$ different. Shew how to find the number of variations of them taken r together? Ex. $n=7$, $p=3$, $r=3$.

3. Of the combinations of n things r at a time; how many of them do not contain any of p of the n things?

4. Of the combinations of n things r at a time, how many of them will contain q out of p of the n things?

5. How many permutations can be made of the letters a^nb^n?

6. Of how many divisors does $a^mb^nc^p$ admit? Find the number of the divisors of 76500?

7. Shew that if two of the n things be alike when they are combined r at a time, the number of combinations will be diminished by

$$\frac{(n-1)(n-2)\,\ldots\,(n-r+1)}{1\,.\,2\,\ldots\,(r-1)}.$$

IV.

1. Of the combinations of 20 different things 6 at a time; find how many of them do not respectively contain 1, 2, 3, 4, 5 of any six specified things?

2. There are 16 things of which 3 are alike; find the number of combinations, and also the number of variations of those things taken 4 at a time.

3. In how many ways could combinations of 6 things be chosen from 3 different sets of n things each, no combination having more than 3 things from any one set?

4. How many of the variations and permutations of the letters a, b, c, d, e, f can be formed, so that the letters a, b may stand first in each variation and permutation?

5. Find the variations of 7 letters taken 4 together, when 3 of the 7 letters are alike. In how many of them will this letter appear twice?

6. In the variations which can be formed from the seven letters a, b, c, d, e, f, g, taken five together, find how many will contain the letter a.

7. How many arrangements can be made out of the first nine letters of the alphabet, all the letters being taken in each arrange-

ment, and no two of the first four letters being together in any arrangement?

8. How many combinations will there be in ten letters, *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *k*, *l*, taken five at a time? In how many of these will *a*, *b*, and *c* occur?

V.

1. Find the number of permutations which can be formed out of the letters composing the words Mississippi, Constantinople, Heliogabalus, misadministration.

2. How many combinations can be formed by taking one vowel with two consonants of the English alphabet for each combination? By how many would the combinations be increased if the vowel was made to occupy the first, the second, and the third place of the three letters?

3. How many different sums could be paid with nine pence, nine shillings, nine sovereigns, and nine ten-pound notes?

4. How many different sums can be formed with a guinea, a half-guinea, a crown, a half-crown, a shilling, and a sixpence?

5. A square is divided into 16 equal squares by vertical and horizontal lines. In how many ways can 4 of these be painted white, 4 black, 4 red, and 4 blue, without repeating the same colour in the same vertical or horizontal row?

6. Shew that if *r* of *n* things be of one kind and all the rest different, the number of permutations of the *n* things is equal to the number of the combinations of *n* different things taken *r* at a time.

7. Find the number of triangles into which a polygon can be divided, of 4, 5, 6, 7 . . *n* sides respectively, by joining the angular points of the polygons.

8. How many triangular pyramids may be formed whose edges are six given straight lines, of which the sum of any three will form a triangle?

9. If a bag contains 8 white balls and 5 black, if 7 be taken out at a venture, how many different sets consisting of 4 white and 3 black balls can be formed?

10. An urn contains 12 black balls, 9 white, and 6 red; how many different sets can be formed by taking 3 black, 2 white balls, and 1 red to form each set?

VI.

1. In how many ways can a party of 2*n* people be arranged in couples?

2. Shew that nine persons can occupy four places at a round table 3024 different ways.

3. Find the number of ways in which *n* men can be placed in a row, so that two given men shall not be next each other.

4. Out of $2n$ men who have to sit down, half of them on each side of a long table, p particular men desire to sit on one side, and q on the other; find the number of ways in which this may be done.

5. There are three candidates for a certain office and twelve electors. In how many different ways is it possible for them to vote, and in how many of these ways will the votes be equally divided between the candidates?

6. Assuming that each of n courts sends an ambassador to each of the others; how many ambassadors would there be, and in how many different ways could they be distributed?

7. From a company of 50 men, four are draughted off every night to guard. On how many different nights can a different guard be posted, and on how many of these will any particular soldier be engaged?

VII.

1. How many permutations can be formed on the musical scale of seven notes?

2. How many different changes can be rung with a peal of nine bells?

3. Find the number of different arrangements that can be made of the prismatic colours, so that the blue and green shall not come together.

4. How many are the combinations in one composition, with ingredients of six different tastes, sweet, pungent, astringent, sour, salt, and bitter, taking them by ones, twos, &c.? (*Lilavati.*)

5. In how many different ways can a volunteer shooting at a target score 16 in 9 shots; a bull's-eye counting 3, a centre 2, an outer 1, and a miss 0?

6. How many squads can be formed of a company of 100 men? How many men must be taken to form the greatest possible number of squads?

7. How many changes can be made by throwing 1, 2, 3, 4, 5, 6 dice respectively?

8. The four faces of a tetrahedron are marked, as a common die, with one, two, three, four points; how many changes can be made in throwing 1, 2, 3, 4, 5, 6 tetrahedrons respectively?

9. There are three boxes severally containing n_1, n_2, n_3 letters; how many different combinations can be formed by taking r_1, r_2, r_3 letters respectively out of each of the three boxes?

VIII.

1. The number of permutations of $2n$ things whereof half are alike, and the other half alike, but different from the first half, is equal to the greatest number of combinations that can be formed of $2n$ things whereof none are alike.

2. The ratio of the number of combinations of $4n$ things taken $2n$ together to that of $2n$ things taken n together is $\frac{1.3.5\ldots(4n-1)}{\{1.3.5\ldots(2n-1)\}^2}$.

3. If ${}^nC_{r-r'} = {}^nC_{r+r''}$, find the value of n.

4. If $\frac{{}^nC_{r-1}}{2} = \frac{{}^nC_r}{3} = \frac{{}^nC_{r+1}}{4}$, then $n = 34$ and $r = 14$.

5. All the combinations of n things, p of one kind, q of another, and r of a third kind is equal to $(1+p)(1+q)(1+r)-1$.

6. Determine the number of combinations and of variations which can be formed with $a^4b^3c^2d$, when five letters are taken to form each combination and each variation.

7. There are n alike of each of the m letters $a, b, c \ldots$; find the whole number of variations which can be made of them without taking more than n letters together.

8. If n points in a plane be joined in all possible ways by indefinite straight lines, and if no two of the straight lines be parallel, and no three pass through the same point (with the exception of the n original points), then the number of points of intersection, exclusive of the n points will be $\frac{n(n-1)(n-2)(n-3)}{8}$.

9. Supposing the three sides of a triangle to be designated by the letters a, b, c, and their several opposite angles by A, B, C; determine (1) the several combinations of these six elements, taken 3 and 3 together. (2) The number of actually different cases which these contain for the determination of any fourth element, and their several forms. (3) The several cases in which this last problem is determinate, limited, or unlimited.

IX.

Verify the truth of the following properties:—

1. ${}^nV_r - {}^{n-1}V_r = r.{}^{n-1}V_{r-1}$.
2. ${}^{n+1}P = (n+1).{}^nP$.
3. ${}^nC_r = \frac{n}{n-r}.{}^{n-1}C_r$.
4. ${}^nV_r = n(n-1)\ldots\ldots(n-r)(n-r+1).{}^nV_{n-r}$ if r be greater than $\frac{n}{2}$.
5. ${}^{2n}C_{2r} = \frac{(2n-1)(2n-3)\ldots\ldots(2n-2r+1)}{1.3.\quad\ldots\ldots(2r-1)}.{}^nC_r$.
6. ${}^{n^2}C_5 = \frac{1.2.3.4}{n(n^2-1)} \times {}^{n^2}C_4 \times {}^{n+2}C_5$.

THE BINOMIAL AND POLYNOMIAL THEOREMS.

Art. 1. The Binomial Theorem is one of the most important theorems in the science of Algebra, as it is equally applicable for the processes of involution and evolution, as well as in many investigations in the higher branches of analysis.

The law of the successive coefficients of any power of a binomial was known to Vieta; and Briggs, in his "Trigonometrica Britannica," appears to have been the first to point out how the terms of any power of a binomial might be successively deduced from each other without any aid of the lower powers. It may be doubted, as Woodhouse has remarked in his "Principles of Analytical Calculation," whether Briggs knew how to express generally the law for the formation of the coefficients of the expansion of a binomial. Newton was the first who discovered that the binomial theorem was as competent to denote evolution, or the extraction of roots, as it was to denote involution, or the raising of the powers of a binomial. From this extension of the theorem many improvements have arisen in the higher departments of analytical science, and especially in logarithmic and other series. The process by which Newton made his discovery has been already described in his own words.* He attempted no general proof of the truth of the theorem, but contented himself by exhibiting its truth from an induction of particular instances. In the examples he adduced he found them all to succeed, and verified the correctness of them.

The first general proof of this theorem when the index is a positive integer was given by James Bernoulli, and was, after his death, printed in his "Ars Conjectandi," in 1713. His proof is based on the theory of combinations. Since that time numerous attempts have been made, with more or less success, to exhibit a general demonstration of this theorem, whether the index be integral or fractional, positive or negative.

Some of them have been conducted by the introduction of increments, or the method of limits, and on this account have not been regarded as satisfactory elementary proofs. Euler gave a proof in which he employed the differential calculus; he also devised another proof depending on elementary principles, which has been generally received, though not without objection.

There is also a proof which is made to depend on the value of the coefficient of the second term of the expansion, which can be shewn to be equal to the index itself.

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* In the third and following pages of Section III. of this work.

The development of the multinomial or polynomial theorem was first exhibited by De Moivre in the "Philosophical Transactions of the Royal Society."

2. *To find an expression for the expansion of* $(x+a)^n$, *or to shew the truth of the binomial theorem, when the index is a positive integer.*

It is obvious that the products of two, of three, and of four binomial factors, can be exhibited in the forms

$$(x+a_1)(x+a_2)=x^2+(a_1+a_2)x+a_1a_2.$$

$$(x+a_1)(x+a_2)(x+a_3)=x^3+(a_1+a_2+a_3)x^2+(a_1a_2+a_1a_3+a_2a_3)x+a_1a_2a_3.$$

$$\begin{aligned}(x+a_1)(x+a_2)(x+a_3)(x+a_4)=x^4&+(a_1+a_2+a_3+a_4)x^3\\&+(a_1a_2+a_1a_3+a_1a_4+a_2a_3+a_2a_4+a_3a_4)x^2\\&+(a_1a_2a_3+a_1a_2a_4+a_1a_3a_4+a_2a_3a_4)x+a_1a_2a_3a_4.\end{aligned}$$

Suppose this to be true for the product of n factors, and denoting the successive coefficients by c_1, c_2, $c_3 \dots c_r \dots c_n$,

$$(x+a_1)(x+a_2)\dots(x+a_n)=x^n+c_1x^{n-1}+c_2x^{n-2}+\dots+x^{n-r}c_r+\dots+c^n,$$

of which product the first term is x^n, and the succeeding terms involve x^{n-1}, x^{n-2}, &c.

Also the coefficient of the first term is unity,
and c_1 the coefficient of the second term is the sum of the n terms, $a_1, a_2, \dots a_n$,

c_2 the coefficient of the third term is the sum of the products of every two of them.

c_3 the coefficient of the fourth term is the sum of the products of every three of them, &c., &c.

c_r the coefficient of the $(r+1)$th term is the sum of the products of every r of them, &c, &c.

And c_n the last term is the product of all the n terms.

By introducing another factor, $x+a_{n+1}$, into this product, it becomes $(x+a_1)(x+a_2)(x+a_3)\dots(x+a_n)(x+a_{n+1})$

$$\begin{aligned}&=x^{n+1}+(c_1+a_{n+1})x^n+(c_2+a_{n+1}c_1)x^{n-1}+\dots+(c_r+a_{n+1}c_{r-1})x^{n-r}\\&\qquad+\dots(c_n+a_{n+1}c_{n-1})x+a_{n+1}c_n.\end{aligned}$$

And it appears that c_1+a_{n+1}, the coefficient of the second term of the product, consists of the sum of the second terms of the $n+1$ binomials; also, the coefficients of the third, fourth, &c., terms consist of the products of every 2, 3, 4, &c., respectively, of the second terms, and that the last term consists of the product of all the $n+1$ terms.

Hence, if the form of the product be true when there are n factors, it will also be true when there are $n+1$ factors.

But by actual multiplication it has been shewn to be true for two, three, and four factors; it is therefore true for 4+1 or five factors, and for 5+1 or six factors, and also for any number of factors, and consequently it is generally true,

that is $(x+a_1)(x+a_2)\dots(x+a_n)=x^n+c_1x^{n-1}+c_2x^{n-2}+c_3x^{n-3}+\dots$
$+c_rx^{n-r}+\dots+c_n$.

Now, when a_1, a_2, a_3, ... a_n become each equal to a, then $c_1=a+a+a+\dots$ to n terms $=na$.

$$c_2=a^2+a^2+a^2+\dots \text{ to } \frac{n(n-1)}{1.2} \text{ terms } =\frac{n(n-1)}{1.2}.a^2.$$

$$c_3=a^3+a^3+a^3+\dots \text{ to } \frac{n(n-1)(n-2)}{1.2.3} \text{ terms } =\frac{n(n-1)(n-2)}{1.2.3}.a^3,$$

&c., &c.

$$c_r=a^r+a^r+a^r+\dots \text{ to } \frac{n(n-1)\dots(n-r+1)}{1.2\dots r} \text{ terms}$$

$$=\frac{n(n-1)\dots(n-r+1)}{1.2\dots r}.a^r, \text{ \&c., \&c.}$$

$c_n=a.a.a\dots$ to n factors $=a^n$.

Making these substitutions for c_1, c_2, c_3, &c., in the expression $(x+a_1)(x+a_2)\dots(x+a_n)=x^n+c_1x^{n-1}-c_2x^{n-2}+\dots+c_rx^{n-r}+\dots+c_n$, it becomes

$$(x+a)^n=x^n+nx^{n-1}a+\frac{n(n-1)}{1.2}.x^{n-2}a^2+\frac{n(n-1)(n-2)}{1.2.3}x^{n-3}a^3+\dots$$
$$+\frac{n(n-1)\dots(n-2+1)}{1.2\dots r}.x^{n-r}a^r\dots+a^n,$$

which is the general expansion of the binomial theorem when the index is a positive integer, and the $(r+1)$th term is

$$\frac{n(n-1)\dots(n+r+1)}{1.2.3\dots r}x^{n-r}a^r.$$

The theorem may also be written, making the last term, and successive preceding terms, the first, second, third, &c., terms; since the combination of any number of things in any case is equal to the complementary combination of the same number of things.

$$(a+x)^n=a^n+na^{n-1}x+\frac{n(n-1)}{1.2}.a^{n-2}x^2+\frac{n(n-1)(n-2)}{1.2.3}a^{n-3}x^3+\dots+x^n.$$

The expansion of $(x-a)^n$ can be deduced from that of the expansion of $(a+x)^n$ by writing in this expansion $-x$ for $+x$,

$$(a-x)^n=a^n-na^{n-1}x+\frac{n(x-1)}{1.2}.a^{n-2}x^2-\frac{n(n-1)(n-2)}{1.2.3}a^{n-3}x^3$$
$$+\dots\frac{n(n-1)\dots(n-r+1)}{1.2\dots r}.a^{n-r}(-x)^r+\dots+(-x)^n,$$

in which the terms are alternately positive and negative.

3. If $x=a=1$,

Then $(x+a)^n$ becomes $(1+1)^n$, or

$$2^n=1+n+\frac{n(n-1)}{1.2}+\frac{n(n-1)(n-2)}{1.2.3}+\dots+\frac{n(n-1)\dots2.1}{1.2\dots(n-1)n},$$

$$\text{and } 2^n-1=n+\frac{n(n-1)}{1.2}+\frac{n(n-1)(n-2)}{1.2.3}+\dots$$

or the total number of the combinations of n things taken 1, 2, 3 . . . n respectively at a time is equal to 2^n-1.

Also $(x-a)^n$ becomes

$$(1-1)^n=1-n+\frac{n(n-1)}{1.2}-\frac{n(n-1)(n-2)}{1.2.3}+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4},\ \&c.,$$

$$\text{and }1+\frac{n(n-1)}{1.2}+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}+\&c.=n+\frac{n(n-1)(n-2)}{1.2.3}+\&c.,$$

or the sum of the coefficients of the terms in the odd places of the expansion of $(x+a)^n$ is equal to the sum of the coefficients in the even places, and each is equal to 2^{n-1}.

4. The number of terms in the expansion of $(x+a)^n$ is $n+1$ when the index is a positive integer.

The series of

$$(a+x)^n=a^n+na^{n-1}x+\frac{n(n-1)}{1.2}.a^{n-2}x^2+\ldots\frac{n(n-1)\ldots 3.2}{1.2\ldots(n-1)}ax^{n-1}+x^n.$$

The first power of x occurs in the second term of the series, and the indices of the powers of x in each successive term are increased by unity, and the series ends with the nth power of x; it is therefore obvious that the series for $(a+x)^n$, when n is a positive integer, consists of $n+1$ terms.

5. The coefficients of all the terms of the series are integers when the index is a positive integer.*

This is also obvious from the fact that the successive coefficients of the expansion are the successive numbers of combinations of n things taken 1, 2, 3, &c., at a time. The coefficients of the terms of the series reckoned from the first term continually increase in value, as also the coefficients of the series reckoned from the last term, and

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* The coefficients of the terms in the expansion of $(a+x)^n$ may be shewn to be integers when the index n is an integer, without assuming them to be the number of combinations of n things taken 1, 2, 3 . . . . n respectively at a time.

Generally the coefficient of the $(r+1)$th term of $(a+x)^n$ is

$$\frac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{1.2.3\ldots\ldots r},$$

in which there are as many factors in the numerator as in the denominator.

First, n must be of one of the forms $2m$ or $2m+1$,
and $\therefore$ one of the first two factors in the numerator is divisible by 2.

Secondly, n must be of one of the forms $3m$, $3m+1$, $3m+2$,
and $\therefore$ one of the first three factors in the numerator is divisible by 3.

Thirdly, n must be of one of the forms $4m$, $4m+1$, $4m+2$, $4m+3$,
and $\therefore$ one of the first four factors in the numerator is divisible by 4.

Similarly, n must be of one of the forms rm, $rm+1$, ... $rm+(r-1)$.
$\therefore$ one of the first r factors in the numerator is always divisible by r.

Hence, since the coefficient of the $(r+1)$th term is a general expression for any term, and it has been shewn that it is divisible by some whole number between 2 and r inclusive, it follows that every coefficient of the expansion of $(a+x)^n$ will be an integer when the index n is an integer.

that when n is an even number, there will be one middle term of the series; but when n is odd, there will be two middle terms.

The coefficients of the second, third, fourth, &c., terms, reckoned from the end of the series, being the complementary combinations of the second, third, fourth, &c., terms, reckoned from the first term of the series, are consequently equal to one another.

Since there are $n+1$ terms in the expansion of $(a+x)^n$, and when n is an even number, the middle term of the series will be the $\left(\frac{n}{2}+1\right)$th term.

The middle term of $(a+x)^n = \dfrac{n(n-1)(n-2)\dots\left(n-\frac{n}{2}+1\right)}{1.2.3\dots\frac{n}{2}} a^{n-\frac{n}{2}} x^{\frac{n}{2}}$

$$= \frac{n(n-1)(n-2)\dots\left(\frac{n}{2}+1\right)}{1.2.3\dots\frac{n}{2}} . a^{\frac{n}{2}} x^{\frac{n}{2}}.$$

This may be exhibited under the equivalent forms

$$\frac{n(n-1)(n-2)\dots.3.2.1}{\left(1.2.3\dots.\frac{n}{2}\right)^2} . a^{\frac{n}{2}} x^{\frac{n}{2}}, \text{ and } \frac{(n-1)(n-3)\dots.5.3.1}{1.2.3\dots.\frac{n-1}{2}.\frac{n}{2}} . 2^{\frac{n}{2}} a^{\frac{n}{2}} x^{\frac{n}{2}}.$$

When n is an odd number, $n+1$ is an even number, and the two middle terms of the expansion of $(a+x)^n$ are the $\frac{n+1}{2}$th and the $\left(\frac{n+1}{2}+1\right)$th terms.

These two terms are $\dfrac{n(n-1)(n-2)\dots.\frac{n+3}{2}}{1.2.3\dots.\frac{n-1}{2}} x^{\frac{n-1}{2}} a^{\frac{n+1}{2}}$, and

$$\frac{n(n-1)(n-2)\dots.\frac{n+3}{2}.\frac{n+1}{2}}{1.2.3\dots.\frac{n-1}{2}.\frac{n+1}{2}} . x^{\frac{n+1}{2}} a^{\frac{n-1}{2}}.$$

These also may be exhibited under the forms

$$\frac{n(n-1)(n-2)\dots.3.2.1}{\left(1.2.3\dots.\frac{n+1}{2}\right)^2} a^{\frac{n-1}{2}} x^{\frac{n+1}{2}}, \text{ and } \frac{n(n-1)(n-2)\dots.3.2.1}{\left(1.2.3\dots.\frac{n+1}{2}\right)^2} a^{\frac{n+1}{2}} x^{\frac{n-1}{2}},$$

and they may be reduced to

$$\frac{(n-1)(n-3)\dots5.3.1}{1.2.3\dots\frac{n+1}{2}} 2^{\frac{n-1}{2}} . a^{\frac{n-1}{2}} . x^{\frac{n+1}{2}}, \text{ and}$$

$$\frac{(n-1)(n-3)\dots5.3.1}{1.2\dots\frac{n+1}{2}}2^{\frac{n-1}{2}}.a^{\frac{n+1}{2}}.x^{\frac{n-1}{2}}.$$

6. When the exponent is a negative integer,

The expansion of $(a+x)^{-n}$ can be derived directly from the expansion of $(a+x)^n$.

$$\text{Since } (a+x)^{-n}=\frac{1}{(a+x)^n}=\frac{1}{a^n+na^{n-1}x+\&c.}$$

The series for $(a+x)^{-n}$ will be found by dividing 1 by $a^n+na^{n-1}x+\&c.$, and as at no step of the division can the remainder be zero, it follows that the number of terms in the expansion will be unlimited, and the expansion will be

$$(a+x)^{-n}=\frac{1}{a^n}-\frac{nx}{a^{n+1}}+\frac{n(n+1)}{1.2}.\frac{x^2}{a^{n+2}}-\frac{n(n+1)(n+2)}{1.2.3}.\frac{x^3}{a^{n+3}}+\dots$$
$$+\frac{n(n+1)\dots(n+r-1)}{1.2\dots r}.\frac{x^r}{a^{n+r}}\dots\&c.$$
$$=a^{-n}-na^{-n-1}x+\frac{n(n+1)}{1.2}a^{-n-2}x^2-\frac{n(n+1)(n+2)}{1.2.3}.a^{-n-3}x^3+\dots$$
$$+\frac{n(n+1)\dots(n+r-1)}{1.2\dots r}a^{-n-r}x^r+\dots.$$

The expansion of $(a-x)^{-n}$ is deduced from this by writing $-x$ for $+x$.

$$(a-x)^{-n}=a^{-n}+na^{-n-1}x+\frac{n(n+1)}{1.2}a^{-n-2}x^2+\frac{n(n+1)(n+2)}{1.2.3}a^{-n-3}x^3+\dots$$
$$+\frac{n(n+1)\dots(n+r-1)}{1.2..r}.a^{-n-r}x^r+\dots$$

7. In the expansions of $(a+x)^n$ and $(a-x)^n$, $(a+x)^{-n}$, $(a-x)^{-n}$, if a be taken equal to unity, the series assumes a form more convenient for effecting expansions, when the exponent is not a positive integer. When a becomes unity, the four expansions are

$$(1+x)^n=1+nx+\frac{n(n-1)}{1.2}.x^2+\frac{n(n-1)(n-2)}{1.2.3}.x^3+\dots.$$
$$+\frac{n(n-1)(n-2)\dots(n-r+1)}{1.2.3\dots r}.x^r+\dots+x^n.$$
$$(1-x)^n=1-nx+\frac{n(n-1)}{1.2}.x^2-\frac{n(n-1)(n-2)}{1.2.3}.x^3+\dots.$$
$$+\frac{n(n-1)\dots(n-r+1)}{1.2\dots r}.x^r(-1)^r+\dots+x^n(-1)^n.$$
$$(1+x)^{-n}=1-nx+\frac{n(n+1)}{1.2}.x^2-\frac{n(n+1)(n+2)}{1.2.3}.x^3+\dots.$$
$$+\frac{n(n+1)(n+2)\dots(n+r-1)}{1.2.3\dots r}.x^r(-1)^r+\dots.$$

$$(1-x)^{-n}=1+nx+\frac{n(n+1)}{1.2}.x^2+\frac{n(n+1)(n+2)}{1.2.3}.x^3+\ldots.$$
$$+\frac{n(n+1)(n+2)\ldots(n+r-1)}{1.2.3\ldots r}x^r+\ldots.$$

8. The expansion of $(a+x)^n$ may be exhibited in an infinite series of the following four forms :—

$$(1)\ a+x=\frac{a(a+x)}{a}=\frac{a}{\frac{a}{a+x}}=\frac{a}{1-\frac{x}{a+x}}=a\left\{1-\frac{x}{a+x}\right\}^{-1},$$

and $$(a+x)^n=a^n\left\{1-\frac{x}{a+x}\right\}^{-n}=a^n\left\{1+\frac{n}{1}.\frac{x}{a+x}+\frac{n(n+1)}{1.2}.\frac{x^2}{(a+x)^2}\right.$$
$$\left.+\frac{n(n+1)(n+2)}{1.2.3}.\frac{x^3}{(a+x)^3}+\ldots.\right\}$$

$$(2)\ a+x=\frac{x(a+x)}{x}=\frac{x}{\frac{x}{a+x}}=\frac{x}{1-\frac{a}{a+x}}=x\left\{1-\frac{a}{a+x}\right\}^{-1},$$

$$(a+x)^n=x^n\left\{1-\frac{a}{a+x}\right\}^{-n}=x^n\left\{1+\frac{n}{1}.\frac{a}{a+x}+\frac{n(n+1)}{1.2}.\frac{a^2}{(a+x)^2}\right.$$
$$\left.+\frac{n(n+1)(n+2)}{1.2.3}.\frac{a^3}{(a+x)^3}+\ldots.\right\}$$

$$(3)\ a+x=\frac{2a(a+x)}{2a}=\frac{2a}{\frac{a+a}{a+x}}=\frac{2a}{1-\frac{x-a}{x+a}}=2a\left\{1-\frac{x-a}{x+a}\right\}^{-1}$$

$$(a+x)^n=2^na^n\left\{1-\frac{x-a}{x+a}\right\}^{-n}=2^na^n\left\{1+\frac{n}{1}.\left(\frac{x-a}{x+a}\right)\right.$$
$$\left.+\frac{n(n+1)}{1.2}.\left(\frac{x-a}{x+a}\right)^2+\frac{n(n+1)(n+3)}{1.2.3}.\left(\frac{x-a}{x+a}\right)^3+\ldots..\right\}$$

$$(4)\ a+x=\frac{2x(a+x)}{2x}=\frac{2x}{\frac{x+x}{x+a}}=\frac{2x}{1-\frac{a-x}{a+x}}=2x\left\{1-\frac{a-x}{a+x}\right\}^{-1}$$

$$(a+x)^n=2^nx^n\left\{1-\frac{a-x}{a+x}\right\}^{-n}=$$
$$2^nx^n\left\{1+\frac{n}{1}.\left(\frac{a-x}{a+x}\right)+\frac{n(n+1)}{1.2}.\left(\frac{a-x}{a+x}\right)^2+\frac{n(n+1)(n+2)}{1.2.3}.\left(\frac{a-x}{a+x}\right)^3+\ldots.\right\}$$

9. *To find the expansion of the polynomial theorem when the index is a positive integer and the polynomial consists of* 3, 4, 5, *or any number of terms.*

The expansion of any power of a polynomial consisting of 3, 4, 5, or n terms respectively, can be directly effected when the index is a positive integer by means of the binomial theorem.

First, to find the expansion of $(a+b+c)^n$ and the general term of the expansion when the polynomial consists of three terms.

Here $a+b$ or $b+c$ may be considered as a single quantity, then $(a+b+c)^n=\{(a+b)+c\}^n$

$$= (a+b)^n + n(a+b)^{n-1}c + \frac{n(n-1)}{1.2}.(a+b)^{n-2}c^2 + \frac{n(n-1)(n-2)}{1.2.3}.(a+b)^{n-3}c^3 + \dots + c^n,$$

and each of the terms $(a+b)^n$, $(a+b)^{n-1}$, &c., can be expanded by the binomial theorem, and all the terms of the expansion determined.

To find the general term of the series involving $a^pb^qc^r$.

Here $p+q+r=n$, and $n-p=q+r$. Let $a+b+c=a+x_1$ or x_1+a.

And the $(p+1)$th term of the expansion of $(x_1+a)^n$

$$= \frac{n(n-1)\dots(n-p+1)}{1.2\dots p}x_1^{n-p}.a^p.$$

$$= \frac{n(n-1)\dots(n-p+1).(n-p)(n-p-1)\dots3.2.1}{1.2\dots p.(n-p)(n-p-1)\dots3.2.1}.x^{n-p}.a^p.$$

$$= \frac{n(n-1)\dots3.2.1}{1.2\dots p.(q+r)(q+r-1)\dots3.2.1}.x_1^{q+r}.a^p.$$

And the $(q+1)$th term of x_1^{q+r} or $(c+b)^{q+r}$

$$= \frac{(q+r)(q+r-1)\dots\{(q+r)-q+1\}}{1.2\dots q}c^r.b^q$$

$$= \frac{(q+r)(q+r-1)\dots(r+1).r.(r-1)\dots3.2.1}{1.2\dots q.r(r-1)\dots3.2.1}.c^r.b^q.$$

Therefore the complete term of the expansion of $(a+b+c)^n$ involving $a^pb^qc^r$

$$= \frac{n(n-1)(n-2)\dots3.2.1}{1.2\dots p.(q+r)(q+r-1)\dots3.2.1}.\frac{(q+r)(q+r-1)\dots3.2.1}{1.2\dots q.1.2\dots r}a^pb^qc^r.$$

$$= \frac{n(n-1)(n-2)\dots3.2.1}{1.2\dots p.1.2\dots q.1.2\dots r}a^p.b^q.c^r.$$

If for b and c there be written bx and cx^2, the general term of the expansion of $(a+bx+cx^2)^n$ will be

$$= \frac{n(n-1)\dots3.2.1}{1.2\dots p.1.2\dots q.1.2\dots r}a^pb^qc^rx^{q+2r}.$$

To find the number of terms in the expansion of $(a+b+c)^n$.

Here $\{(a+b)+c\}^n = (a+b)^n + n(a+b)^{n-1}c + \frac{n(n-1)}{1.2}.(a+b)^{n-2}c^2 + \dots$

$$+ n(a+b).c^{n-1} + c^n.$$

The number of terms in the expansion of $(a+b)^n = n+1$.
„ „ „ $(a+b)^{n-1} = n$
„ „ „ $(a+b)^{n-2} = n-1$
&c. &c.
„ „ „ $c^n - 1$

and $(n+1)+n+(n-1)+\dots+2+1 = \frac{(n+1)(n+2)}{1.2}$.

∴ The number of terms in the expansion of $(a+b+c)^n$ is $\frac{(n+1)(n+2)}{1.2}$.

Next, to find the expansion of $(a+b+c+d)^n$, and the general term of the expansion, when the polynomial consists of four terms.

Here $(a+b+c+d)^n = \{(a+b+c)+d\}^n$

$$=(a+b+c)^n+n(a+b+c)^{n-1}d+\frac{n(n-2)}{1.2}(a+b+c)^{n-2}d^2+\ldots+d^n,$$

and when $(a+b+c)^n$, $(a+b+c)^{n-1}$, &c., are expanded, all the terms of the expansion of $(a+b+c+d)^n$ can be determined.

To find the general term of the series which involves $a^p b^q c^r d^s$.

Here $p+q+r+s=n$, then $n-p=q+r+s$, $n-p-q=r+s$.

Also let $a+b+c+d=x_1+a$, $d+c+b=x_2+b$.

Then the $(p+1)$th term of $(x_1+a)^n = \frac{n(n-1)\ldots(n-p+1)}{1.2\ldots p}.x_1^{n-p}.a^p$

$$=\frac{n(n-1)\ldots(n-p+1)}{1.2\ldots p}.\frac{(n-p)(n-p-1)\ldots 3.2.1}{(n-p)(n-p-1)\ldots 3.2.1}.x_1^{n-p}.a^p.$$

$$=\frac{n(n-1)\ldots 3.2.1}{1.2\ldots p.(n-p)(n-p-1)\ldots 3.2.1}.x_1^{n-p}.a^p.$$

Next, the $(q+1)$th term of x_1^{n-p} or $(x_2+b)^{n-p}$

$$=\frac{(n-p)(n-p-1)\ldots\{(n-p)-q+1\}}{1.2\ldots q}.x_2^{n-p-q}.b^q.$$

$$=\frac{(n-p)(n-p-1)\ldots(r+s+1)}{1.2\ldots q}.\frac{(r+s)(r+s-1)\ldots 3.2.1}{(r+s)(r+s-1)\ldots 3.2.1}x_2^{r+s}.b^q.$$

Thirdly, the $(r+1)$th term of x_2^{r+s} or $(d+c)^{r+s}$

$$=\frac{(r+s)(r+s-1)\ldots\{(r+s)-r+1\}}{1.2\ldots r}d^s c^r$$

$$=\frac{(r+s)(r+s-1)\ldots(s+1)}{1.2\ldots r}.\frac{s.(s-1)\ldots 3.2.1}{s.(s-1)\ldots 3.2.1}d^s c^r.$$

Hence the complete general term of the expansion of $(a+b+c+d)^n$ involving $a^p b^q c^r d^s$, is

$$=\frac{n(n-1)(n-2)\ldots.3.2.1}{1.2.3\ldots p.(n-p)(n-p-1)\ldots 3.2.1}.\frac{(n-p)(n-p-1)\ldots 3.2.1}{1.2\ldots q.(r+s)(r+s-1)\ldots 3.2.1}$$

$$.\frac{(r+s)(r+s-1)\ldots 3.2.1}{1.2\ldots r.1.2\ldots s}.a^p.b^q.c^r.d^s.$$

$$=\frac{n(n-1)(n-2)\ldots 3.2.1}{1.2.3\ldots p.1.2.3\ldots q.1.2.3\ldots r.1.2.3\ldots s}.a^p b^q c^r d^s.$$

If for b, c, d there be written bx, cx^2, dx^3, the general term of the expansion of $(a+bx+cx^2+dx^3)^n$ will be

$$\frac{n(n-1)(n-2)\ldots 3.2.1}{1.2.3\ldots p.1.2.3\ldots q.1.2.3\ldots r.1.2.3\ldots s}.a^p b^q c^r d^s x^{q+2r+3s}.$$

To find the number of terms in the expansion of $(a+b+c+d)^n$.

Here $\{(a+b+c)+d\}^n = (a+b+c)^n+n(a+b+c)^{n-1}.d$

$$+\frac{n(n-1)}{1.2}(a+b+c)^{n-2}d^2+\ldots+n(a+b+c)d^{n-1}+d^n.$$

The number of terms in the expansion of $(a+b+c)^n = \frac{(n+1)(n+2)}{1.2}$

,, ,, ,, $(a+b+c)^{n-1} = \frac{n(n+1)}{1.2}$

The number of terms in the expansion of $(a+b+c)^{n-2} = \frac{(n-1)n}{1.2}$

&c. &c.

,, ,, ,, $(a+b+c) = 3$

,, ,, ,, $d^n = 1$

$$\text{And } \frac{(n+1)(n+2)}{1.2}+\frac{n(n+1)}{1.2}+\frac{(n-1)n}{1.2}+\dots+\frac{3.2}{1.2}+\frac{2.1}{1.2}$$

$$=\frac{1}{1.2}\left\{(n+1)(n+2)+n(n+1)+(n-1)n+\dots+3.2+2.1\right\}$$

$$=\frac{(n+1)(n+2)(n+3)}{1.2.3}.$$

And by a similar process may be found the expansion of a polynomial and the general term when it consists of 5, 6 ... m terms respectively, as also the number of terms in each series.

The general term of the expansion of $(a+b+c+d+\dots)^n$ can be put into the form

$$\frac{n(n-1)\dots3.2.1}{1.2\dots p.1.2\dots q.1.2\dots r.1.2\dots s.1.2\dots t,\ \&c.}.a^p.b^q.c^r.d^s.e^t,\ \&c.$$

subject to the condition that $p+q+r+s+t+\dots=n$.

If for b, c, d, e, &c., be written bx, cx^2, dx^3, ex^4, &c.

The general term in the expansion of

$$(a+bx+cx^2+dx^3+ex^4+\dots)^n$$

may be put into the form

$$n(n-1)(n-2)\dots3.2.1.\frac{a^p}{1.2\dots p}.\frac{b^q}{1.2\dots q}.\frac{c^r}{1.2\dots r}\dots x^{q+2r+3s\dots}.\text{*}$$

Subject to the conditions, $p+q+r+s+t+\dots.=n$ and $q+2r+3s+4t+\dots.=$ the index of x in the particular term.

To find the number of terms in the expansion of $(a+b+c+d\dots)^n$ when the polynomial consists of m terms.

The process for finding the number of terms in the expansion of $(a+b+\dots)^n$ is evidently the same as that for finding the number of terms in the expansion of $(a+b+c+d)^n$; the only difference is that 4 in the expression for the number of terms in the latter becomes m in the former, and the expression for the whole number of terms in the expansion of $(a+b+c+\dots)^n$ is

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[*] The general term of the expansion of any polynomial can be employed to find the successive terms of the expansion.

To find the first five terms of the expansion of $(a+bx+cx^2+dx^3+ex^4+\dots.)^n$.

The general term of the series is

$$1.2.3\dots.n.\frac{a^p}{1.2\dots p}.\frac{b^q}{1.2\dots q}.\frac{c^r}{1.2\dots r}.\frac{d^s}{1.2\dots s}.\frac{e^t}{1.2\dots t}\dots x^{q+2r+3s+4t\dots}$$

subject to the conditions $p+q+r+s+t+\dots=n$, and $q+2r+3s+\dots.=$ Index of x.

The first term of the expansion involving no power of x is obviously a^n.

To find the second term involving the first power of x.

In this case $p+q=n$ and $q=1$; $\therefore p=n-1$,

$$\frac{(n+1)(n+2)(n+3)\ldots(n+m-1)}{1.2.3\ldots(m-1)m},$$

which number is obviously the $(n+1)$ term of the mth order of the figurate numbers.

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and the term involving $x = 1.2.3\ldots n.\left\{\frac{a^{n-1}}{1.2\ldots(n-1)}.\frac{b}{1}\right\}x = na^{n-1}bx.$

To find the third term involving x^2.

Here $p+q+r=n$, and $q+2r=2$, which can be satisfied in two ways, when $q=0$, $r=1$; and when $q=2$, $r=0$.

$\therefore\ p+0+1=n$, and $p=n-1$, $p+2+0=n$, $p=n-2$.

Hence the term involving x^2

$$=1.2.3\ldots n.\left\{\frac{a^{n-1}}{1.2\ldots(n-1)}.\frac{c}{1}+\frac{a^{n-2}}{1.2\ldots(n-2)}.\frac{b^2}{1.2}\right\}x^2$$

$$=\left\{na^{n-1}c+\frac{n(n-1)}{1.2}.a^{n-2}b^2\right\}x^2.$$

To find the fourth term involving x^3.

Here $q+2r+3s=3$, and $p+q+r+s=n$.

The condition $q+2r+3s=3$ can be satisfied in three ways,

$q=0,\ r=0,\ s=1,\ \therefore\ p+1=n,$ and $p=n-2.$
$q=1,\ r=1,\ s=0,\quad p+1+1=n,$ and $p=n-2.$
$q=3,\ r=0,\ s=0,\quad p+3=n,$ and $p=n-3.$

Hence the term involving x^3

$$=1.2\ldots n\left\{\frac{a^{n-1}}{1.2\ldots(n-1)}.\frac{d}{1}+\frac{a^{n-2}}{1.2\ldots(n-2)}.\frac{b}{1}.\frac{c}{1}+\frac{a^{n-3}}{1.2\ldots(n-3)}.\frac{b^3}{1.2.3}\right\}x^3$$

$$=\left\{na^{n-1}d+n(n-1)a^{n-2}bc+\frac{n(n-1)(n-2)}{1.2.3}a^{n-3}b^3\right\}x^3.$$

To find the fifth term involving x^4.

Here $q+2r+3s+4t=4$, and $p+q+r+s+t=n$.

The equation $q+2r+3s+4t=4$ can be satisfied in five ways,

$q=0,\ r=0,\ s=0,\ t=1,\ \therefore\ p+1=n,$ and $p=n-1.$
$q=0,\ r=2,\ s=0,\ t=0,\ \therefore\ p+2=n,$ and $p=n-2.$
$q=1,\ r=0,\ s=1,\ t=0,\ \therefore\ p+1+1=n,$ and $p=n-2.$
$q=2,\ r=1,\ s=0,\ t=0,\ \therefore\ p+2+1=n,$ and $p=n-3.$
$q=4,\ r=0,\ s=0,\ t=0,\ \therefore\ p+4=n,$ and $p=n-4.$

Hence the fifth term involving x^4

$$=1.2.3\ldots n.\left\{\frac{a^{n-1}}{1.2\ldots(n-1)}.\frac{e}{1}+\frac{a^{n-2}}{1.2\ldots(n-2)}.\frac{c^2}{1.2}+\frac{a^{n-2}}{1.2\ldots(n-2)}.\frac{b}{1}.\frac{d}{1}\right.$$
$$\left.+\frac{a^{n-3}}{1.2\ldots(n-3)}.\frac{b^2}{1.2}.\frac{c}{1}+\frac{a^{n-4}}{1.2\ldots(n-4)}.\frac{b^4}{1.2.3.4}\right\}x^4$$

$$=\left\{na^{n-1}e+\frac{n(n-1)}{1.2}.a^{n-2}c^2+n(n-2)a^{n-2}bd+\frac{n(n-1)(n-2)}{1.2}.a^{n-3}b^2c\right.$$
$$\left.+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}.a^{n-4}b^4\right\}x^4.$$

If any term of a polynomial be negative, it will be found more convenient to write it positive, placing the negative sign over the term. Instead of $(a+bx-cx^2)^n$, write $(a+bx+\bar{c}x^2)^n$, and the expansion can be effected considering all the terms positive. In the final reduction of the terms so affected, the odd powers will be negative and the even powers positive.

10. *To prove the truth of the binomial theorem generally, whether the index be integral or fractional, positive or negative, and independently of the theory of combinations.*

First $(a+b)^n$ may be put under the form $a^n(1+x)^n$.

For $a+b=a\left(1+\frac{b}{a}\right)=a(1+x)$, if $\frac{b}{a}=x$; and $(a+b)^n=a^n(1+x)^n$.

In the following proof the form $(1+x)^n$ will be employed as being more simple than $(a+b)^n$:—

By involution, evolution, and division, it may be shewn that

$$(1+x)^2=1+2x+x^2,\ (1+x)^3=1+3x+3x^2+x^3,$$
$$(1+x)^{-2}=1-2x+3x^2-\text{ \&c.},\ (1+x)^{-3}=1-3x+6x^2-\text{ \&c.},$$
$$(1+x)^{\frac{1}{2}}=1+\tfrac{1}{2}x-\tfrac{1}{8}x^2+\text{ \&c.},\ (1+x)^{\frac{1}{3}}=1+\tfrac{1}{3}x-\tfrac{1}{9}x^2+\text{ \&c.},$$
$$(1+x)^{-\frac{1}{2}}=1-\tfrac{1}{2}x+\tfrac{3}{8}x^2+\text{ \&c.},\ (1+x)^{-\frac{1}{3}}=1-\tfrac{1}{3}x+\tfrac{2}{9}x^2+\text{ \&c.},$$

It appears that whether the index be integral or fractional, positive or negative, the expansions proceed regularly by increasing powers of x, and that in every case the coefficient of the second term of each expansion is equal to the index with its proper sign.

Hence it may be assumed generally, whatever be the value of the index, that $(1+x)^n=1+nx+Ax^2+Bx^3+Cx^4+$ &c., when B, C, D, &c., are to be determined. In this expression for x write $x+z$.

$$\text{Then } \{1+(x+z)\}^n=1+n(x+z)+A(x+z)^2+B(x+z)^3+C(x+z)^4+\text{ \&c.}$$
$$=1+nx+Ax^2+Bx^3+Cx^4+\text{ \&c.}$$
$$+nz+2Azx+3Bz^2x+4Cz^3x+\text{ \&c. omitting higher powers of } z.$$
$$\text{And } \{(1+x)+z\}^n=(1+x)^n+n(1+x)^{n-1}z+A(1+x)^{n-2}z^2+\text{ \&c.}$$
$$=1+nx+Ax^2+Bx^3+Cx^4+\text{ \&c.}$$

$+nz\{1+(n-1)x+A'x^2+B'x^3+C'x^4+$ &c.$\}+$ terms involving z^2, z^3, &c., and writing A', B', C', &c., for A, B, C, &c., when n becomes $n-1$.

$$=1+nx+Ax^2+Bx^3+Cx^4+\text{ \&c.},$$
$$+nz+n(n-1)zx+nA'zx^2+nB'zx^3+\text{ \&c.}$$

Since the two series which result from the two expansions of $(1+x+z)^n$ are equal; omitting the terms which are common in the two series, there results the identity,

$$2Azx+3Bzx^2+4Czx^3+\text{ \&c.},\ =n(n-1)zx+nA'zx^2+nB'zx^3+\text{ \&c.}$$

And equating the coefficients of the corresponding powers of x,

$$2A=n(n-1),\text{ and } A=\frac{n(n-1)}{1.2}.$$

$$\text{And putting } n-1 \text{ for } n,\ A'=\frac{(n-1)(n-2)}{1.2}.$$

$$3B=nA',\text{ and } B=\frac{nA'}{3}=\frac{n(n-1)(n-2)}{1.2.3}.$$

$$\text{And putting } n-1 \text{ for } n,\ B'=\frac{(n-1)(n-2)(n-3)}{1.2.3}$$

$$4C=nB',\text{ and } C=\frac{nB'}{4}=\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}.$$

and so on for the successive coefficients.

And lastly, substituting the values of A, B, C, &c., in the series, $(1+x)^n = 1+nx+Ax^2+Bx^3+Cx^4+$ &c., there results

$$(1+x)^n = 1+nx+\frac{n(n-1)}{1.2}.x^2+\frac{n(n-1)(n-2)}{1.2.3}x^3$$
$$+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}.x^4+\text{ \&c.}$$

in which the index n may be integral or fractional, positive or negative, and since $a^n(1+x)^n = a^n\left(1+\frac{b}{a}\right)^n = (a+b)^n$,

$$\therefore\ (a+b)^n = a^n\left\{1+\frac{b}{a}\right\} = a^n\left\{1+n\frac{b}{a}+\frac{n(n-1)}{1.2}.\frac{b^2}{a^2}+\frac{n(n-1)(n-2)}{1.2.3}.\frac{b^3}{a^3}\right.$$
$$\left.+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}.\frac{b^4}{a^4}+\text{ \&c.}\right\}$$
$$= a^n+na^{n-1}b+\frac{n(n-1)}{1.2}.a^{n-2}b^2+\frac{n(n-1)(n-2)}{1.2.3}a^{n-3}b^3+\text{ \&c.};$$

and the series for the expansion of $(a+b)^n$, terminates with the $(n+1)$th term, when the index n is a positive integer; * but the series

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* Since the successive coefficients of the expanded binomial $(1+x)^n$ denote the numbers of the combinations of n things taken 1, 2, 3, . . . n at a time; by means of the products of two or more binomial series some curious relations may be found between the combinations of different sets of things, when taken 1, 2, 3, &c., at a time.

To find the relations between the corresponding coefficients of the two series which arise from the expansions of $(1+x)^n \times (1+x)^n = (1+x)^{2n}$.

Let $C_1, C_2, C_3 \dots C_n$ denote the coefficients of the expansion of $(1+x)^n$.
and $D_1, D_2, D_3, \dots D_{2n}$,, ,, ,, $(1+x)^{2n}$.
Then $(1+C_1x+C_2x^2+\dots C_{n-1}x^{n-1}+C_nx^n)(1+C_1x+C_2x^2+\dots C_{n-1}x^{n-1}+C_nx^n)$
$= 1+D_1x+D_2x^2+\dots+D_{2n-1}x^{2n-1}+D_{2n}x^{2n}$,
or $1+2C_1x+(C_1^2+2C_2)x^2+(2C_3+2C_1C_2)x^3+(2C_4+2C_1C_3+C_2^2)x^4$
$+(2C_5+C_1C_4+2C_2C_3)x^5+\text{&c.} = 1+D_1x+D_2x^2+D_3x^3+D_4x^4+D_5x^5+\text{&c.}$,
and each side of this identity consists of $2n+1$ terms, and by equating the coefficients of the corresponding powers of x on each side, a series of properties will be found which connect the combinations of $2n$ and of n things taken 1, 2, 3, &c., at a time. For instance,

The coefficient of x^n in the product on the left side, restoring the value of n in C_1, C_2, &c., and bearing in mind the property of complementary combinations, is

$$1^2+n^2+\left\{\frac{n(n-1)}{1.2}\right\}^2+\left\{\frac{n(n-1)(n-2)}{1.2.3}\right\}^2+\text{ \&c.}$$

And the coefficient of x^n in the expansion of $(1+x)^{2n}$, restoring the value of $2n$ in D_n, is

$$\frac{2n(2n-1)(2n-2)\dots(n+1)}{1.2.3\dots n}.$$

Therefore $1^2+n^2+\left\{\frac{n(n-1)}{1.2}\right\}^2+\left\{\frac{n(n-1)(n-2)}{1.2.3}\right\}^2+$ &c.

$$=\frac{2n(2n-1)(2n-2)\dots(n+1)}{1.2.3\dots n}$$

Or the sum of the squares of the combinations of n things taken 1, 2, 3, . . . , n at a time, is equal to the number of combinations of $2n$ things taken n at a time.

And since the product of the two equal series on the left side is the square of does not terminate when the index is a negative integer, or a positive or negative fraction.

The series for the expansions of $(a+b+c)^n$, $(a+b+c+d)^n$, &c., $(a+b+c+d+e+\ldots)^n$, and of $(a+bx+cx^2)^n$, $(a+bx+cx^2+dx^3)^n$, &c., $(a+bx+cx^2+dx^3+ex^4+\ldots\ldots)^n$, when the index is a negative integer, or a fraction positive or negative, may be exhibited in the same forms as when the index is a positive integer; with this difference, that the numbers of terms in the expansions do not terminate when the index is not a positive integer.

Also the general terms of each series is subject to the same equations of condition as when the index is a positive integer.

If a series arising from the expansion of a binomial be neither wholly convergent, nor wholly divergent, but begin to converge from some term of the expansion, the term of the expansion from which the converging begins can be determined.

Different cases will arise according as the index of the binomial is integral or fractional, and positive or negative.

If the index be a positive integer the number of terms in the expansion is limited to $n+1$, and the coefficient of the middle term is the maximum of all the coefficients; but if the index be a negative integer or a positive or negative fraction, the number of terms of the expansion will be unlimited.

11. *To determine that term in the expansion of $(a+x)^n$ at which the succeeding terms begin to converge, when n is a positive integer.*

Let T_p, T_{p+1} denote the pth and $(p+1)$th terms of the expansion of $(a+x)^n$,

$$\text{Then } T_p=\frac{n(n-1)\ldots(n-p+2)}{1.2\ldots(p-1)}.a^{n-p+1}x^{p-1},$$

$$\text{and } T_{p+1}=\frac{n(n-1)\ldots(n-p+2)(n-p+1)}{1.2\ldots(p-1).p}a^{n-p}x^p,$$

$$\therefore\ \frac{T_{p+1}}{T_p}=\frac{n-p+1}{p}.\frac{x}{a}=\frac{(n-p+1)x}{pa}.$$

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one of them, which consists of the sum of the squares of every term together with twice the product of every two consecutive terms; the sum of the products of every two consecutive coefficients of these terms may be shewn to be equal to

$$\frac{n.2^n}{n+1}.\frac{(2n-1)(2n-3)\ldots.5.3.1}{1.2\ldots.(n-1)n}.$$

In the same manner, if there be given the product of three equal binomial expressions, $(1+x)^n(1+x)^n(1+x)^n=(1+x)^{3n}$, and these expressions be expanded and arranged according to powers of x, there will be formed two identical series, of which the coefficients of corresponding powers of x are equal. These equivalents will express relations between the combinations of n and $3n$ things when taken 1, 2, 3, &c., at a time.

Also if the identical expressions

$$(1+x)^m(1+x)^n=(1+x)^{m+n} \text{ and } (1+x)^m(1+x)^n(1+x)^p=(1+x)^{m+n+p}$$

be expanded, relations between the combinations of m, n, and $m+n$ things, and of m, n, p, and $m+n+p$ things may be determined.

Now the pth term will be greater than the $(p+1)$th term

when $pa > (n-p+1)x$,

or when $p(a+x) > (n+1)x$,

and $\therefore$ when $p > \frac{(n+1)x}{a+x}$.

But as p must be necessarily a positive integer,

The pth term will be greater than the $(p+1)$th term when p is the integer next greater than $\frac{(n+1)x}{a+x}$.

And this obviously holds good so long as n is a positive integer. The coefficient of the middle term of the expansion, being the maximum number of n things, is *the greatest coefficient;* but it does not follow that *the middle term itself* is the greatest term of the series. This will depend on the relative values of a and x in any particular series. If x be less than a, then $\frac{x}{a+x}$ is less than $\frac{a}{a+x}$, and it is manifest that this multiplier in particular instances, when the values of a and x are assigned, may diminish or increase the numerical value of p.

But if $n = -1$, then $p > \frac{(-1+1)x}{a+x} = 0$, and the method fails to determine the point of convergence when $n = -1$.

Next. If n be a negative integer different from unity, the series does not terminate, but the term of the expansion of $(a+x)^{-n}$, at which the convergence commences, can be found.

For $T_p = \frac{n(n+1)(n+2)\dots(n+p-2)}{1.2.3\dots(p-1)}.a^{-n-p+1}x^{p-1}.(-1)^{p-1}$,

and $T_{p+1} = \frac{n(n+1)(n+2)\dots(n+p-2)(n+p-1)}{1.2.3\dots(p-1).p}.a^{-n-p}.x^{p}.(-1)^{p}$.

$$\therefore \frac{T_{p+1}}{T_p} = \frac{n+p-1}{p}.\frac{x(-1)}{a} = \frac{-(n+p-1)x}{pa}.$$

The pth term will be greater than the $(p+1)$th term numerically

when $pa > (n+p-1)x$, neglecting the negative sign,

or when $p(a-x) > (n-1)x$,

$\therefore$ when $p > \frac{(n-1)x}{a-x}$.

But as p must be necessarily a positive integer,

The pth term will be greater than the $(p+1)$th term

when p is the integer next greater than $\frac{(n-1)x}{a-x}$.

If n be a fraction, proper or improper, positive or negative, the series does not terminate with the $(n+1)$th term of the expansion, and p may be less or greater than $n+1$.

12. *From the expansion of the binomial theorem to deduce the exponential theorem* $a^x = 1 + Ax + \frac{A^2x^2}{1.2} + \frac{A^3x^3}{1.2.3} + \text{\&c.},$

$$\text{where } A = (a-1) - \tfrac{1}{2}(a-1)^2 + \tfrac{1}{3}(a-1)^3 - \text{\&c.}$$

Here $a = 1 + a - 1,$

And $a^x = \{1 + (a-1)\}^x$

$$= 1 + x(a-1) + \frac{x(x-1)}{1.2}(a-1)^2 + \frac{x(x-1)(x-2)}{1.2.3}(a-1)^3$$

$$+ \frac{x(x-1)(x-2)(x-3)}{1.2.3.4}.(a-1)^4 + \text{\&c.}$$

$$= 1 + x(a-1) + \left\{\frac{x^2}{2} - \frac{x}{2}\right\}.(a-1)^2 + \left\{\frac{x^3}{6} - \frac{x^2}{2} + \frac{x}{3}\right\}.(a-1)^3$$

$$+ \left\{\frac{x^4}{24} - \frac{x^3}{4} + \frac{11x^2}{24} - \frac{x}{4}\right\}(a-1)^4 + \text{\&c.}$$

$$= 1 + \left\{(a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \text{\&c.}\right\}x + Bx^2 + Cx^3$$

$$+ Dx^4 + \text{\&c.},$$

by finding the terms of the coefficient of x, and denoting the coefficients of x^2, x^3, &c., by B, C, D, &c., which involve only powers of a and numerical quantities,

$$\text{and if } (a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{3} + \text{\&c.} = A.$$

$$\text{Then } a^x = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \text{\&c.},$$

and this expression is true whatever be the value of x. Write z for x, then $a^z = 1 + Az + Bz^2 + Cz^3 + Dz^4 + \text{\&c.}$

And $a^x.a^z = a^{x+z} = 1 + A(x+z) + B(x+z)^2 + C(x+z)^3 + D(x+z)^4 + \text{\&c.},$

but $a^xa^z = a^z\{1 + Ax + Bx^2 + Cx^3 + Dx^4 + \text{\&c.}\}$

Equating the coefficients of x in these identical values of a^xa^z,

$\therefore\ A + 2Bz + 3Cz^2 + 4Dz^3 + \text{\&c.} = Aa^z = A\{1 + Az + Bz^2 + Cz^3 + Dz^4 + \text{\&c.}\}$

Next equating the coefficients of the corresponding powers of z in these identical series,

$$2B = A^2,\ 3C = AB,\ 4D = AC,\ \text{\&c.},$$

$$\text{whence } B = \frac{A^2}{1.2},\ C = \frac{AB}{3} = \frac{A^3}{1.2.3},\ D = \frac{AC}{4} = \frac{A^4}{1.2.3.4} + \text{\&c.},$$

$$\therefore\ a^x = 1 + Ax + \frac{A^2x^2}{1.2} + \frac{A^3x^3}{1.2.3} + \frac{A^4x^4}{1.2.3.4} + \text{\&c.},$$

$$\text{in which } A = (a-1) - \frac{(a-1)^2}{2} + \frac{(a-1)^3}{3} - \frac{(a-1)^4}{4} + \text{\&c.}$$

If e denote that value of a which renders the series for A equal to unity,

$$\text{then } e^x = 1 + x + \frac{x^2}{1.2} + \frac{x^3}{1.2.3} + \frac{x^4}{1.2.3.4} + \text{\&c.}\quad \text{And if } x = 1,$$

$$e = 1 + 1 + \frac{1}{1.2} + \frac{1}{1.2.3} + \frac{1}{1.2.3.4} + \text{\&c.} = 2{\cdot}7182818\ldots$$

EXERCISES.

I.

Expand by the binomial theorem the following expressions:—

1. $(a+3x)^3$, $(a+nb)^4$, $(1-2x)^5$, $(2-5y)^6$, $(x^2+x^2)^7$, $(2x+3y)^8$.

2. $(ax+by)^{10}$, $(ax-by)^8$, $(2x+3y)^{12}$, $(2x-3y)^{14}$.

3. $(2+\sqrt{3})^5$, $(\sqrt{2}-\sqrt{3})^6$, $(a+b\sqrt{-1})^6$, $(a-b\sqrt{-1})^7$, $(2+\sqrt{-3})^8$.

4. $(x+x^{-1})^4$, $(x-x^{-1})^6$, $(x+x^{-1})^{11}$, $(x-x^{-1})^{12}$.

II.

Expand the following expressions, each to six terms, and write the pth term of each series.

1. $(1\pm x)^{-1}$, $(1\pm x)^{-2}$, $(1\pm x)^{-3}$, $(1\pm x)^{-4}$, $(1\pm x)^{-5}$, $(1\pm x)^{-6}$.

2. $(1\pm x)^{\frac{1}{2}}$, $(1\pm x)^{\frac{1}{3}}$, $(1\pm x)^{\frac{1}{4}}$, $(1\pm x)^{\frac{1}{5}}$, $(1\pm x)^{\frac{1}{6}}$.

3. $(1\pm x)^{-\frac{1}{2}}$, $(1\pm x)^{-\frac{1}{3}}$, $(1\pm x)^{-\frac{1}{4}}$, $(1\pm x)^{-\frac{1}{5}}$, $(1\pm x)^{-\frac{1}{6}}$.

4. $(x\pm x^{-1})^n$, $(x\pm x)^{-n}$.

5. $(a^2+x^2)^{\frac{m}{n}}$, $(2a-3x)^{\frac{1}{2}}$, $(2x+3y)^{-\frac{1}{2}}$, $(x^2+2ax)^{\frac{1}{2}}$, $(x^2-2ax)^{-\frac{1}{2}}$, $a^{\frac{1}{2}}(a^2+x^2)^{-\frac{1}{2}}$, $a^{-1}(a^3-a^2x)^{-\frac{1}{3}}$, $(a^{\frac{1}{2}}x^{\frac{1}{3}}+a^{\frac{1}{3}}x^{\frac{1}{2}})^{-\frac{1}{2}}$.

6. $(a+b\sqrt{-1})^{\frac{m}{n}}$, $(ax-bx^2\sqrt{-1})^{\frac{1}{2}}$, $(1+x\sqrt{-1})^{n\sqrt{-1}}$, $\{x+(1+x)^{-1}\}^{\frac{1}{2}}$.

III.

Verify the correctness of the following expansions, and find the value of each to four places of decimals:—

1. $5^{\frac{1}{2}}=2\left\{1+\frac{1}{2}\cdot\frac{1}{4}-\frac{1.1}{2.4}\cdot\frac{1}{4^2}+\frac{1.1.3}{2.4.6}\cdot\frac{1}{4^3}-\frac{1.1.3.5}{2.4.6.8}\cdot\frac{1}{4^4}+\text{ \&c.}\right\}$

2. $8^{\frac{1}{2}}=3\left\{1-\frac{1}{2}\cdot\frac{1}{9}-\frac{1.1}{2.4}\cdot\frac{1}{9^2}-\frac{1.1.3}{2.4.6}\cdot\frac{1}{9^3}-\frac{1.1.3.5}{2.4.6.8}\cdot\frac{1}{9^4}-\text{ \&c.}\right\}$

3. $(1010)^{\frac{1}{3}}=10\left\{1+\frac{1}{3}\cdot\frac{1}{10^2}-\frac{1.2}{3.6}\cdot\frac{1}{10^4}+\frac{1.2.5}{3.6.9}\cdot\frac{1}{10^6}-\frac{1.2.5.8}{3.6.9.12}\cdot\frac{1}{10^8}+\text{\&c.}\right\}$

4. $(250)^{\frac{1}{5}}=3\left\{1+\frac{1}{5}\cdot\frac{7}{243}-\frac{1.4}{5.10}\cdot\left(\frac{7}{243}\right)^2+\frac{1.4.9}{5.10.15}\cdot\left(\frac{7}{243}\right)^3+\text{ \&c.}\right\}$

5. $(130)^{\frac{1}{7}}=2\left\{1+\frac{1}{7}\cdot\frac{1}{2^7}-\frac{1.6}{7.14}\cdot\frac{1}{2^{14}}+\frac{1.6.7}{7.14.21}\cdot\frac{1}{2^{21}}-\text{ \&c.}\right\}$

6. $(27)^{\frac{1}{3}}=2+\frac{5}{1.2}\cdot\frac{1}{3}+\frac{5.7}{1.2.3}\cdot\frac{1}{3^2}+\frac{5.7.9}{1.2.3.4}\cdot\frac{1}{3^3}+\ldots$ and

7. Show that $1+\frac{1}{3}+\frac{1.3}{1.2}\cdot\frac{1}{3^2}+\frac{1.3.5}{1.2.3}\cdot\frac{1}{3^3}+\ldots=3^{\frac{1}{2}}$,

and $\frac{1}{2}+\frac{1.3}{2.4}\cdot\left\{\frac{1}{4}\right\}+\frac{1.3.5}{2.4.6}\cdot\left\{\frac{1}{4}\right\}^2+\ldots.=\frac{4}{3}(2-\sqrt{3})\sqrt{3}$.

IV.

Verify the correctness of the following expansions:—

1. $(1+x)^{\frac{1}{2}}=1+\frac{1}{2}\cdot\frac{x}{1+x}+\frac{3}{2^2.2}\cdot\frac{x^2}{(1+x)^2}+\frac{3.5}{2^3.2.3}\cdot\frac{x^3}{(1+x)^3}$
$+\frac{3.5.7}{2^4.2.3.4}\cdot\frac{x^4}{(1+x)^4}+$ &c.

2. $(a+b)^{\frac{1}{2}}=a^{\frac{1}{2}}\left\{1+\frac{1}{2}\cdot\frac{b}{a+b}+\frac{1.3}{2.4}\cdot\frac{b^2}{(a+b)^2}+\frac{1.3.5}{2.4.6}\cdot\frac{b^3}{(a+b)^3}+\right.$ &c.

3. $(a+x)^{\frac{a}{x}}=a^{\frac{a}{x}}\left\{2+\frac{a(a-x)}{1.2}\cdot\frac{1}{a^2}+\frac{a(a-x)(a-2x)}{1.2.3}\cdot\frac{1}{a^3}\cdots\right\}$

4. $\left\{\frac{a+x}{a-x}\right\}^{\frac{1}{2}}=1+\frac{x}{a+x}+\frac{3}{2}\cdot\frac{x^2}{(a+x)^2}+\frac{5}{2}\cdot\frac{x^3}{(a+x)^3}+$ &c.

5. $\left\{\frac{1+x^n}{1-x^{2n}}\right\}^{\frac{1}{3}}=1+\frac{1}{3}x^n+\frac{1.4}{3.6}.x^{2n}+\frac{1.4.7}{3.6.9}.x^{3n}+$ &c.

6. $\left\{\frac{1+\alpha x}{1-\beta x}\right\}^{\frac{1}{2}}=1+\frac{\alpha}{2}.x+\left\{\frac{4\beta-\alpha^2}{2.4}\right\}.x^2+\frac{\alpha}{2}\left\{\frac{\alpha^2+4\beta}{2.4}\right\}.x^3+$ &c.

7. $(x^2-2)^{\frac{1}{2}}=\frac{x^2-1}{x}\left\{1-\frac{1}{2}\cdot\frac{1}{(x^2-1)^2}-\frac{1}{8}\cdot\frac{1}{(x^2-1)^4}-\frac{1}{16}\cdot\frac{1}{(x^2-1)^6}-\text{\&c.}\right\}$.

8. $\left\{\frac{1+x}{1-x}\right\}^n=1+n.\left(\frac{2x}{1+x}\right)+\frac{n(n+1)}{1.2}\cdot\left(\frac{2x}{1+x}\right)^2$
$+\frac{n(n+1)(n+2)}{1.2.3}\cdot\left(\frac{2x}{1+x}\right)^3+\ldots$

9. $\left\{\frac{1+2x}{1+x}\right\}^n=1+n.\left(\frac{x}{1+2x}\right)+\frac{n(n+1)}{1.2}\cdot\left(\frac{x}{1+2x}\right)^2$
$+\frac{n(n+1)(n+2)}{1.2.3}\cdot\left(\frac{x}{1+2x}\right)^5+\ldots$

10. $(1+x)(1-x)^{-2}=1+3x+5x^2+7x^3+9x^4+$ &c.

11. $(1+x)(1-x)^{-3}=1+2^2x+3^2x^2+4^2x^3+5^2x^4+$ &c.

12. $(1+4x+x^2)(1-x)^{-4}=1+2^3x+3^3x^2+4^3x^3+5^3x^4+$ &c.

V.

1. Expand $(1+x)^{1+x}$ in a series of ascending powers of x.

2. Shew that $(1+x)^n+n(1+x)^{n-1}x+\frac{n(n+1)}{1.2}.(1+x)^{n-2}x^2$
$+$&c. ... $=(1+x)^{2n}$.

3. The series $1+\frac{n}{1}.\frac{1}{2}+\frac{n(n+1)}{1.2}\cdot\frac{1}{2^2}+\frac{n(n+1)(n+2)}{1.2.3}\cdot\frac{1}{2^3}+$&c. $=2^n$.

4. Shew that $\left\{\frac{n}{n-1}\right\}^{\frac{1}{n}}=1+\frac{1}{n^2}+\frac{n+1}{1.2.n^4}+\frac{(n+1)(2n+1)}{1.2.3.n^6}$
$+\frac{(n+1)(2n+1)(3n+1)}{1.2.3.4.n^8}+\ldots$

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5. Shew that $1+x+\left(1+\frac{1}{n}\right)\frac{x^2}{2}+\left(1+\frac{1}{n}\right)\left(\frac{1}{2}+\frac{1}{n}\right)\frac{x^3}{3}+\ldots\ldots=\left(1-\frac{x}{n}\right)^{-n}$.

6. $$\frac{1+n\frac{a}{a-b}+\frac{n(n+1)}{1.2}.\left(\frac{a}{a-b}\right)^2+\ldots}{1-n\frac{b}{a-b}+\frac{n(n+1)}{1.2}.\left(\frac{b}{a-b}\right)^2+\ldots}=(-1)^n.\frac{a^n}{b^n}.$$

7. $$\frac{1+n^2+\left(\frac{n.(n-1)}{1.2}\right)^2+\left(\frac{n.(n-1).(n-2)}{1.2.3}\right)^2+\ldots+1}{n\frac{(n-1)}{1.2}+n^2.\frac{(n-1).(n-2)}{2.3}+\left(\frac{n.(n-1)}{1.2}\right)^2\frac{(n-2).(n-3)}{3.4}+\ldots\text{ \&c.}+\frac{n(n-1}{1.2}}$$
$$=1+\frac{2(2n+1}{n.(n-1)}.$$

VI.

Determine the coefficient of x^n in the expansion of each of the following expressions:—

1. $(1+x+2x^2+3x^3+\ldots)^2$.
2. $(1+2x+3x^2+\ldots)^2$.
3. $(1.2+2.3x+3.4x^2+\ldots)^2$.
4. $(1+x)^2.(1-x)^{-4}$.
5. $(1-2x)^2.(1-x)^{-4}$.
6. $(1+x)^2.(1-x)^{-3}$.
7. $x(x-a)^{-1}.(x-b)^{-1}$.

And the coefficient of x^{2n} in the expansions of

8. $(1+x^2)^4.(1-x)^{-3}$.
9. $(1-x)(1+x)^{-2}$.
10. $(1+x)^{-2}.(1-x)^{-4}$.

VII.

1. If x be very small compared with unity, shew that
$$\frac{1}{1+\sqrt{(1-x)}}-\frac{1}{1+\sqrt[3]{(1-x)}}=\frac{x}{24}\text{ nearly.}$$

2. If x and y be nearly equal, shew that
$$\left(\frac{y}{x}\right)^{\frac{1}{2}}=\frac{y}{x+y}+\frac{x+y}{4x},\text{ and }\left(\frac{x}{y}\right)^{\frac{1}{3}}=\frac{2x+y}{x+2y}.$$

3. If x be nearly equal to unity, shew that
$$\frac{mx^m-nx^n}{m-n}=x^{m+n}\text{ nearly.}$$

4. If $c=a-b$, and is small when compared with a and b; shew that $a^2b^2(a^2-a^2x^2+b^2x^2)^{-\frac{3}{2}}=a-2c+3cx^2$ nearly.

5. The approximate value of the ratio of $\left\{\frac{a+x}{b-x}\right\}^2$ to $\left\{\frac{a}{b}\right\}^2$ is the same as the approximate ratio of $(1+x)^{\frac{2(a+b)}{ab}}$ to 1, when x is very small.

6. The expression $\dfrac{(1-3x)^{-\frac{2}{3}}+(1-4x)^{-\frac{3}{4}}}{(1-3x)^{-\frac{1}{3}}+(1-4x)^{-\frac{1}{4}}}=1+\dfrac{3}{2}x$ approximately, if x be a small fraction.

7. Shew that $\dfrac{(1-x)^{-\frac{1}{2}}+(1-y)^{-\frac{1}{2}}}{1+(1-x)^{\frac{1}{2}}-(1-y)^{\frac{1}{2}}}=1+\frac{1}{2}(x+y)+\frac{1}{8}(3x^2+xy+3y^2)$, rejecting x^3, y^3, &c.

8. Expand $\dfrac{(1-x)^{-\frac{1}{3}}+(1+x)^{-\frac{1}{2}}}{(1+x)^{-\frac{1}{2}}+(1-x)^{-\frac{1}{3}}}$ in a series to four terms ascending by integral powers of x.

9. If a and n be integers, then the expansion of $\{a+(a^2+1)^{\frac{1}{2}}\}^n$ has its integral part odd when n is even, and even when n is odd.

VIII.

1. Explain how Newton discovered that the square root of $1-x^2$ was identical to the expansion of $(1-x^2)^{\frac{1}{2}}$.

2. Determine the ratio of the two terms which involve x^{n+1} and x^n in the expansion of $(1-x)^{-n}$.

3. If A_r, A_{r+1}, be respectively the coefficients of the rth and $(r+1)$th terms of the expansion of $(x+a)^n$, and B_{r+1} be that of the $(r+1)$th term of the expansion of $(x+a)^{n+1}$; prove that

$$B_{r+1}=A_r+A_{r+1}.$$

4. Expand $(1-4x)^{-\frac{1}{2}}$ in a series ascending by integral powers of x and write the nth term.

Shew that the $(n+1)$th term of $(1-4x)^{-\frac{1}{2}}$ is equal to the middle term of the expansion of $(1+x)^{2n}$.

Prove that the sum of the middle terms of $(1+x)^{2n}$ for all integral values of n, including zero, is equal to $(1-4x)^{-\frac{1}{2}}$.

5. Assuming the binomial theorem for a positive integral index; does the truth of the equation

$$\left\{1+\frac{m}{n}x+\frac{m}{n}\left(\frac{m}{n}-1\right)\frac{x^2}{1.2}+\text{ \&c.}\right\}^n=(1+x)^m,$$

x being a quantity between 0 and 1, depend on the interpretation of fractional indices?

IX.

1. If P denote the sum of the odd terms and Q the sum of the even terms of the expansion of $(a+b)^n$, shew that $P^2-Q^2=(a^2-b^2)^n$.

2. Shew that 4 times the product of the odd and even terms of an expanded binomial $(a+b)^n=(a+b)^{2n}-(a-b)^{2n}$.

3. Expand $\left\{\dfrac{a+b\sqrt{-1}}{a-b\sqrt{-1}}\right\}^n$, and express the result in the form $A+B\sqrt{-1}$.

4. Determine the middle term or terms in the expansions of $(x\pm x^{-1})^n$, when n is a number of the forms $4m$, $4m+1$, $4m+2$, $4m+3$.

5. $(1+\sqrt{-1})^{2n}+(1-\sqrt{-1})^{2n}=0$ or $\pm 2^{n+1}$, according as n is odd or even.

6. Which is rational of the expressions
$(a+b\sqrt{-1})^{2n}+(a-b\sqrt{-1})^{2n}$ and $(a+b\sqrt{-1})^{2n}-(a-b\sqrt{-1})^{2n}$?

7. Find the values of $(1+x)^{\frac{1}{1-x}} \pm (1-x)^{\frac{1}{1+x}}$ in a series of ascending powers of x.

8. Shew that the expansion of $(1+\sqrt{2})^{2n}$ may be exhibited in the following forms,

$$1+2n.2^{\frac{1}{2}}+\frac{2n(2n-1)}{1.2}.2+\frac{2n(2n-1)(2n-2)}{1.2.3}.2^{\frac{3}{2}}+\ldots+2^{n}.$$

$$3^{n}+n.3^{n-1}.2^{\frac{3}{2}}+\frac{n(n-1)}{1.2}.3^{n-2}.2^{3}+\frac{n(n-1)(n-2)}{1.2.3}.3^{n-3}.2^{\frac{9}{2}}+\ldots\ 2^{\frac{3n}{2}}.$$

9. To determine the integral and the decimal parts of the expansion of $(2+\sqrt{3})^{n}$, in which n is a positive integer.

X.

1. Find the term from which the series for $\left(1+\frac{3}{2}\right)^{12}$ becomes convergent, and find the limits of the error introduced by neglecting the remaining terms of the series.

2. If in the expansion of $(1-x)^{-n}$, the symbols x and n be respectively replaced by $\frac{1}{10}$ and 40, find the general term of the new series.

3. Find the first negative coefficient in the expansion of $(1+3x)^{\frac{13}{3}}$.

4. Find the value of $\left\{1+\frac{1}{99}\right\}^{\frac{1}{100}}$ correctly to 10 places of decimals.

5. After how many terms will each of the series for

$$\left(1+\frac{7}{8}\right)^{10},\ \left(1+\frac{9}{10}\right)^{-2},\ \left(1+\frac{99}{100}\right)^{\frac{5}{3}},\ \left(1+\frac{1}{2}\right)^{\frac{2}{17}},\ \left(1+\frac{8}{9}\right)^{\frac{-7}{2}},\ \left(1+\frac{11}{12}\right)^{\frac{-3}{7}}.$$

$(\sqrt{2}+\sqrt{3})^{12\sqrt{5}}$, respectively begin to converge?

6. Find the remainder after n terms of the expansion of $\frac{1+x}{(1-x)^{3}}$ in a series of ascending powers of x, &c.

XI.

1. Shew that the coefficient of x^{n} in the expansion of $(1-x)^{-n}$ is equal to the sum of all the preceding coefficients.

2. Shew that the coefficient of x^{n} in the expansion of $(1-x)^{-r}$ is equal to the coefficient of x^{n} in the expansion of $(1+x)^{n+r-1}$.

3. If a_r be the coefficient of x^{r} in the expansion of $(1+x)^{n}$,

then shall $a_0-a_1+a_2-a_3+\ldots(-1)^{r}.a_r=(-1)^{r+1}.\frac{r+1}{n}.a_{r+1}$.

4. If a_r, a_{r-1}, be the coefficients of x^r, x^{r-1} respectively in the expansion of $(1-x)^{-5}$, prove that $a_r = a_{r-1} + \frac{(r+1)(r+2)(r+3)(r+4)}{1.2.3.4}$.

5. Prove that the difference of the coefficients of the $(r+1)$th and rth terms of the expansion of $\frac{1}{(1-x)^{n+1}}$ is equal to the coefficient of the $(r+1)$th term of the expansion of $\frac{1}{(1-x)^n}$.

6. Prove that the difference of the coefficients of x^{r+1} and x^r in the expansion of $(1+x)^{n+1}$, is equal to the difference of the coefficients of x^{r+1} and x^{r-1} in the expansion of $(1+x)^n$.

7. The coefficient of x^m in the expansion of $\frac{1}{(1-x)^{n+1}}$ is equal to the coefficient of x^n in the expansion of $\frac{1}{(1-x)^{m+1}}$.

8. In the expansion of $\left\{\frac{a+x}{a-x}\right\}^{\frac{1}{2}}$, shew that the coefficients of the $(2r-1)$th and of the $(2r)$th terms are the same.

9. If a_r be the coefficient of x^r in the expansion of $\left\{\frac{1+x}{1-x}\right\}^n$,

shew that $(r+1)a_{r+1} - 2na_r - (r-1)a_{r-1} = 0$.

10. If $(1+x)^n.(1-x)^{-2}$, be expanded in ascending powers of x, the coefficient of x^{n+r-1} is $(n+2r)2^{n-1}$.

11. The coefficient of x^{n+r-1} in the expansion of $\frac{(1+x)^n}{(1-x)^3}$ in ascending powess of x is $2^{n-3}\{n+(n+2r)(n+2r+2)\}$, r being any integral number.

XII.

1. If 1, C_1, C_2, &c., be the coefficients of the expansion of $(1+x)^n$, n being an integer,

$$\text{then } 1+\frac{1}{2}C_1+\frac{1}{3}C_2+\ldots=\frac{1}{n+1}\left\{2^{n+1}-1\right\}.$$

2. If P and Q be the nth terms of the expansions of

$$(a^2-x^2)^{-\frac{1}{2}} \text{ and } (a^2-x^2)^{-\frac{3}{2}}; \text{ then } \frac{P}{Q}=\frac{a^2}{2n-1}.$$

3. If a_r be the coefficient of x^r in the expansion of $(1+x)^n$, n being a positive integer, prove that

$$(1)\ \frac{a_1}{a_0}+\frac{2a_2}{a_1}+\frac{3a_3}{a_2}+\ldots+\frac{na_n}{a_{n-1}}=\frac{n(n+1)}{1.2}.$$

$$(2)\ (a_0+a_1)(a_1+a_2)\ldots(a_{n-1}+a_n)=a_1a_2a_3\ldots a_n.\frac{(n+1)^n}{1.2.3\ldots n}.$$

4. If a_1, a_2, a_3, a_4 be any four consecutive coefficients of an

expanded binomial, prove that $(a_1a_4+a_2a_3)(a_2-a_3)=2(a_1a_3^2-a_4a_2^2)$, and shew that this expression may be put under the form

$$\frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}=\frac{2a_2}{a_2+a_3}.$$

5. If C_r be the coefficient of the rth term of the expansion of $(1+x)^n$, n being an integer, shew that

$$C_1^2-C_2^2+C_3^2-\ldots\pm C^2_{n+1} \text{ is equal to zero, if } n \text{ be odd,}$$

$$\text{and equal to } \frac{1.3.5\ldots(n-1)}{1.2.3\ldots\frac{n}{2}}.2^{\frac{n}{2}}, \text{ if } n \text{ be even.}$$

6. If $1, C_1, C_2\ldots C_n$ be the coefficients of the expanded binomial $(1+x)^n$ when n is a positive integer, prove that

$$(1)\quad 1^2+C_1^2+C_2^2+C_3^2+\ldots+C_n^2=\frac{2n(2n-1)(2n-2)\ldots 3.2.1}{(1.2.3\ldots n)^2}.$$

$$(2)\quad 1.C_1+C_1.C_2+C_2.C_3+\ldots+C_{n-1}C_n=\frac{2n(2n-1)\ldots 3.2.1}{n(n+1)\{1.2.3\ldots(n-1)\}^2}.$$

7. If ${}^nC_r$ be the coefficient of x^r in the expansion of $(1+x)^n$, and ${}^nD_r$ that of x^r in the expansion of $(1-x)^{-n}$:

$$\text{then shall } \frac{n+r}{n}.\frac{1.2.3.4\ldots r}{(r+1)(r+2)\ldots 2r}.{}^nC_r.{}^nD_r={}^{n+r}C_{2r}.$$

8. There are p quantities $a+b$, $a+2b$, $a+2^2b$, &c., each raised to the power $2p$. Shew that the sum of the $(p+1)$th terms is equal to

$$\frac{(p+1)(p+2)\ldots 2p}{1.2\ldots p}.a^p.b^p.\frac{2^{p^2}-1}{2^p-1}.$$

XIII.

1. Find the terms in the expansions of

$$(a+b+c)^n,\ (a-b+c)^n,\ (a+b-c)^n,\ (a-b-c)^n,$$

which respectively contain the factor $a^{n-p}b^{p-q}c^q$.

2. Find the terms in the expansions of

$$(a+b+c+d)^n,\ (a+b-c-d)^n,\ (a-b+c-d)^n,\ (a-b-c-d)^n,$$

which respectively contain the factor $a^{n-p}b^{p-q}c^{q-r}d^r$.

3. The coefficient of x^{3r} in the expansion of $(9a^2+6ax+4x^2)^{-1}$ is $2^{3r}.(3a)^{-3r-2}$.

4. Write the coefficient of x^{10} in each of the expansions

$$(1+x^2+x^4)^6, (1+2x+3x^2)^{10}, (1+3x^2+5x^3)^{21}, (2-3x-4x^2)^5, (1+x+x^2)^{-3},$$
$$(a+bx+cx^2)^{-1}, (1-x-x^2)^{-\frac{1}{2}}, (x^2-5x+2x^{-1})^{-5}, \text{ respectively.}$$

5. In the expansion of $(a+bx+cx^2)^n$, find the coefficient of x^8.

6. Write the term in the expansion of $(a+b+c)^{n+1}$ which involves $a^{n-2}b^2c$.

7. The coefficient of $a^nb^nc^n$ in the expansion of $(a+b+c)^{3n}$ is equal to $\dfrac{3n(3n-1)\ldots.(n+1)}{(1.2\ldots..n)^2}$.

8. The middle term of the expansion of $(1+x+x^2)^n$ is

$$\left\{1+n(n-1)+\frac{n(n-1).(n-2)(n-3)}{1.2.1.2}+\ \&c.\dots\right\}x^n$$

$$+\frac{n(n-1)\dots3.2.1}{\left((1.2\dots\dots\frac{n}{2}\right)^2}x^n,\text{ if } n \text{ be even,}$$

$$\text{or }+\frac{n(n-1)\dots3.2.1}{\left(1.2\dots\frac{n-1}{2}\right)^2.\frac{n+1}{2}}.x^n,\text{ if } n \text{ be odd.}$$

9. The coefficient of the middle term in the expansion of $(1+x+x^2)^{12}$ is $73789x^{12}$.

10. What is the 12th term of the expansion of $(a-bx+cx^2-dx^3+ex^4)^{\frac{1}{2}}$.

11. Write the complete terms in the expansion of $(a+b+c+d)^{14}$ which contain $a^2b^3c^4d^5$, $a^5b^4c^3d^2$, $a^2b^3c^5d^4$, $a^3b^2c^4d^5$, respectively.

12. The complete term in the expansion of $(a+b+c+d)^n$ which involves d^{n-3} is $\frac{n(n-1)(n-2)}{1.2.3}.\{a^3+b^3+c^3+3a^2b+3ab^2+3a^2c+3ac^2+3b^2c$
$+3bc^2+6abc\}$.

13. Write the nth term of the expansion of $(1-3x+5x^2-7x^3)^n$.

14. Find the number of terms in the expansion of $(a+b+c)^{21}$ and of $(a+b+c+d)^{21}$.

XIV.

1. State the principal steps in the proof of the binomial theorem, and how the series for a^x is derived from it.

2. Expand a^x by means of an indeterminate index in a series of ascending powers of x (*La Grange, Calcul des Fonctions*).

3. Expand e^x in a series of ascending powers of x.

4. Express $e^{x\sqrt{-1}}\pm e^{-x\sqrt{-1}}$, in series of ascending powers of x.

5. Find the coefficient of x^{2n} in the expansion of $(1-x)^{-2n}e^x$.

6. Shew that $e^{-1}=\frac{1}{1.3}+\frac{1}{1.2.3.5}+\frac{1}{1.2.3.4.5.7}+\dots$

7. $\left\{1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\dots\right\}^x=1+x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\dots$

8. Expand $\left(1+\frac{1}{nx}\right)^{nx}$ by the binomial theorem, and shew that the value of the series when nx becomes indefinitely great is $1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\&c.=e$.

9. Shew that $2^n-(n-1)2^{n-2}+\frac{(n-2)(n-3)}{1.2}.2^{n-4}$

$$-\frac{(n-3)(n-4)(n-5)}{1.2.3}.2^{n-6},\ \&c.=n+1.$$

THE THEORY OF PROBABILITIES.

ART. 1. A popular writer some years ago declared that the theory of probabilities, as being devoid of all certainty, was nothing more than "an elaborate delusion." Although it is not possible to determine with certainty whether a specific event will or will not happen, the language of ordinary life implies that there are considerations on this subject which ought not altogether to be disregarded. The ordinary events of human life are spoken of as possible or impossible, probable or improbable, certain or uncertain; and sometimes with qualifying epithets, as "rather likely," "very unlikely," or "as likely as not." These and similar expressions imply a comparison of the evidence greater or less in favour of events which are uncertain. As whenever an event of one kind is observed to happen more frequently than an event of another kind, the former is expected rather than the latter. And if an event recur constantly, and regularly, there must be some intelligible cause for the constant and regular recurrence of the like events. Such events are called *morally certain*, while other events which do not so happen are named *uncertain*. There are also some truths which are universally admitted, and which are called *necessary truths*, which cannot be denied without absurdity; as, for instance, every adult human being in a healthy condition is *certain* of his own existence, of the perceptive power of his senses, and the reflective power of his mind, but he is *uncertain* how long his senses may retain their integrity, or for how many years his life may be continued. By *evidence* is meant whatever facts or arguments can be adduced which tend to dispose the mind to admit the affirmative or the negative of a proposition, and the certainty or uncertainty of an event.

The belief and disbelief of individuals are not always regulated by evidence. Persons under the influence of feeling or other bias frequently believe what is true to be false, and what is false to be true, with a wonderful degree of sincerity, and without the least suspicion that there might be evidence on one side or the other. With the majority of mankind, and womankind too, *confident assertion* has far greater influence than sound reason or legitimate authority.

Even at the present time, with all the vaunted intelligence of the nineteenth century, the most absurd fictions when pronounced with confidence are believed as truths. It occasions less trouble to take for granted emphatic, plausible assertions than to examine the grounds of evidence by which such assertions can be rendered credible or incredible.

In ordinary language the words *probability* and *chance* are employed in reference to the secondary causes of future events. The word *chance* does not signify anything as really existing, and chance cannot be the cause of any event whatever; it merely serves to signify human ignorance of the true cause. In mathematical language these words are generally used as synonymous.

2. A distinction has been drawn between *moral* and *mathematical chance*, the latter being such as can be made subject to numerical calculation, the former such as may be derived from experience and observation. Mathematical chances are such as in no way depend on experience or observation, but simply on the principles of combination. Moral chances can only be treated in the same way, bearing in mind that the same uncertainty is involved in the mathematical results, as in the assumptions made in the estimation of the number of facts. The former may be exemplified by considering the chance of throwing an ace with a single die;* the latter by the average yearly deaths of a given number of persons born until the whole number is extinct. As for example, suppose a thousand children born, and the numbers which survive in successive years be recorded until there are none left alive. Here a comparison may be instituted between the numbers of survivors in successive years and the original number of births; and arithmetical relations can be expressed between them.

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* When a common die is thrown from the hand, by the force of gravity we know it will fall *towards* the earth and not *from* it; and we also know from the form of its cubical figure, having six equal squares as its faces, if it fall upon a plane horizontal surface, it will rest with one of its faces upon that surface and the opposite face turned upwards. But as the six faces of the die are marked with one, two, three, four, five, six points respectively, there appears to be no particular circumstances in the form of the die, nor in the motion given to it, which can give any assistance to assert that any one particular face rather than any other of the six faces shall be turned up. All that can be certainly known is, that one of the six faces will be turned up. Again, if two dice be thrown, all the possible combinations which can be made with each throw is 36, for each number of points on the faces of one of the dice can be combined with the numbers of points on each of the six faces of the other. The number of points which can be thrown with two dice are 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12.

And these numbers can be made up of the following sets of points of one die, taken with those of the other:—

2	points can be thrown	in	1	way,	1+1
3	,,	,,	2	ways,	1+2, 2+1
4	,,	,,	3	,,	1+3, 3+1, 2+2
5	,,	,,	4	,,	1+4, 4+1, 2+3, 3+2
6	,,	,,	5	,,	1+5, 5+1, 3+3, 4+2, 2+4
7	,,	,,	6	,,	1+6, 6+1, 2+5, 5+2, 4+3, 3+4
8	,,	,,	5	,,	2+6, 6+2, 3+5, 5+3, 4+4
9	,,	,,	4	,,	3+6, 6+3, 5+4, 4+5
10	,,	,,	3	,,	4+6, 6+4, 5+5
11	,,	,,	2	,,	5+6, 6+5
12	,,	,,	1	,,	6+6

The probability of an event happening, or of a proposition being true, means the degree of credit which ought to be accorded to it. This depends not only on the evidence adduced, but also on the fact that the mind is free to receive impartially and admit that evidence. Evidence which ought to be the most conclusive and convincing, when offered to a mind enslaved by superstition, prejudice, credulity, or scepticism, is about as effective as the submission of colours to a blind man for discrimination.

It is scarcely correct to speak of an event in itself as probable or improbable, as the same event may appear probable to one person and improbable to another; and under those circumstances, different, and even contradictory conclusions may be drawn respecting the same event by different minds. The human mind is so constituted as to be capable of confounding fictions with facts and taking one for the other, as well as mistaking sophisms for sound reasons, and the contrary; so that the mind is not unlikely, unless great caution be taken, to mistake the false for the true and the wrong for the right, or the converse; and so be led both into errors of judgment as well as of practice.*

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* "Among the early sophisms with which the Greeks tried the power of logic as a formal mode of detecting fallacies, was the construction of what we may call *suicidal* propositions, assertions the truth of which would be their own falsehood. If a man should say 'I lie,' he speaks neither truth nor falsehood; for if he says true, he lies, and if he lie, he speaks truth. Such a speech cannot be interpreted. Again, the Cretan, Epimenides, said that all the Cretans were incredible liars; is he to be believed or not? If we believe him, we must, he being a Cretan, disbelieve him. Some stated it thus:—'If we believe him, then the Cretans are liars, and we should not believe him, then there is no evidence against the Cretans, or we may believe him, so that the evidence against the Cretans revives,' &c. &c. &c. Refer such a proposition to the theory of probabilities, and the difficulty immediately disappears. Whatever the credit of Epimenides as a witness may be, that is, whatever, upon his word, the odds may be for his proposition, the same odds are there against him from the proposition itself. These equal conflicting testimonies balance one another, and leave the effect of other testimonies to the same point unaltered."—*Formal Logic* (*p.* 210), *by Augustus De Morgan, of Trinity College, Cambridge. London*, 1847.

It may be observed that the words of Epimenides,

Κρῆτες ἀεὶ ψεῦσται, κακὰ θηρία, γαστέρες ἀργαί,

can scarcely be extended to mean that

"All the Cretans were incredible liars."

St. Paul quotes the hexameter of the poet (whom he calls προφήτης), in his Epistle to Titus (ch. i. v. 12). In the two preceding verses many (πολλοί), especially of the Jews, are censured, and St. Paul quotes the verse as a true testimony that the vice of lying was common among the Cretans, but does not affirm that all the Cretans were liars. Both Eustatius and Ovid express a more just opinion; the former states:—'Εισι βλασφημίαι καὶ ἀπὸ ἐθνῶν καὶ πόλεων καὶ δήμων πολλὰι ρηματικῶς πεποιημέναι, as, for instance, κρητίζειν τὸ ψευδεσθαι; and the latter, "Cretes erunt testes, nec fingunt omnia Cretes." Another example may be read in the 51st chapter of the "Adventures of Don Quixote."

It is also more probable that a person of greater intelligence and experience, if unbiased, will be capable of estimating the circumstances of evidence rightly, than one whose knowledge and intellectual power is circumscribed within narrower limits. And consequently the former, with due attention, may naturally be expected to arrive at a more distinct conception of the measure of probability of any uncertain event. But here a difficulty arises, how the force or value of evidence can be made a subject of measurement. It is usual to speak of persons as having enlarged benevolence, deep insight, or high intellectual power. And the common sense of mankind generally awards these and other suitable epithets to persons who by some intelligible acts or courses of conduct have exhibited evidences that they possess such qualities.

The difference of intellectual power in different individuals will occasion some difference in the estimation of the probability of events into which the element of *skill* may enter.* The relative skill of two persons would naturally be defined by the ratio of the average number of events of the same kind favourable to one and to the other. If this ratio be constant, the comparative skill of the two persons is a constant number which may be employed in the same manner as chance in calculation. But if the ratio be not constant, any approximate value may be used, and it must be remembered that the final result is only attained as an approximation.

But the equality of intellect and experience does not always enable men to agree in what is the truth and the untruth respecting questions at issue, even in cases where their honesty and impartiality are above all suspicion. This is seen sometimes in the opinion of judges in the courts of appeal on questions where the evidence is complicated and conflicting. The like differences of judgment occur between men of ability and impartiality on questions connected with theology and morality. These and other differences of judgment may be referred rather to the data and the difficulties which skill and experience encounter in attempting to separate the true from the untrue, and the right from the wrong.†

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* This may be exemplified in the case of two students, both of whom, it is supposed, have an equal knowledge of the principles involved in a paper of 24 problems. One can solve 12 problems out of the 24 in the same time that the other can solve 8. Here $\frac{12}{24}$ and $\frac{8}{24}$, or $\frac{1}{2}$ and $\frac{1}{3}$, will denote the value of their respective knowledge and skill combined. And supposing their knowledge equal, then $\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$ will denote the measure of the skill of one above the other.

† One of the most singular instances of this kind is to be found in the following judgment of the Holy Inquisition pronounced on the heresy of Galileo in 1633 :—

"The proposition that the sun is in the centre of the world and immoveable from its place, is absurd, philosophically false, and formally heretical, because it is expressly contrary to the Holy Scriptures."

"The proposition that the earth is not the centre of the world, nor immoveable, As it is almost impossible to discover all the circumstances which make up the evidence for an event happening or not happening; it is obvious that an event may admit of different degrees of evidence, from that of the lowest degree possible to that of the highest degree probable, or that degree of evidence which produces the conviction of what is called *moral certainty*.

3. The force of evidence, or the value of it as influencing belief, constitutes the subject of the theory of probabilities.

This theory consists in the reduction to some numerical estimation the relations which exist between the number of ways in which an event can and cannot happen. Although it is impossible to determine with certainty whether any specified event will or will not happen, it is possible to ascertain what degree of probability may exist for the expectation of an event happening, by comparing the number of ways in which it can happen, with the number of all the ways in which it can and cannot happen. It may be added that scarcely any event ever falls out as it is determined by calculation, except there be an indefinite number of repetitions, which, one with another, will at length always produce a definite result.

This remark is important; for scarcely ever any event comes off at the first trial; but in repeated trials, it will be found that the number of favourable and unfavourable events will continually approximate nearer to some constant ratio as the number of trials is repeated. As for example in throwing a common die, the ace, on one of the six faces, is as likely to turn up as any one of the other five. There is therefore only one way in which the ace can turn up, and five ways in which it cannot. And in 6 throws it is not certain that the ace will turn up once, though it is possible that in the 6 throws it may turn

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but that it moves, and also with a diurnal motion, is also absurd, philosophically false, and, theologically considered, at least erroneous in faith."

The infallible Papal decree, on which this judgment is founded, is recognised in the following declaration, printed on the seventh page of the third volume of an edition of Newton's "Principia," which was edited by the Jesuit fathers Le Seur and Jacquier, and published at Geneva in 1742:

"Newton, in this third book, assumes the hypothesis of the motion of the earth. The propositions of the author could not be explained otherwise than by making the same hypothesis. From this circumstance we have been compelled to personate the character of another; but we profess to obey the decrees made by the supreme pontiffs against the motion of the earth."

It cannot be maintained, without perverting the plain sense of Scripture, that the motion of the earth is "expressly contrary to the Holy Scriptures." Natural phenomena are described in the Scriptures *as they appear*, and no claims or pretences are therein found to explain the true system of the universe.

When a body at rest is seen from another body in motion, the former *appears* to move in a direction *contrary* to that in which the latter body is *actually* moving. The infallible decree of the supreme pontiff appears to have been issued without any regard to the distinction between *real* and *apparent* motion.

up more than once. Nor is it certain that the ace will turn up twice in 12 throws, nor three times in 18 throws. But it may be reasonably expected that the ratio of the number of times the ace will turn up, to the number of times that it will not turn up, will continually approximate to the ratio of 1 to 5, as the number of trials is indefinitely increased.

In the same manner, if a coin be thrown up, as it has only two faces, and one of them is as likely to turn up as the other in each throw; as the number of throws is increased, it may be expected that in an even number of throws that one face will turn up as many times as the other face, when the number of throws is increased indefinitely.*

Events are said to be *independent* when they have no connection directly or indirectly with one another, or when the success or failure of one does not affect the success or failure of another.

Two or more events are said to be *dependent* when the probability of their success or failure is altered by the success or failure of some other event.

In the case of independent events, as there may be several ways in which a possible event may or may not happen; the probability of any event happening is measured or defined by the ratio which exists between the number of ways in which the event can happen, and the whole number of ways in which it both can and cannot happen. If a denote the number of ways in which an event can happen, and b the number of ways in which it cannot happen, then the fraction or the ratio $\frac{a}{a+b}$ will denote the probability of the event happening, and $\frac{b}{a+b}$ the probability of the event not happening. And the ratio of the probability of success to that of failure, and of failure to that of success, or, as it is called, *the odds for and against*, will be denoted by the ratios of a to b and b to a respectively.

And it is obvious that the sum of the probability of an event happening and not happening, is equal to unity.

In the computation of the chances of success or failure, it is supposed that all the chances in any case are of the same kind and are equal to each other.

4. *If several events be independent of each other, the probability that all of them will happen is the product of their separate probabilities.*†

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* Professor de Morgan, in his "Formal Logic," p. 185, states that one of his pupils tried the experiment on 2,048 sets, and the result was that the ratio of the number of tails to heads was 2,044 to 2,048.

† This proposition may be otherwise stated.

If there be two independent events, whose separate chances of happening are $\frac{1}{m}$ and $\frac{1}{n}$, the chance of both of them happening is $\frac{1}{mn}$.

First, suppose there be two events.

Let a_1, b_1; a_2, b_2 denote the numbers of ways in which the two events can happen and fail respectively;

then $\frac{a_1}{a_1+b_1}$ denotes the probability of the first event happening,

and $\frac{a_2}{a_2+b_2}$ that of the second;

hence each of the cases of a_1+b_1 can be combined with each of the cases of a_2+b_2,

$(a_1+b_1)(a_2+b_2)=a_1a_2+a_1b_2+a_2b_1+b_1b_2$ denotes all the cases of success and failure of the two combined events, of which a_1a_2 denotes the cases of success, and b_1b_2 the cases of failure of both events; also a_1b_2 denotes the cases in which the first event happens and the second fails, and a_2b_1 those in which the first fails and the second happens,

$\therefore \frac{a_1a_2}{(a_1+b_1)(a_2+b_2)}$ is the probability of both events happening,

and $\frac{b_1b_2}{(a_1+b_1)(a_2+b_2)}$ the probability of both events not happening.

Also $\frac{a_1b_2}{(a_1+b_1)(a_2+b_2)}$ is the probability of the first happening and the second failing, and $\frac{a_2b_1}{(a_1+b_1)(a_2+b_2)}$ that of the first failing and the second happening.

Next let there be three events whose respective probabilities are $\frac{a_1}{a_1+b_1}$, $\frac{a_2}{a_2+b_2}$, $\frac{a_3}{a_3+b_3}$.

The chance that either one or the other will happen is $\frac{m+n-1}{mn}$.

For the chance that the first will not happen is $1-\frac{1}{m}=\frac{m-1}{m}$, and the chance that the second will not happen is $1-\frac{1}{n}=\frac{n-1}{n}$.

And therefore the chance that neither will happen is $\frac{(m-1)(n-1)}{mn}$, and the chance that either one or the other will happen is $1-\frac{(m-1)(n-1)}{mn}=\frac{m+n-1}{mn}$.

In a similar manner, if there be three events whose separate chances are $\frac{1}{m}$, $\frac{1}{n}$, $\frac{1}{p}$.

Then $\frac{1}{mnp}$ will denote the chance of all of them happening,

and $\left(1-\frac{1}{m}\right)\left(1-\frac{1}{n}\right)\left(1-\frac{1}{p}\right)$ the chance of all of them failing.

Therefore the chance of some of them happening is

$$1-\left(1-\frac{1}{m}\right)\left(1-\frac{1}{n}\right)\left(1-\frac{1}{p}\right)=\frac{mnp+m+n+p-(mn+mp+np)}{mnp};$$

and $\frac{p-1}{mnp}$ is the chance of the first and second happening and the third failing, and so for other events.

Then each of the cases of a_3+b_3 can be combined with each of the cases of $(a_1+b_1)(a_2+b_2)$, and thus form $(a_1+b_1)(a_2+b_2)(a_3+b_3)$ combinations of cases, of which $a_1a_2a_3$ and $b_1b_2b_3$ will respectively denote the number of cases of success and failure of the three events, $\therefore \dfrac{a_1a_2a_3}{(a_1+b_1)(a_2+b_2)(a_3+b_3)}$ will denote the probability of the success of three events, and $\dfrac{b_1b_2b_3}{(a_1+b_1)(a_2+b_2)(a_3+b_3)}$ that of their failure.

In the same manner may be found the probability of four or any greater number of events. And generally, if there be n independent events, the probability of the n events happening is

$$\frac{a_1a_2a_3 \ldots . a_n}{(a_1+b_1)(a_2+b_2) \ldots (a_n+b_n)}.$$

The probability can be determined when the events are *dependent*. Let the several events be named the first, second, third, &c., events. Then the probability of the first event happening must be considered as independent.

The probability of the second event must be determined on the supposition that the first event has happened in some way.

Next, the probability of the third event is to be determined on the supposition that both the first and second events have happened, and so on for the rest of the events.

And, lastly, the probability of all of them happening will be the product of the several probabilities thus determined.

5. *Having given the probability of an event happening and failing in one trial, to find the probability of its happening and failing once, twice, thrice, &c., in n trials.*

Let $\dfrac{a}{a+b}$ denote the probability of the event happening, and $\dfrac{b}{a+b}$ of its failing in one trial.

Then since the probability of the event happening in any one particular trial is $\dfrac{a}{a+b}$; and the probability of it failing in the remaining $n-1$ trials is $\dfrac{b^{n-1}}{(a+b)^{n-1}}$; therefore the probability of the event happening in one particular trial and failing in all the rest, is $\dfrac{ab^{n-1}}{(a+b)^n}$; and since there are n trials, the probability that the event will happen in some one of these and fail in the rest, is n times as great, or $\dfrac{nab^{n-1}}{(a+b)^n}$.

Next, since the probability of the event happening in any two particular trials and failing in the remaining $n-2$ trials, is $\dfrac{a^2b^{n-2}}{(a+b)^n}$; and since there are $\frac{n(n-1)}{1.2}$ ways in which the particular event may happen twice in n trials and fail in all the rest; therefore the probability that it will happen twice exactly in n trials is $\frac{n(n-1)a^2b^{n-2}}{1.2.(a+b)^n}$.

Similarly, the probability that the event will happen exactly three times in n trials is $\frac{n(n-1)(n-2)a^3b^{n-3}}{1.2.3.(a+b)^n}$, and so on.

And generally, the probability that the event will happen exactly r times in n trials is $\frac{n(n-1)(n-2)\ldots(n-r+1)a^rb^{n-r}}{1.2.3\ldots r.(a+b)^n}$.

Also the probability of the event failing r times exactly in n trials may be shewn to be $\frac{n(n-1)(n-2)\ldots(n-r+1)a^{n-r}b^r}{1.2.3\ldots r.(a+b)^n}$.

And the probability of the event happening *at least* r times, and failing *at most* $n-r$ times in n trials will be

$$\frac{a^n+na^{n-1}b+\frac{n(n-1)}{1.2}a^{n-2}b^2+\ldots\ldots\frac{n(n-1)\ldots\ldots(n-r+1)}{1.2\ldots\ldots r}a^rb^{n-r}}{(a+b)^n}.$$

For if the event happen in every one of the n trials; or if the event happen $n-1$ times, and fail once; or if it happen $n-2$ times and fail twice, and so on; or lastly, if it happen r times, and fail $n-r$ times, the event happens r times. Therefore the whole probability of the event happening *at least* r times in n trials is equal to the sum of these separate probabilities.

6. *To find the value of an expectation of several prizes, when each prize is subject to different chances.*

If there be two prizes denoted by p, q, and a chances in favour of p, and b in favour of q; then the probability of the first prize is $\frac{a}{a+b}$, and of the second prize, $\frac{b}{a+b}$. Therefore the expectation for the first prize is $\frac{pa}{a+b}$, and for the second $\frac{qb}{a+b}$. And the value of the whole expectation is the sum $\frac{pa+qb}{a+b}$.

Similarly, if there were a third prize r, and c chances in its favour, the probability of the first is $\frac{a}{a+b+c}$, of the second $\frac{b}{a+b+c}$, and of the third $\frac{c}{a+b+c}$. And the expectation for the first is $\frac{pa}{a+b+c}$, for the second $\frac{qb}{a+b+c}$, and for the third $\frac{rc}{a+b+c}$, and the whole value of the expectation is $\frac{pa+qb+rc}{a+b+c}$, and so on for more than three prizes.

EXERCISES.

I.

1. Explain the meaning of the terms certain and uncertain, probable and improbable, as they are employed in the theory of probabilities.

2. Define the mathematical meaning of the term chance, and explain how the chance, favourable or unfavourable, of an event is denoted.

If out of 1,000 persons born, there were found alive at the ages of
10, 20, 30, 40, 50, 60, 70 years respectively,
590, 539, 470, 393, 315, 230, 143 persons;
find the probability that a person alive at the beginning of each of the six decades will not survive to the end of it.

3. Explain when events are considered dependent and independent; and prove that the chance of an event contingent upon other independent events is the continued product of the chances of the separate events.

4. If there be a chances of an event happening in one trial, b of another, c of a third; find the probability of the first event's happening p times, the second q times, and the third r times, in $p+q+r$ trials.

5. If there be a chances of an event happening, and b of its failing in one trial; find the probability of its happening t times at least in n trials.

6. If a be the number of chances for the happening of an event, and b the number for its failure in each single trial; find the probability of its happening p times and failing q times in $p+q$ trials; and determine how many trials are necessary to make it an even chance whether the event will happen or not.

7. Out of $a+b$ statements by the witness A, only b are true; and out of $a'+b'$ statements made by B, only b' are found true. What is the probability of the truth of a story in which they both agree?

8. The probability of A speaking truth is p, and of B q. B reports that A affirms a certain event took place; what is the probability that it actually did take place?

II.

1. How many different throws can be made with 1, 2, 3 … n dice respectively?

2. Compare the chances of throwing an ace in two trials with one die, and in one trial with two dice.

3. To find the chance of throwing a given number of points in the first of two successive throws with a single die.

4. Find the probability of casting an ace in three throws with one die, or at one throw with three dice.

5. A and B play with a single die, each making three throws. In how many ways can A score 8 points and B 7 points?

6. How many successive throws must be made with a tetrahedron to give an even chance of throwing an ace, the faces of the tetrahedron being marked 1, 2, 3, 4, and the lower face reckoned in each throw?

7. In how many throws of a single die is there an even chance of throwing an ace?

8. Required the number of throws with a single die in which a person may undertake, for an even wager, to throw three aces.

9. Find the probability of casting 1 ace, 2 aces, and 3 aces only respectively with 10 dice in one throw, or with 10 throws of one die.

10. Four polyhedrons of 4, 6, 8, 12 faces respectively are marked in the same manner as a die; find the chance of the polyhedrons, when thrown together, falling on any one of the four faces marked 1, 2, 3, 4.

III.

1. If seven balls be drawn from a bag containing eleven in all, five of which are white and six black; what is the probability that three white balls will be drawn?

2. If an urn contain 10 white balls, 12 black, and 14 red; how many ways can 2 balls be drawn? How many pairs contain a white and a black ball? What are the odds for or against?

3. If an urn contain 20 balls, 5 white, 7 black, and 8 red; what is the chance of drawing a red ball?

4. A bag contains 12 balls of four different colours, 3 of each colour; what is the probability of taking one of each sort, if 4 be taken out at once?

5. Two bags each contain 4 black and 3 white balls. A person draws a ball from each bag; and if the ball be white, he puts it into the second bag, and draws a ball from it. Find the chance of his drawing two white balls.

6. There are three urns, the first contains 7 white balls and 11 black, the second 9 white and 9 black, and the third 10 white and 8 black; what is the chance of drawing from the first urn a white ball, from the second a black ball, and from the third a white ball?

7. If 9 balls, 3 white, 3 red, and 3 black, be distributed at random in three urns, 3 balls in each; find the chance that there will be a red ball in each urn.

8. An urn contains 20 balls, 4 of which are white. If a person draw out 5 at a venture, find (1) the probability of drawing only one white ball; (2) the probability of drawing at least one white ball.

9. There are two urns, A and B, the former contains 3 white and the latter 3 black balls; a ball is taken from each at the same time and put into the other, and this operation is repeated three times; what is the probability that the urn A will contain 3 black balls and the urn B 3 black?

10. If an urn contain 26 balls, of which 5 are white, 6 black, 7 red, and 8 blue; what is the chance of drawing, when 10 are drawn at a time, 1 blue ball, 2 white, 3 black, and 4 red balls?

11. A bag contains balls of different colours, of which 11 are red, and such a number white that if 4 balls be drawn out together, the chance that they are all red is equal to the chance that 2 are red and 2 white. Required the number of white balls.

12. If there be 10 tickets in a bag, of which 5 are blanks and the other 5 are marked as prizes 1, 2, 3, 4, 5; what is the chance of drawing 10 in three separate trials, the ticket drawn each time being replaced?

13. If a bag contain eight tickets, marked 1, 2, 3, 4, 5, 6, 7, 8; what is the chance that the number 17 will be made up in four drawings of one ticket, each ticket being replaced when drawn?

14. From two bags, one of which contains 12 tickets, marked 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, and the other 10 tickets, marked 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, two tickets are drawn, one from each bag; what is the probability that they have both the same number?

IV.

1. In a game of pure skill, if A's skill be to B's as 3 to 1, what is the chance of A's winning 3 games out of 4?

2. The probability of A solving a problem is $\frac{1}{4}$, and of its being solved when both A and B try it is $\frac{3}{4}$. Find the probability of B solving it.

3. A's skill is to B's as 1 to 2, to C's as 3 to 2, and to D's as 4 to 3; what is the probability that A in 3 trials of skill, one with each person, will be successful, (1) twice *exactly*; (2) twice *at least*?

4. One side of a die is black, two are red, and the remaining three are white; what is the chance of a red face being uppermost three times, the black twice, and a white once exactly in three throws?

5. If on an average 9 ships out of 10 return safe after a voyage to port, find the chance that out of 5 ships expected 3 at least will arrive.

6. If A speak truth 3 times out of 4, and B 5 times out of 6, find the chances that they will agree in their statements with respect to an event that has taken place.

7. It is 3 to 1 that A speaks the truth, 4 to 1 that B does, and 6 to 1 that C does. What is the probability that an event took place which A and B assert to have happened, but which C denies?

8. A speaks truth 3 times out of 4, B 4 times out of 5; they agree in asserting that from a bag containing 9 balls, all of different colours, a white ball has been drawn. Show that the probability that this is true is $\frac{96}{97}$.

9. What is the chance of winning the first game, and losing the second and third; winning the third, and losing the first and second; or winning the second, and losing the first and third; supposing the chances of winning and losing are equal? The chance of losing three games in succession is also $\frac{1}{8}$.

10. In a lottery if the prizes are to the blanks as 1 to 20; find the chance of gaining at least two prizes in six trials.

11. If in the French Lottery there are 100 numbers, 5 of which are drawn at a time; what is the chance that 2 only of 5 specified numbers will be drawn?

V.

1. A bag contains 10 sovereigns, 5 florins, 3 shillings, two sixpences, and 1 penny. Two coins are drawn out; prove that the worth of the expectation is £1 0*s.* 4$\frac{2}{3}$*d.*

2. If there be 3 chances for 4*s.*, 6 chances for 8*s.*, and five chances for 5*s.*; what is the value of the expectation?

3. In a bag are eight bank-notes, namely, one of £20, two of £5, and five of £1; a person is allowed to take out three indiscriminately. What is the value of his expectation?

4. There are four equal chances for drawing 3*s.*, 5*s.*, 8*s.*, 10*s.*, and six blanks; what is the value of the expectation?

5. If a bag contain 3 guineas, 2 sovereigns, and 4 shillings, and 3 coins be drawn out; what is the value of the expectation?

6. In a lottery of 100 tickets there are 4 prizes of £100 each, 10 of £30, and 20 of £5, and 66 blanks; what ought to be the price of a ticket?

7. A bag contains a number of tickets, of which one is marked 1, four marked 2, nine marked 3, 16 marked 4, and so on, and a person draws a ticket at random, for which he is to receive as many shillings as the number marked on the drawn ticket; required the value of his expectation?

8. The lottery at Paris consists of 90 numbers, out of which 5, to be drawn at hazard, are the winning numbers. A person staking his money upon naming *one* of the five winning numbers, if he succeed, receives 15 times his stake; if upon naming *two* out of the five, he receives 270 times his stake; show the disadvantages in both cases under which money is risked on these conditions. If a ticket of 100 francs were saleable in either case, what would be its real value?

RESULTS, HINTS, ETC., FOR THE EXERCISES ON VARIATIONS, PERMUTATIONS, AND COMBINATIONS.

I.

1. See Art. 1.

2. The number ${}^5V_2=5.4=20$, and the variations are

ab, ba, ac, ca, ad, da, ae, ea, bc, cb, bd, db, be, ed, cd, dc, ce, ec, de, ed.

And ${}^5C_3=\frac{5.4.3}{1.2.3}=10$, the combinations are *abc, abd, abc, acd, ace, ade, bcd, bce, bde, cde.*

3. The different combinations of the six letters taken 4 at a time are

aaab, aaac, aabb, aabc, abbc.

aacb admits of 4 variations, *aaab, aaba, abaa, baaa.*

aaac admits of the same, *aaac, aaca, acaa, caaa.*

aabb admits of 6 variations, *aabb, bbaa, abab, baba, abba, baab.*

also a^2bc, admits of 12 variations, *aabc, cbaa, aacb, bcaa, caab, baac, abca, acba, abac, caba, acab, baca,*

and ab^2c „ „ *bbac, cabb, bbca, acbb, cbba, abbc, bacb, bcab, babc, cbab, bcba, abcb.*

4. Generally ${}^nP=1.2.3.4.5.6\ldots(n-1)n$. Here $120=1.2.3.4.5$.

Hence ${}^nP=1.2.3.4.5$. $\therefore$ $n=5$ things.

5. See Art. 8; the number is 20.

7. First find the number of combinations of the six digits taken 3 at a time. Secondly, the number of permutations of each of these combinations, and then discriminate the *integral* and the *decimal* numbers of three digits.

8. First, if 1 be made the numerator, and the successive digits the denominators, there will be formed the fractions $\frac{1}{1}$, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, $\frac{1}{5}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{8}$, $\frac{1}{9}$, which are nine in number. The same number may be formed with the rest of the nine digits as numerators, and with the same denominators. Hence there will be formed 9 sets and 9 fractions in each set, or 81 fractions. Then separate those identical in value, and next arrange in pairs, fractions with their reciprocals.

9. See note, Art. 2.

II.

1. $n=10$, and the question requires the number of permutations of n things when $n=10$.

2. $n=8$ and -9, $n=8$ will give the answer.

3. $n=7$. 4. $n=12$ and -7, $n=12$ answers the question.

5. Here ${}^nV_r=272$, ${}^nC_r=136$, $\therefore$ ${}^nC_r=\frac{{}^nV_r}{1.2}$, which is only true when $r=2$, then ${}^nV_r=n(n-1)=272$, and $n=17$ and -16, $n=17$ answers the question.

6. $r=4$, $n=12$. 7. $n=7$. 8. $n=6$.

III.

1. $n=12$ and -7.

2. See Art. 3.

3. If the p things be removed, $n-p$ things remain, which when combined r at a time will give the number required, supposing that $n-p$ is greater than r.

4. If the p like things be removed, there remain $n-p$ unlike things. If the p like things be combined q at a time, then ${}^pC_q$ indicates the number of the combinations of p things taken q at a time. And if the $n-p$ unlike things be combined $r-q$ at a time, then ${}^{n-p}C_{r-q}$ denotes the combinations of the unlike things.

$\therefore$ ${}^{n-p}C_{r-q}\times{}^pC_q$ will give the number of combinations taken r at a time, each of which contains q of the p like things.

5. By Art. 3. The number is $\dfrac{2n(2n-1)\ldots(n+1)}{1.2\ldots n}$.

6. The divisors of a^m are $1, a, a^2, a^3, \ldots a^m$, in number $1+m$.
" " b^n " $1, b, b^2, b^3, \ldots b^n$, " $1+n$.
" " c^p " $1, c, c^2, c^3, \ldots c^p$, " $1+p$.

$\therefore$ The number is $(1+m)(1+n)(1+p)$, or excluding unity, $(1+m)(1+n)(1+\nu)-1$.
Here $76500=2^2.3^2.5^3.17$. The number of divisors is $(1+2)(1+2)(1+3)(1+1)=72$.

7. ${}^nC_r=\dfrac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3\ldots(r-1).r}$, ${}^{n-1}C_r=\dfrac{(n-1)(n-2)\ldots(n-r+1)(n-r)}{1.2\ldots(r-1).r}$

$$\therefore\ {}^nC_r-{}^{n-1}C_r=\frac{(n-1)(n-2)\ldots(n-r+1)}{1.2\ldots(r-1)}\cdot\left\{\frac{n}{r}-\frac{n-r}{r}\right\}=\frac{(n-1)(n-2)\ldots(n-r+1)}{1.2\ldots(r-1)}.$$

IV.

1. When one of the 20 things is left out, there remain 19 things to be combined 6 at a time. And when 2 are left out there are 18; and so for the rest.

2. There are 13 things different and 3 alike. First, the 13 unlike things can be combined 4 at a time. Secondly, the 13 unlike with one of the like things make 14 different things; these can be combined 4 at a time. Thirdly, the 13 unlike and two like things can be combined; and, fourthly, the 13 unlike with the three like things can be combined 4 at a time. The sum of these will be the number of combinations. And, lastly, each combination of 4 things can be varied, and the sum of all the variations will be the number required.

3. There are six ways in which the required combinations from three sets can be formed 6 at a time.

(1) By combining 3 things from the first set, 2 from the second, 1 from the third.
(2) " " 3 " " 1 " " 2 " "
(3) " " 2 " " 1 " " 3 " "
(4) " " 2 " " 3 " " 1 " "
(5) " " 1 " " 2 " " 3 " "
(6) " " 1 " " 3 " " 2 " "

and then taking the sum of these combinations.

4. The four letters c, d, e, f may be arranged taking 1, 2, 3, 4 respectively to form each arrangement; and if ab be placed first in each of these variations, the whole number of variations and permutations will be found.

6. The number of variations of the 6 letters b, c, d, e, f, g, taken 4 at a time, is 360, and if a be placed before each of these variations, the result will give 360 variations of 7 letters taken 5 at a time, each containing the letter a.

8. Let a, b, c be left out, and d, e, f, g, h, k, l be combined 2 at a time; then let a, b, c be added to each of these combinations, and the result will give the combinations of the 10 letters in each of which a, b, c occur.

V.

1. See Art. 3.

2. Suppose 21 consonants and five vowels,

$${}^{21}C_2=\frac{21.20}{1.2}=210,\quad {}^5C_1=\frac{5}{1}=5.\quad \text{Then by Art. 8, } {}^{21}C_2\times{}^5C_1=1050.$$

But as each combination of 3 letters admits 6 permutations,
The increased number is $1050 \times 6 = 6300$.

3. Nine different payments can be made with 9 pence, and the same number with each of the other sums. Besides these 4 sets of 9 payments, each can be combined in sets of 2, 3, and 4 payments. The sum of the number of sets will be the whole number of payments.

4. One, two, three, &c., coins can be respectively taken to form the different sums; the total number is 63.

6. See Art. 3 and 4.

7. Taking the general case, a polygon of n sides, and of n angular points. Three points must be joined to form a triangle. The number of triangles is therefore equal to the number of combinations of n things taken 3 at a time. This may be verified by the particular cases.

VI.

1. See Art. 3.

3. As a particular instance 5 men may be taken, two of whom must not be placed together in the variation of places the 5 men can occupy. The whole number of variations of 5 things is 5.4.3.2.1 or 120. If the 2 men be removed 3 remain, and these 3 admit of 3.2.1 or 6 permutations, and the 2 omitted can occupy the first, second, third, and fourth place in each permutation of 3 men. If 24, the number of permutations in which two specified things occur together, be subtracted, the remainder will give the number required, namely, 96. Other cases may be considered, and then the general case of n men.

4. See Art. 3.

5. Each voter may vote for one, two, or all three of the candidates.

6. As n courts send $n-1$ ambassadors, the number of ambassadors will be $n(n-1)$. The number of ways in which they may be distributed will be equal to the number of the permutations of $n-1$ different things.

7. A different guard can be posted on 230,300 nights; 18,424 is the number of them in which any particular man will be engaged.

VII.

1. See Art. 2. 2. See Art. 2. 4. See Art. 4.

5. Here are 9 sets of 4 things of the respective values 3, 2, 1, 0, and the question requires 9 to be taken, one out of each set of the four things, so that the sum of the 9 things shall be 16. As for instance, 8 centres and a miss will give 16.

7. See Art. 4. note.

8. The only difference of this from the preceding question, is that there are four faces of a tetrahedron, but six in a die.

9. See Art. 8.

VIII.

1. See Art. 3, and Art. 4.

3. Since ${}^nC_{r-r'} = {}^nC_{r+r'}$, the number $r+r'$ must be complementary to $r-r'$,

$$\therefore\ r+r' = n-(r-r') = n-r+r',\ \therefore\ n = 2r.$$

5. See Art. 4, note.

6. $a^4b^3c^2d$ consists of 10 letters, which are to be combined 5 at a time; 5 may be made up by $4+1$, $3+2$, $3+1+1$, $2+2+1$, and $2+1+1+1$.

Hence the following different combinations can be formed :—

a^4b, a^4c, a^4d.
a^3b^2, a^3c^2, b^3a^2, b^3c^2.

$a^3bc,\ a^3bd,\ a^3cd,\ b^3ac,\ b^3ad,\ b^3cd.$
$a^2b^2c,\ a^2b^2d,\ a^2c^2d,\ a^2c^2b,\ b^2c^2a,\ b^2c^2d.$
$a^2bcd,\ b^2acd,\ c^2abd.$

The variations of these combinations by Art. 3 will be found to be 415.

8. First let three points be taken and joined in every way by straight lines, next four points in the same manner, then five points, and so on. Lastly find the number of intersections when there are n points.

IX.

1. $${}^nV_r = n(n-1)(n-2)\ldots\ldots(n-r+1),$$
$$\text{and } {}^{n-1}V_r = (n-1)(n-2)\ldots(n-r+1)(n-r);$$
$$\therefore\ {}^nV_r - {}^{n-1}V_r = \{n-(n-r)\}.(n-1)(n-2)\ldots\ldots(n-r+1)$$
$$= r.{}^{n-1}V_{r-1}.$$

3. $${}^nC_r = \frac{n(n-1)\ldots(n-r+1)}{1.2\ldots(r-1)r},\ {}^{n-1}C_r = \frac{(n-1)(n-2)\ldots(n-r+1)(n-r)}{1.2\ldots(r-1)r};$$
$$\therefore\ \frac{{}^nC_r}{{}^{n-1}C_r} = \frac{n}{n-r},\ \text{and } {}^nC_r = \frac{n}{n-r}.{}^{n-1}C_r.$$

5. Here $${}^{2n}C_{2r} = \frac{2n(2n-1)(2n-2)\ldots(2n-2r+2)(2n-2r+1)}{1.2.3\ldots(2r-1).(2r)}$$
$$= \frac{(2n-1)(2n-3)\ldots\ldots(2n-2r+1)}{1.3\ldots\ldots(2r-1)}.\frac{2n(2n-2)\ldots\ldots(2n-2r+2)}{2.4\ldots\ldots(2r)}.$$
$$= \frac{(2n-1)(2n-3)\ldots\ldots(2n-2r+1)}{1.3\ldots\ldots(2r-1)}.\frac{n(n-1)\ldots\ldots(n-r+1)}{1.2\ldots\ldots r}.$$
$$= \frac{(2n-1)(2n-3)\ldots\ldots(2n-2r+1)}{1.3\ldots\ldots(2r-1)}{}^nC_r.$$

6. Here $$\frac{1.2.3.4}{n(n^2-1)}\times{}^{n^2}C_4\times{}^{n+2}C_5$$
$$= \frac{1.2.3.4}{n(n^2-1)}\times\frac{n^2(n^2-1)(n^2-2)(n^2-3)}{1.2.3.4}\times\frac{(n+2)(n+1)n(n-1)(n-2)}{1.2.3.4.5}$$
$$= \frac{n^2(n^2-1)(n^2-2)(n^2-3)(n^2-4)}{1.2.3.4.5}.$$
$$= {}^{n^2}C_5.$$

RESULTS, HINTS, ETC., FOR THE EXERCISES ON THE BINOMIAL AND POLYNOMIAL THEOREMS.

I.

1. $a^3+9a^2x+27ax^2+27x^3$. $a^4+4na^3b+6n^2a^2b^2+4n^3ab^3+n^4b^4$.

$$1-10x+40x^2-80x^3+80x^4-32x^5.$$

$$64-960y+6000y^2-20000y^3+37500y^4-37500y^5+15625y^6.$$

$$a^{14}+7a^{12}x^2+21a^{10}x^4+35a^8x^6+35a^6x^8+21a^4x^{10}+7a^2x^{12}+x^{14}.$$

$$256x^8+3062x^7y+48384x^6y^2+48384x^5y^3+90720x^4y^4+108864x^3y^5+81648x^2y^6 +34972xy^7+6561y^8.$$

The other examples offer no difficulty.

The expansion of $(2+\surd 3)^5$ can be exhibited in the form $362+209\surd 3$.

Also the expansion of $(x+x^{-1})^4$ can be put into the form $(x^4+x^{-4})+4(x^2+x^{-2})+6$, as also the expansions of the other powers of $x\pm x^{-1}$.

II.

The following are the series which arise from the first set of examples.

1. $(1\pm x)^{-1}=1\mp x+x^2\mp x^3+x^4\mp x^5+x^6\mp\ldots\mp x^{p-1}+\ldots$

$$(1\pm x)^{-2}=1\mp 2x+3x^2\mp 4x^3+5x^4\mp 6x^5+7x^6\mp\ldots\mp px^{p-1}+\ldots$$

$$(1\pm x)^{-3}=1\mp 3x+6x^2\mp 10x^3+15x^4\mp 21x^4+\ldots+\frac{p(p+1)}{1.2}.x^{p-1}+\ldots$$

$$(1\pm x)^{-4}=1\mp 4x+10x^2\mp 20x^3+35x^4\mp 56x^5+\ldots\mp\frac{p(p+1)(p+2)}{1.2.3}.x^{p-1}+\ldots$$

$$(1\pm x)^{-5}=1\mp 5x+15x^2\mp 35x^3+70x^4\mp 126x^5+\ldots\mp\frac{p(p+1)(p+2)(p+3)}{1.2.3.4} x^{p-1}+\ldots$$

$$(1\pm x)^{-6}=1\mp 6x+21x^2\mp 56x^3+126x^4\mp 252x^5+\ldots \mp\frac{p(p+1)(p+2)(p+3)(p+4)}{1.2.3.4.5}.x^{p-1}+\ldots$$

The examples under 2 and 3 offer no difficulty, as

$$(1-x)^{-\frac{1}{2}}=1+\frac{1}{2}.x+\frac{1.3}{2.4}.x^2+\frac{1.3.5}{2.4.6}.x^3+\frac{1.3.5.7}{2.4.6.8}.x^4+\ldots\frac{1.3.5\ldots(2p-1)}{2.4.6\ldots 2p}x^{p-1}+\ldots$$

The first example under 4 is

$$(x+x^{-1})^n=(x^n+x^{-n})+\frac{n(n-1)}{1.2}(x^{n-2}+x^{-n+2})+\frac{n(n-1)(n-2)}{1.2.3}.(x^{n-4}+x^{-n+4})+\ldots$$

$$+\frac{n(n-1)(n-2)\ldots\left(\frac{n}{2}+1\right)}{1.2.3\ldots\frac{n}{2}},\text{ when } n \text{ is even,}$$

$$\text{or }+\frac{n(n-1)(n-2)\ldots\frac{n+1}{2}}{1.2.3\ldots\frac{n-1}{2}}.(x+x^{-1})\text{ when } n \text{ is odd.}$$

5. When the index of the binomial is not a positive integer, the expansion will be more readily effected by reducing the binomial into such a form that the first term shall be unity, as

$$(a^2+x^2)^{\frac{m}{n}}=\left\{a^2\left(1+\frac{x^2}{a^2}\right)\right\}^{\frac{m}{n}}=a^{\frac{2m}{n}}\left(1+\frac{x^2}{a^2}\right)^{\frac{m}{n}}$$

$$=a^{\frac{2m}{n}}\left\{1+\frac{m}{n}.\frac{x^2}{a^2}+\frac{\frac{m}{n}\left(\frac{m}{n}-1\right)}{1.2}.\frac{x^4}{a^4}+\frac{\frac{m}{n}\left(\frac{m}{n}-1\right)\left(\frac{m}{n}-2\right)}{1.2.3}.\frac{x^6}{a^6}+\ldots\right\}$$

$$=a^{\frac{2m}{n}}\left\{1+\frac{m}{n}.\frac{x^2}{a^2}+\frac{m(m-n)}{1.2.n^2}.\frac{x^4}{a^4}+\frac{m(m-n)(m-2n)}{1.2.3.n^3}.\frac{x^6}{a^6}+\ldots\right\}$$

and if required, $a^{\frac{2m}{n}}$ can be multiplied into each term of the expansion.

In the same manner

$$(a^{\frac{2}{3}}n^{\frac{1}{3}}+a^{\frac{1}{3}}x^{\frac{2}{3}})^{-\frac{4}{7}}=\left\{a^{\frac{2}{3}}x^{\frac{1}{3}}\left(1+\frac{x^{\frac{1}{3}}}{a^{\frac{1}{3}}}\right)\right\}^{-\frac{4}{7}}=a^{-\frac{8}{21}}x^{-\frac{4}{21}}\left(1+\frac{x^{\frac{1}{3}}}{a^{\frac{1}{3}}}\right)^{-\frac{4}{7}}$$

$$=a^{-\frac{8}{21}}x^{-\frac{4}{21}}\left\{1-\frac{4}{7}.\frac{x^{\frac{1}{3}}}{a^{\frac{1}{3}}}+\frac{\frac{4}{7}\left(\frac{4}{7}+1\right)}{1.2}.\frac{x^{\frac{2}{3}}}{a^{\frac{2}{3}}}-\frac{\frac{4}{7}\left(\frac{4}{7}+1\right)\left(\frac{4}{7}+2\right)}{1.2.3}.\frac{x}{a}+\ldots\right\}$$

$$=a^{-\frac{8}{21}}x^{-\frac{4}{21}}\left\{1-\frac{4}{7}.\frac{x^{\frac{1}{3}}}{x^{\frac{1}{3}}}+\frac{4.11}{7.14}.\frac{x^{\frac{2}{3}}}{a^{\frac{2}{3}}}-\frac{4.11.18}{7.14.21}.\frac{x}{a}+\ldots\right\}.$$

III.

The binomial theorem may be sometimes advantageously employed in finding approximations to the roots of numbers in cases where the given number is not an exact power. The method can be applied to the fifth, seventh, or any other roots as readily as to the square and cube roots.

When the number whose root is required does not differ greatly in excess or defect of the same power; in such cases, the convergency being rapid, the required root can be determined to six or seven places of decimals, by means of a few terms of the expansion.

There are numerous forms of series which result from the expansion of expressions by the binomial theorem, and the sums of such series may be determined by discovering the forms of the binomial expressions of which the given series are the developments. Where no direct methods can be discovered, approximations can always be made when the series are convergent.

1. 2·2360. 2. 2·8284. 3. 10·0332. 4. 3·6170. 5. 2·0044.
6. 5·1961.

IV.

1. See Art. 8, p. 20.

$$1+x=\frac{1+x}{1}=\frac{1}{\frac{1}{1+x}}=\frac{1}{1-\frac{x}{1+x}}=\left(1-\frac{x}{1+x}\right)^{-1},\text{ and }(1+x)^{\frac{1}{2}}=\left(1-\frac{x}{1+x}\right)^{-\frac{1}{2}}.$$

$$2.\ a+b=\frac{a(a+b)}{a}=\frac{a}{\frac{a}{a+b}}=\frac{a}{1-\frac{b}{a+b}}=a\left(1-\frac{b}{a+b}\right)^{-1},\text{ and }(a+b)^{\frac{1}{2}}=a^{\frac{1}{2}}\left(1-\frac{b}{a+b}\right)^{-\frac{1}{2}}.$$

$$4.\ \frac{n}{n-1}=\frac{1}{\frac{n-1}{n}}=\frac{1}{1-\frac{1}{n}}=\left(1-\frac{1}{n}\right)\text{, and }\left\{\frac{n}{n-1}\right\}^{\frac{1}{n}}=\left\{1-\frac{1}{n}\right\}^{-\frac{1}{n}}.$$

$$7.\ (x^2-2)^{\frac{1}{2}}=\left\{\frac{x^4-2x^2}{x^2}\right\}^{\frac{1}{2}}=\left\{\frac{(x^4-2x^2+1)-1}{x^2}\right\}^{\frac{1}{2}}=\left\{\frac{(x^2-1)^2-1}{x^2}\right\}^{\frac{1}{2}}$$

$$=\frac{x^2-1}{x}.\left\{1-\frac{1}{(x^2-1)^2}\right\}^{\frac{1}{2}}.$$

8. $\dfrac{1+x}{1-x}=\dfrac{1}{\dfrac{1-x}{1+x}}=\dfrac{1}{1-\dfrac{2x}{1+x}}=\left(1-\dfrac{2x}{1+x}\right)^{-1}$.

9. $\dfrac{1+2x}{1+x}=\dfrac{1}{\dfrac{1+x}{1+2x}}=\dfrac{1}{1-\dfrac{x}{1+2x}}=\left(1-\dfrac{x}{1+2x}\right)^{-1}$.

V.

1. $(1+x)^{1+x}=1+x^1+x^2+\dfrac{x^3}{2}+\dfrac{x^4}{3}+\dfrac{x^5}{5}+$ &c.

2. $(1+x)^n+n(1+x)^{n-1}x+\dfrac{n(n+1)}{1.2}(1+x)^{n-2}x^2+\dots$

$$=(1+x)^n.\left\{1+n\frac{x}{1+x}+\frac{n(n+1)}{1.2}.\left(\frac{x}{1+x}\right)^2+\frac{n(n+1)(n+2)}{1.2.3}.\left(\frac{x}{1+x}\right)^3+\dots\right.$$

$$=(1+x)^n.\left\{1-\frac{x}{1+x}\right\}^{-n}, \text{ by Art. 8, p. 20.}$$

$$=(1+x)^n.\frac{1}{(1+x)^{-n}}=(1+x)^n.(1+x)^n=(1+x)^{2n}.$$

3. $2=1+1=\dfrac{1(1+1)}{1}=\dfrac{1}{\dfrac{1}{1+1}}=\dfrac{1}{1-\dfrac{1}{2}}=\left(1-\dfrac{1}{2}\right)^{-1}$, $\therefore\ 2^n=\left(1-\dfrac{1}{2}\right)^{-n}$.

8. The given expression arises from

$$\frac{\left\{1-\dfrac{a}{a-b}\right\}^{-n}}{\left\{1-\dfrac{b}{a-b}\right\}^{-n}}=\frac{\left\{\dfrac{-b}{a-b}\right\}^{-n}}{\left\{\dfrac{a}{a-b}\right\}^{-n}}=\left\{\frac{a}{a-b}\times\frac{a-b}{b}\right\}^n.(-1)^n=\left\{\frac{a}{b}\right\}^n.(-1)^n.$$

VI.

1. Here $(1+x+2x^2+3x^3+\dots)^2=(1-x+x^2)^2(1-x)^{-4}$. And the coefficients of x^n in the product of the two expansions will be found to be $\dfrac{n(n^2+11)}{1.2.3}$.

2. $\left(\dfrac{n}{1}+1\right)\left(\dfrac{n}{2}+1\right)\left(\dfrac{n}{3}+1\right)$.

3. Let $S=1.2+2.3x+3.4x^2+4.5x^3+\dots(n+1)(n+2)x^n$,

then $Sx=1.2x+2.3x^2+3.4x^3+\dots+(n+1)(n+2)x^{n+1}$

$\therefore\ S(1-x)=2(1+2x+3x^2+4x^3+\dots)-(n+1)(n+2)x^{n+1}$.

4. $\frac{1}{3}\{2(n+1)^3+(n+1)\}$. 5. $\frac{1}{6}(n-6)(n^2-1)$. 6. $\frac{1}{3}\{2(n+1)^3+(n+1)\}$.

7. $\dfrac{a^n-b^n}{a-b}.\dfrac{b}{a^nb^n}$. 8. $2^{n-1}(n^2+4n+2)$. 9. $4n+1$.

10. $1^2+2^2+3^2+\dots+(n+1)^2$.

VII.

1. $(1-x)^{\frac{1}{2}}=1-\frac{1}{2}x+\text{\&c.}=1-\frac{1}{2}x$ nearly. Similarly $(1-x)^{\frac{1}{3}}=1-\frac{1}{3}x$ nearly,

and $\dfrac{1}{1+\sqrt{(1-x)}}-\dfrac{1}{1+\sqrt[3]{(1-x)}}=\dfrac{1}{2-\frac{1}{2}x}-\dfrac{1}{2-\frac{1}{3}x}=\dfrac{x}{24}$ nearly.

3. Let $x=1+h$, when h is small.

Then $x^m=(1+h)^m=1+mh+\text{\&c.}=1+mh$ nearly, and $mx^m=m+m^2h$ nearly, and $x^n=(1+h)^n=1+nh+\text{\&c.}=1+nh$ nearly, and $nx^n=n+n^2h$ nearly.

$\therefore\ mx^m-nx^n=(m-n)\{1+(m+n)h\}$

and $\dfrac{mx^m-nx^n}{m-n}=1+(m+n)h$ nearly $=(1+h)^{n+m}=x^{m+n}$ nearly.

5. The ratio of $\left\{\frac{a+x}{b-x}\right\}^2$ to $\frac{a^2}{b^2}=\frac{\left(1+\frac{x}{a}\right)^2}{\left(1-\frac{b}{a}\right)^2}=1+\frac{2(a+b)}{ab}.x+\&c.$

$$=1+\frac{2(a+b)}{ab}.x \text{ nearly}$$

$$=(1+x)^{\frac{2(a+b)}{ab}}$$

$$=\text{the ratio of } (1+x)^{\frac{2(a+b)}{ab}} \text{ to } 1$$

nearly when x is very small.

8. $\frac{(1-x)^{-\frac{1}{3}}+(1+x)^{-\frac{1}{3}}}{(1+x)^{-\frac{2}{3}}+(1-x)^{-\frac{2}{3}}}=\frac{2\left\{1+\frac{2}{9}.x^2+\frac{35}{243}.x^4+\frac{728}{9261}.x^6+\&c.\right.}{2\left\{1+\frac{5}{9}x^2+\frac{110}{243}x^4+\frac{2618}{9261}.x^6+\&c.\right.}$

$$=1-\frac{x^2}{3}-\frac{10x^4}{81}-\frac{50x^6}{729}-\&c.$$

VIII.

1. See Section III., pp. 3—6.

IX.

3. $\frac{a+b\sqrt{-1}}{a-b\sqrt{-1}}=\frac{a+b\sqrt{-1}}{a-b\sqrt{-1}}\times\frac{a+b\sqrt{-1}}{a+b\sqrt{-1}}=\frac{a^2-b^2+2ab\sqrt{-1}}{a^2+b^2}=\frac{a^2-b^2}{a^2+b^2}+\frac{2ab\sqrt{-1}}{a^2+b^2}$,

and $\therefore \left\{\frac{a+b\sqrt{-1}}{a-b\sqrt{-1}}\right\}^n=\left\{\frac{a^2-b^2}{a^2+b^2}+\frac{2ab\sqrt{-1}}{a^2+b^2}\right\}^n=\left\{\frac{a^2-b^2}{a^2+b^2}\right\}^n.\left\{1+\frac{2ab\sqrt{-1}}{a^2-b^2}\right\}^n$.

6. Expand the expressions, and find their sum and difference.

8. The expression $(1+\sqrt{2})^{2n}=\{(1+\sqrt{2})^2\}^n=(3+2\sqrt{2})^n$.

9. Here $(2+\sqrt{3})^n=2^n+\frac{n(n-1)}{1.2}2^{n-2}.3+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}.2^{n-4}.3^2\ldots$

$$+\sqrt{3}\left\{n.2^{n-1}+\frac{n(n-1)(n-2)}{1.2.3}.2^{n-3}.3+\ldots\right\}$$

and $(2-\sqrt{3})^n=2^n+\frac{n(n-1)}{1.2}.2^{n-2}.3+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}.2^{n-4}.3^2\ldots$

$$-\sqrt{3}\left\{n.2^{n-1}+\frac{n(n-1)(n-2)}{1.2.3}2^{n-3}.3+\ldots\right\}.$$

And $2+\sqrt{3}$ being greater than 3 but less than 4, is partly integral and partly decimal.

Also $2-\sqrt{3}$ being greater than 0 but less than 1, is wholly decimal.

$\therefore (2+\sqrt{3})^n$ is partly integral and partly decimal, and $(2-\sqrt{3})^n$ is wholly decimal.

If I and D_1 denote the integral and decimal parts of the former, and D_2 the decimal of the latter,

then $(2+\sqrt{3})^n+(2-\sqrt{3})^n=I+D_1+D_2$.

$$\therefore I+D_1+D_2=2\left\{2^n+\frac{n(n-1)}{1.2}.2^{n-2}.3+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}.2^{n-4}.3^2+\ldots\right\}$$

which is wholly integral, as all the coefficients are integral.

$\therefore I+D_1+D_2$ is an integer, and I being an integer, D_1+D_2 must be an integer or zero.

But since D_1, D_2 are decimals each less than unity, D_1+D_2 lies between 0 and 2, and 1 is the only integer between 0 and 2, $\therefore D_1+D_2=1$.

$$\therefore\ I=2\left\{2^n-\frac{n(n-1)}{1.2}2^{n-2}.3+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}.2^{n-4}.3^2+\ldots\right\}-1$$

is the integral part of $(2+\sqrt{3})^n$,

and $D_1=1-D_2=1-(2-\sqrt{3})^n$ the decimal part of $(2+\sqrt{3})^n$.

The truth of the general form may be illustrated by taking n equal to 2, 3, 4, &c.

XI.

2. The coefficient of x^n in the expansion of $(1-x)^{-r}=\frac{r(r+1)(r+2)\ldots(r+n-1)}{1.2.3\ldots n}$,

and the coefficient of x^n in the expansion of $(1+x)^{n+r-1}$

$$=\frac{(n+r-1)(n+r-2)\ldots\{(n+r-1)-n+1\}}{1.2.3\ldots n}$$

$$=\frac{(n+r-1)(n+r-2)\ldots(r+1)r}{1.2\ldots n}.$$

Hence it appears that the two expressions are identical.

5. Let ${}^nT_r$ denote the coefficient of the $(r+1)$th term of $\frac{1}{(1-x)^n}$, and ${}^{n+1}T_r$, ${}^{n+1}T_{r-1}$ the $(r+1)$th and rth term of $\frac{1}{(1-x)^{n+1}}$.

Then $${}^{n+1}T_r-{}^{n+1}T_{r-1}=\frac{(n+1)(n-2)\ldots(n+r-1)(n+r)}{1.2\ldots(r-1).r}-\frac{(n+1)(n+2)\ldots(n+r-1)}{1.2\ldots(r-1)}$$

$$=\frac{(n+1)(n+2)\ldots(n+r-1)}{1.2\ldots(r-1)}.\left\{\frac{n+r}{r}-1\right\}=\frac{n(n+1)(n+2)\ldots(n+r-1)}{1.2.3\ldots r}={}^nT_r.$$

XII.

1. Here $$1+\frac{1}{2}C_1+\frac{1}{3}C_2+\frac{1}{4}C_3+\ldots.=1+\frac{n}{1.2}+\frac{n(n-1)}{1.2.3}+\frac{n(n-1)n-2)}{1.2.3}$$

$$+\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}+\ldots.\}$$

$$=\frac{1}{n+1}\left\{(n+1)+\frac{(n+1)n}{1.2}.+\frac{(n+1)n(n-1)}{1.2.3}+\ldots.\right\}$$

$$=\frac{1}{n+1}\left\{1+\frac{(n+1)n}{1.2}+\frac{(n+1)n(n-1)}{1.2.3}+\ldots.-1\right\}$$

$$=\frac{1}{n+1}\left\{2^{n+1}-1\right\}$$

3. (1) Here $$(1+x)^n=1+nx+\frac{n(n-1)}{1.2}x^2+\frac{n(n-1)(n-2)}{1.2.3}.x^3+\ldots$$

$$+\frac{n(n-1)\ldots(n-r+1)}{1.2\ldots.r}x^r+\ldots$$

And $\frac{a_1}{a_0}=n$, $\frac{2a_2}{a_1}=n-1$, $\frac{3a_3}{a_2}=n-2$, $\frac{4a_4}{a_3}=n-3$, &c.

$$\therefore\ \frac{a_1}{a_0}+\frac{2a_2}{a_1}+\frac{3a_3}{a_2}+\ldots\frac{na^n}{a_{n-1}}=n+(n-1)+(n-2)+(n-3)+\text{\&c., to } n \text{ terms}=\frac{n(n+1)}{1.2}.$$

6. See p. 26, note.

7. ${}^nC_r=\frac{n(n-1)\ldots(n-r+1)}{1.2\ldots r}$, and ${}^nD_r=\frac{n(n+1)\ldots(n+r-1)}{1.2\ldots r}$

$$\therefore\ \frac{n+r}{n}.\frac{1.2.3\ldots.r}{(r+1)(r+2)\ldots.2r}.{}^nC_r.{}^nD_r=\frac{n+r}{n}.\frac{1.2.3\ldots r}{(r+1)(r+2)\ldots2r}.$$

$$\frac{n(n-1)\ldots(n-r+1)}{1.2\ldots.r}.\frac{n(n+1)\ldots.(n+r-1)}{1.2\ldots.r}$$

$$=\frac{(n+r)(n+r-1)\ldots.(n+1)n(n-1)\ldots.(n-r+1)}{1.2.3\ldots.r.(n+1)(r+2)\ldots.2r}={}^{n+r}C_{2r}.$$

8. The coefficients of the $(p+1)$th terms of the expansions of $(a+b)^{2p}$, $(a+2b)^{2p}$ $(a+2^2b)^{2p}$, &c., will be identical, and the sum of these terms will be

$$\frac{2p(2p-1)\ldots\ldots(p+1)}{1.2\ldots\ldots p}.a^p.b^p.\left\{1+2^p+2^{2p}+2^{3p}+\text{to } p \text{ terms}\right\}$$

$$=\frac{2p(2p-1)\ldots(p+1)}{1.2\ldots\ldots p}.a^p.b^p.\frac{1-2^{p^2}}{1-2^p}.$$

XIII.

. The exercises under this number offer no difficulties. Both the expansions and the particular terms of expansions required may be readily effected by means of the general forms given in the preceding pages 20—24.

10. When the index is fractional or negative the same rule applies, as in the case of finding the 12th term of $(a-bx+cx^2-dx^3+ex^4)^{\frac{1}{2}}$: the equations of condition to be fulfilled are $p+q+r+\ldots=\frac{1}{2}$, $q+2r+3s+\ldots=11$.

XIV.

2. Here $a=1+(a-1)$. Let n be any variable number, then $a^n=\{1+(a-1)\}^n$

$$=\left\{1+n(a-1)+\frac{n(n-1)}{1.2}(a-1)^2+\frac{n(n-1)(n-2)}{1.2.3}(a-1)^3+\ldots\right.$$

$$=1+\left\{(a-1)-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\ldots\right\}n+Bn^2+Cn^3+\ldots$$ by collecting the terms of the coefficient of n, and putting B, C, &c., for the coefficients of n^2, n^3, &c.

$$\text{Let } (a-1)-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\text{\&c.}=A, \text{ and } Bn^2+Cn^3+\text{\&c.}=Pn^2,$$

$$\text{then } a^n=1+An+Pn^2=1+(A+Pn)n, \text{ and } a^x=a^{\frac{nx}{n}},$$

$$\therefore\ a^x=\{a^n\}^{\frac{x}{n}}=\{1+(A+Pn)n\}^{\frac{x}{n}},$$

$$=1+\frac{x}{n}(A+Pn)n+\frac{\frac{x}{n}\left(\frac{x}{n}-1\right)}{1.2}(A+Pn)^2n^2+\frac{\frac{x}{n}\left(\frac{x}{n}-1\right)\left(\frac{x}{n}-2\right)}{1.2.3}(A+Pn)^3n^3+\ldots$$

$$=1+(A+Pn)+\frac{x(x-n)}{1.2}(A+Pn)^2+\frac{x(x-n)(x-2n)}{1.2.3}(A+Pn)^3+\ldots$$

Now this is true whatever may be the value of n, for a^x is obviously independent of the value of n. Let $n=0$,

$$\therefore\ a^x=1+Ax+\frac{A^2x^2}{1.2}+\frac{A^3x^3}{1.2.3}+\ldots$$

$$\text{in which } A=(a-1)-\frac{(a-1)^2}{2}+\frac{(a-1)^3}{3}-\text{\&c.}\ldots$$

3. Here $e^x=1+x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\ldots$

$$\therefore\ e^{e^x}=e^{1+x+\frac{x^2}{2}+\frac{x^3}{6}+\ldots}$$

$$=e.e^x.e^{\frac{x^2}{2}}.e^{\frac{x^3}{6}}\ldots$$

$$=\left(1+1+\frac{1}{1.2}+\ldots\right)\left(1+x+\frac{x^2}{1.2}+\ldots\right)\left(1+\frac{x^2}{2}+\frac{x^4}{8}+\ldots\right)\left(1+\frac{x^3}{6}+\frac{x^6}{216}+\ldots\right)\ldots$$

Find the first six terms of the series.

5. In the product of the expansions of $(1-x)^{-2n}$ and e^x, find the aggregate of the products of every two terms which contain x^{2n}.

7. This expression may be deduced from the identity $\left(1+\frac{1}{n}\right)^{nx}=\left\{\left(1+\frac{1}{n}\right)^{n}\right\}^{x}$, and taking n greater than any assignable quantity.

$$\text{For } \left(1+\frac{1}{n}\right)^{nx}=1+nx\frac{1}{n}+\frac{nx(nx-1)}{1.2}\cdot\frac{1}{n^2}+\frac{nx(nx-1)(nx-2)}{1.2.3}\cdot\frac{1}{n^3}+\&c.$$

$$=1+x+\frac{x\left(x-\frac{1}{n}\right)}{1.2}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{1.2.3}\cdot+\&c.$$

$$\text{And } \left(1+\frac{1}{n}\right)^{n}=1+n.\frac{1}{n}+\frac{n(n-1)}{1.2}\cdot\frac{1}{n^2}+\frac{n(n-1)(n-2)}{1.2.3}\cdot\frac{1}{n^3}+\&c.$$

$$=1+1+\frac{1.\left(1-\frac{1}{n}\right)}{1.2}+\frac{1.\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{1.2.3}+\&c.$$

$$\therefore\ 1+x+\frac{x\left(x-\frac{1}{n}\right)}{1.2}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{1.2.3}+\&c.$$

$$=\left\{1+1+\frac{1.\left(1-\frac{1}{n}\right)}{1.2}+\frac{1.\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{1.2.3}+\&c.\right\}^{x}$$

Let n become greater than any assignable quantity, then $\frac{1}{n}, \frac{2}{n}$, &c., respectively become less than any assignable quantity, and therefore may be rejected, and there results $1+x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\&c.=\left\{1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\&c.\right\}^{x}$.

8. $$\left(1+\frac{1}{nx}\right)^{nx}=1+nx\frac{1}{nx}+\frac{nx(nx-1)}{1.2}\cdot\frac{1}{n^2x^2}+\frac{nx(nx-1)(nx-2)}{1.2.3}\cdot\frac{1}{n^3x^3}+\ldots$$

$$=1+1+\frac{1.\left(1-\frac{1}{nx}\right)}{1.2}+\frac{1.\left(1-\frac{1}{nx}\right)\left(1-\frac{2}{nx}\right)}{1.2.3}+\ldots$$

when x becomes indefinitely great, $\frac{1}{nx}$ becomes indefinitely small.

And series becomes $=1+1+\frac{1}{1.2}+\frac{1}{1.2.3}+\ldots$

9. The successive terms of the given series will be the coefficients of x^n in the respective expansions of $\frac{2^n}{1-x}$, $-\frac{2^{n-2}x^2}{(1-x)^2}$, $\frac{2^{n-4}x^4}{(1-x)^3}$, $-\frac{2^{n-6}x^6}{(1-x)^4}$, &c.

But these terms constitute a geometrical series continued indefinitely, the first term being $\frac{2^n}{1-x}$, and the common ratio $-\frac{2^{-2}x^2}{(1-x)^2}$.

$$\text{Their sum is } \frac{2^{n+2}}{4-4x+x^2}=\frac{2^{n+2}}{(2-x)^2}=\frac{2^n}{\left(1-\frac{x}{2}\right)^2}=2^n\left(1-\frac{x}{2}\right)^{-2}.$$

$$\text{And } 2^n\left(1-\frac{x}{2}\right)^{-2}=2^n\left\{1+2.\frac{x}{2}+3.\frac{x^2}{2^2}+4.\frac{x^3}{2^3}+\ldots+(n+1)\frac{x^n}{2^n}+\ldots\right\}$$

The nth term of which is $2^n.(n+1)\frac{x^n}{2^n}=(n+1)x^n$.

$$\text{Hence } 2^n-(n-1)2^{n-2}+\frac{(n-2)(n-3)}{1.2}.2^{n-4}-\&c.=n+1.$$

RESULTS, HINTS, ETC., FOR THE EXERCISES ON PROBABILITIES.

I.

2. The probabilities of a person alive at the beginning of each of the following six decades, not surviving 10 years:—

1. At the age of 10 years is $\frac{51}{590}=\frac{1}{11\frac{29}{51}}$, or between $\frac{1}{12}$ and $\frac{1}{11}$.
2. ,, 20 ,, $\frac{69}{539}=\frac{1}{6\frac{46}{69}}$, ,, $\frac{1}{7}$,, $\frac{1}{6}$.
3. ,, 30 ,, $\frac{77}{470}=\frac{1}{6\frac{8}{77}}$, ,, $\frac{1}{7}$,, $\frac{1}{6}$.
4. ,, 40 ,, $\frac{78}{393}=\frac{1}{5\frac{3}{78}}$, ,, $\frac{1}{6}$,, $\frac{1}{5}$.
5. ,, 50 ,, $\frac{85}{315}=\frac{1}{3\frac{60}{85}}$, ,, $\frac{1}{4}$,, $\frac{1}{3}$.
6. ,, 60 ,, $\frac{87}{230}=\frac{1}{2\frac{56}{87}}$, ,, $\frac{1}{3}$,, $\frac{1}{2}$.

Dr. Carpenter, the eminent physiologist, has remarked:—"It has been ascertained that from about the age of 18 to 28 the mortality is much greater in males than in females, being at its maximum at 25, when the viability (or probability of life) is only half of what it is at puberty. This fact is a very striking one, and shows most forcibly that the indulgence of the passions not only weakens the health, but in a great number of instances is the cause of a very premature death."

II.

3. Suppose the 3 the number of points,

then the chance of throwing 3 in the first throw is $\frac{1}{6}$.
And in the second throw the chance of failure is $\frac{5}{6}$.
Therefore the required chance is $\frac{1}{6}\times\frac{5}{6}=\frac{5}{36}$.

4. By Art. 5, the probability is $\frac{a^3+3a^2b+3ab^2}{(a+b)^3}$. And when $a=1$ and $b=5$

$$\therefore\ \frac{1+15+75}{216}=\frac{91}{216}\text{ the probability required.}$$

Otherwise the probability of failure is $\frac{5^3}{6^3}=\frac{125}{216}$, and of success $1-\frac{125}{216}=\frac{91}{216}$.

5. In each of his three throws *A* may throw 1, 2, 3, 4, 5, or 6 points, and the number 8 can be made up by the following throws:—

1, 1, 6; 2, 2, 4; 3, 3, 2; 1, 2, 5; 1, 3, 4.

Next *B* in three throws can make up 7 points by the throws

1, 1, 5; 2, 2, 3; 3, 3, 1; 1, 2, 4.

6. One throw gives 4 chances, 3 unfavourable, and 1 favourable.

Two throws give 16 ,, 9 ,, ,, 7 ,,
Three ,, 64 ,, 27 ,, ,, 37 ,,

Therefore two throws give the odds against, 9 to 7;
three throws give the odds for, 37 to 27, greater than an even chance by $\frac{5}{32}$.

7. With one throw there is	1 chance favourable and		5 unfavourable.	
With two throws there are	11 chances	,,	25	,,
three ,, ,,	91 ,,	,,	125	,,
four ,, ,,	671 ,,	,,	625	,,
five ,, ,,	3651 ,,	,,	3125	,,
	&c.		&c.	

With three throws the odds *against* are 125 to 91;
but with four throws the odds *for* are 671 to 625.

There is less than an even chance in three throws, but greater than an even chance in four throws, and still greater than an even chance in five throws, &c.

8. Since in one throw there is one chance of succeeding and five chances of failing, Suppose x the number of throws, then $\left\{\frac{5}{6}\right\}^x$ = probability of failing in x throws. But as the wager is the same, failing and succeeding, this probability of success and failure will be represented by $\frac{1}{2}$, $\therefore \left\{\frac{5}{6}\right\}^x = \frac{1}{2}$, taking the logarithms of this equation $x\{\log 5 - \log 6\} = -\log 2$, $\therefore x = \frac{\log 2}{\log 6 - \log 5} = \frac{\cdot 30103}{\cdot 07918} = 3{\cdot}82$ nearly; that is, the chance is greater than 3, but less than 4, and nearer 4 than 3.

In 4 throws there is also an equal chance of throwing any one of the other five faces of the die.

There is an even chance of throwing two aces with two dice when $\left(\frac{35}{36}\right)^x = \frac{1}{2}$, from which $x = 24{\cdot}605$ nearly.

9. First suppose one ace to be cast, then setting aside one die, there remain 9 dice, of each of which only 5 points must be taken.

The number of combinations of these will be 5^9, which when combined with all the aces in the 10 dice, will give 10.5^9; and the number of the combinations is 6^{10}.

$\therefore$ The probability of casting one ace is $\frac{10.5^9}{6^{10}} = \frac{19531250}{60466176} = \frac{1}{3}$ nearly.

Next if two aces be cast, then setting aside two dice there remain 8 dice, of each of which as before 5 points are to be taken.

The number of combinations of these will be 5^8, which when combined with all the aces of the 10 dice taken 2 at a time will give $\frac{10.9}{1.2}.5^8$, and the whole number of combinations is 6^{10}.

$\therefore$ The probability of casting two aces $= \frac{5.9.5^8}{6^{10}} = \frac{17578125}{60466176} = \frac{17}{60}$ nearly.

Thirdly, suppose three aces to be cast,

Then ${}^{10}C_3 = \frac{10.9.8}{1.2.3}$ the number of favourable chances of 3 aces with 10 dice, and 5^7 is the number of combinations of the other 5 points.

Therefore the probability of casting three aces is $\frac{10.9.8}{1.2.3}.\frac{5^7}{10^6} = \frac{9375000}{60466176} = \frac{2}{13}$ nearly.

10. First, the chance of the tetrahedron falling on some one of four faces marked 1, 2, 3, 4, is $\frac{4}{4} = 1$.

The chance of the cube falling on some one of four faces marked 1, 2, 3, 4, is $\frac{4}{6} = \frac{2}{3}$.

The chance of the octahedron falling on some one of four faces marked 1, 2, 3, 4, is $\frac{4}{8} = \frac{1}{2}$.

The chance of the dodecahedron falling on some one of four faces marked 1, 2, 3, 4, is $\frac{4}{12} = \frac{1}{3}$.

$\therefore$ The chance of all four falling on some one of four faces is $1\times\frac{2}{3}\times\frac{1}{2}\times\frac{1}{3}=\frac{1}{9}$.
The same result will appear from the consideration.
All the possible cases that can occur with the four polyhedrons is

$$4\times6\times8\times12=2304.$$

And the cases of the four specified numbers is $4\times4\times4\times4=256$.
$\therefore$ The probability required is $\frac{256}{2304}=\frac{1}{9}$.

III.

1. Here ${}^{11}C_7={}^{11}C_4=\frac{11.10.9.8}{1.2.3.4}$ = all the combinations of the black and white balls, 7 at a time; and ${}^5C_3=\frac{5.4.3}{1.2.3}$ = all the combinations of the 5 white balls, 3 at a time.

The chance required is $\frac{5.4.3}{1.2.3}\div\frac{11.10.9.8}{1.2.3.4}=\frac{10}{330}=\frac{1}{33}$.

2. Here ${}^{36}C_2=\frac{36.35}{1.2}=630$ the number of pairs which can be drawn.

Of these ${}^{10}C_2=\frac{10.9}{1.2}=45$ pairs of white balls. Also

${}^{12}C_2=\frac{12.11}{1.2}=66$ pairs of black balls.

${}^{14}C_2=\frac{14.13}{1.2}=91$ pairs of red balls.

$10\times12=120$ pairs, one white and one black.
$10\times14=140$ pairs, one white and one red.
$12\times14=168$ pairs, one black and one red.

And since there are 120 favourable chances out of 630 chances, the probability of success is $\frac{120}{630}=\frac{4}{21}$, and the probability of failure is $1-\frac{4}{21}=\frac{17}{21}$. Hence the odds against are as 17 to 4.

3. There are 8 red balls and 12 not red, or 8 events in favour out of 20. The chance of drawing a red ball is $\frac{8}{20}$ or $\frac{2}{5}$.

4. Here ${}^{12}C_4=\frac{12.11.10.9}{1.2.3.4}=495$, all the possible combinations favourable and unfavourable of the 12 things 4 at a time.

Next ${}^3C_1\,.\,{}^3C_1\,.\,{}^3C_1\,.\,{}^3C_1=3.3.3.3=81$, the number of favourable combinations of the 12 things 4 at a time.

$\therefore$ The probability required is $\frac{81}{495}=\frac{9}{55}$.

6. The separate probabilities are $\frac{7}{18}$, $\frac{9}{18}$, $\frac{10}{18}$ respectively.

$\therefore$ The probability in favour of the event is $\frac{7}{18}\times\frac{1}{2}\times\frac{5}{9}=\frac{35}{324}$; the probability against is $1-\frac{35}{324}=\frac{289}{324}$ and the odds against is 289 to 35, or greater than 8 to 1.

9. Here is a case of an event being dependent on another event. After the first exchange the urn *A* contains 1 black and 2 white balls; and *B*, 1 white and 2 black balls.

In the second exchange the probability of a white ball taken from *A* is $\frac{2}{3}$, also the probability of a black ball taken from B is also $\frac{2}{3}$: and the probability that both these events will happen is $\frac{2}{3}\times\frac{2}{3}=\frac{4}{9}$.

On this hypothesis, that A now contains 1 white and 2 black balls, and B contains 1 black and 2 white balls,

In the third exchange, the probability of a white ball being taken from A is $\frac{1}{3}$; and a black ball from B is also $\frac{1}{3}$: and the probability that both these events will happen is $\frac{1}{3}\times\frac{1}{3}=\frac{1}{9}$.

Now since the probability in the second exchange is $\frac{4}{9}$ that A contains 2 black and B 2 white balls. Also since the probability in the third exchange is $\frac{1}{9}$ that A contains 1 black and B 1 white ball,

Therefore the probability of 3 black balls in A and 3 white in B is $\frac{4}{9}\times\frac{1}{9}=\frac{4}{81}$

10. Here ${}^{26}C_{10}=\frac{26.25.24.23.22....17}{1.2.3.4....10}$ all the possible combinations of 26 balls, 10 at a time.

$$\text{And } {}^{8}C_1=\frac{8}{1} \text{ sets of blue balls, } {}^{5}C_2=\frac{5.4}{1.2} \text{ sets of white balls,}$$

$$ {}^{6}C_3=\frac{6.5.4}{1.2.3} \text{ sets of black balls, } {}^{7}C_4=\frac{7.6.5.4}{1.2.3.4} \text{ sets of red balls ;}$$

$$\therefore\ {}^{8}C_1.{}^{5}C_2.{}^{6}C_3.{}^{7}C_4=\frac{8}{1}.\frac{5.4}{1.2}.\frac{6.5.4}{1.2.3}.\frac{7.6.5.4}{1.2.3.4};$$

$$\therefore \text{ The chance is } \frac{8}{1}.\frac{5.4}{1.2}.\frac{6.5.4}{1.2.3}.\frac{7.6.5.4}{1.2.3.4}\div\frac{26.25....17}{1.2....10}=\frac{11200}{1062347}=\frac{7}{664} \text{ nearly.}$$

11. Let x denote the number of white balls, $x+11$ is the whole number of balls.

$$\text{Then } {}^{x+11}C_4=\frac{(x+11)(x+10)(x+9)(x+8)}{1.2.3.4}, \text{ and } {}^{11}C_4=\frac{11.10.9.8}{1.2.3.4}.$$

$\therefore\ \frac{{}^{11}C_4}{{}^{x+11}C_4}=\frac{11.10.9.8}{(x+11)(x+10)(x+9)(x+8)}$ is the probability that the 4 balls drawn are red.

$$\text{Next } {}^{11}C_2=\frac{11.10}{1.2}, \text{ and } {}^{x}C_2=\frac{x(x-1)}{1.2}.$$

$\therefore\ {}^{11}C_2.{}^{x}C_2=\frac{11.10}{1.2}.\frac{x(x-1)}{1.2}$ is the number of ways in which 4 balls can be drawn, 2 red and 2 white,

$$\text{and } \frac{{}^{11}C_2.{}^{x}C_2}{{}^{x+11}C_4}=\frac{11.10.2.3.x.(x-1)}{(x+11)(x+10)(x+9)(x+8)}.$$

$$\text{Hence } \frac{11.10.9.8}{(x+11)(x+10)(x+9)(x+8)}=\frac{11.10.2.3.x.(x-1)}{(x+11)(x+10)(x+9)(x+8)}$$

or $12=x(x-1)$, $x=4$ and -3. The number of white balls is 4.

12. The number 10 can be made up in four ways by taking three numbers—

1, 4, 5; 2, 3, 5; 3, 3, 4; 4, 4, 2.

$${}^{10}C_3=\frac{10.9.8}{1.2.3}=120,\ {}^{4}C_3=\frac{4.3.2}{1.2.3}=4,\ \therefore \text{ the chance is } \frac{4}{120}=\frac{1}{30}.$$

13. First find the number of combinations of all the 8 numbers taken 4 at a time. Next all the number of combinations of four of the numbers whose sum is 17, bearing in mind that every one of the 8 numbers is likely to be drawn each time.

IV.

1. Since *A*'s skill is to *B*'s as 3 to 1. This implies that *A* wins 3 games while *B* wins 1. $\therefore$ *A*'s chance of winning one game is $\frac{3}{4}$, and *B*'s $\frac{1}{4}$.

6. The chance of the correctness of *A*'s affirmation is $\frac{3}{4}$, and of his negation $\frac{1}{4}$.

And the chance of *B*'s affirmation $\frac{5}{6}$, of his negation is $\frac{1}{6}$.

$\therefore$ The chance of *A* and *B*'s affirmation is $\frac{3}{4}\times\frac{5}{6}=\frac{5}{8}$, and

of *A*'s and *B*'s negation is $\frac{1}{4}\times\frac{1}{6}=\frac{1}{24}$.

And these two agreements are exclusive.

$\therefore$ The chance of one or the other is $\frac{5}{8}+\frac{1}{24}=\frac{16}{24}$.

That is they agree in 16 out of 24 of their statements.

8. The chance of a white ball being drawn is $\frac{1}{9}$; of its not being drawn $\frac{8}{9}$.

The chance of *A* speaking truth is $\frac{3}{4}$, and of his failure $\frac{1}{4}$.

" " *B* " " $\frac{4}{5}$ " " $\frac{1}{5}$.

The chance of *A* and *B* both speaking truth is $\frac{3}{4}\times\frac{4}{5}=\frac{3}{5}$,

and the chance of their failure is $\frac{1}{4}\times\frac{1}{5}=\frac{1}{20}$.

$$\text{The chance required is } \frac{\frac{3}{5}\cdot\frac{1}{9}}{\frac{3}{5}\cdot\frac{1}{9}+\frac{1}{20}\cdot\frac{8}{9}}=\frac{3}{5}.$$

9. Here the chance of winning is equal to the chance of losing a game, and therefore is equal to $\frac{1}{2}$. And each of the three simple chances is $\frac{1}{2}$.

Therefore in each of the three cases the chance required is $\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$.

10. By reference to the general formula, Art. 5. $n=6$, $r=2$, and in this case the chance is $\dfrac{a^6+6a^5b+15a^4b^2}{(a+b)^6}$, where $a=1$ and $b=20$, its numerical value is $\dfrac{6121}{21^6}$.

11. First $^{100}C_5=\dfrac{100.99.98.97.96}{1.2.3.4.5}$, $^{5}C_2=\dfrac{5.4}{1.2}$, $^{95}C_3=\dfrac{95.94.93}{1.2.3}$.

$$\text{Then } {}^{5}C_2.{}^{95}C_3=\frac{95.94.93}{1.2.3}.\frac{5.4}{1.2}.$$

$$\text{Hence the chance required is } \frac{100.99.98.97.96}{1.2.3.4.5}\div\left\{\frac{95.94.93}{1.2.3}.\frac{5.4}{1.2}\right\}.$$

V.

1. There are $10+5+3+2+1=21$ coins, and two of them are to be drawn. The chance of any two being drawn is $\frac{2}{21}$, and the expectation is $\frac{2}{21}\times$(£10 14s. 1d.).

2. The chance for 4s. is $\frac{3}{14}$; for 8s. $\frac{6}{14}$, and for 5s. $\frac{5}{14}$.

$$\therefore \text{ The value of the expectation is } \frac{12+48+25}{14}=\frac{85}{14} \text{ shillings.}$$

4. Here $4+6=10$, and $\frac{1}{10}$ is the equal chance for drawing each prize.

$$\text{The value of the expectation is } \frac{3+5+8+10}{10}=\frac{26}{10}=2\tfrac{3}{5} \text{ shillings.}$$

5. The bag contains 9 coins; if 3 be drawn, the chance is $\frac{3}{9}$, or $\frac{1}{3}$ of the value of the coins; that is $\frac{1}{3}$ of 107 shillings, or £1 15s. 8d.

6. The price of a ticket will be the sum of the expectations for each prize.

There are 4 tickets out of 100 for a prize of £100;

$$\text{the chance of a prize of £100 is } \frac{4}{100} \text{ or } \frac{1}{25},$$

$$\text{and the value of the expectation of this prize is } \frac{£100}{25}=£4.$$

$$\text{Similarly, the value of the expectation of a prize of £50 is } \frac{£50}{10}=£5.$$

$$\text{And the value of the expectation of a prize of £5 is } \frac{£5}{5}=£1.$$

The sum of the expectations is £10, the value of a ticket.

ELEMENTARY ALGEBRA,
WITH BRIEF NOTICES OF ITS HISTORY.

SECTION XII.

CUBIC AND BIQUADRATIC EQUATIONS.

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CUBIC AND BIQUADRATIC EQUATIONS.

Art. 1. In the theory of equations are considered equations involving one unknown quantity, which is supposed to admit of as many values as the highest power of that quantity involved in the equation.*

The notations p, q, r, &c., and p_1, p_2, p_3, &c., employed as the coefficients of equations, are to be considered as denoting generally the second, third, &c., coefficients of the terms of an equation in order, and not any definite numbers.

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* When two equations are given involving two unknown quantities, either of the unknown quantities can be eliminated, and the resulting equation will involve only one unknown quantity, and the highest power involved in it will depend on the powers of the unknown quantities in the given equations.

In the same manner, if three equations be given involving three unknown quantities, any two of the unknown quantities can be eliminated, and an equation found involving only one of them. In like manner, if four equations be given with four unknown quantities, any three of them may be eliminated, and the resulting equation will involve only one of the unknown quantities.

This may be exemplified by the two following examples :—

1. Any two of the unknown quantities x, y, z can be eliminated between the equations $x+y+z=p$, $xy+xz+yz=q$, $xyz=r$, and the resulting equation will involve only one unknown quantity.

Let y and z be eliminated, then $y+z=p-x$, $yz=\frac{r}{x}$. Substitute these values of $y+z$ and yz in $x(y+z)+yz=q$, and $x(p-x)+\frac{r}{x}=q$, or $x^3-px^2+qx-r=0$, an equation which involves x only.

In the same manner, if x, z were eliminated, the resulting equation would be $y^3-py^2+qy-r=0$; and if x, y were eliminated, the result would be $z^3-pz^2+qz-r=0$. And as x, y, z are similarly involved in the three given equations, any one of them as x must represent all three quantities or in other words, x will have three values in the equation $x^3-px^2+qx-r=0$.

2. Any three of the unknown quantities x, y, z, v can be eliminated between the equations $x+y+z+v=p$, $xy+xz+xv+yz+yv+zv=q$, $xyz+xyv+xzv+yzv=r$, $xyzv=s$, and the resulting equation will involve only one unknown quantity.

Let y, z, v be eliminated,

Here $x^2+xy+xz+xv=px$, multiplying the first equation by x,
and $xy+xz+xv+yz+yv+zv=q$, the second equation
$-x^2+yz+yv+zv=q-px$, by subtraction.
$-x^3+xyz+xyv+xzv=qx-px^2$, multiplying by x,
$xyz+xyv+xzv+yzv=r$ the third equation,
$x^3+yzv=r-qx+px^2$ by subtraction.
$x^4+xyzv=rx-qx^2+px^3$ multiplying by x.
$xyzv=s$ the fourth equation,
$x^4=px^3-qx^2+rx-s$ by subtraction,

or $x^4-px^3+qx^2-rx+s=0$, is the resulting equation involving x only. By the same process may be eliminated also x, z, v ; x, y, v ; x, y, z, respectively; and the resulting equations will be of the same form. And since the four quantities x, y, z, v are similarly involved in the four given equations, any one of them as x must represent all the four quantities, or x will have four values in the equation $x^4-px^3+qx^2-rx+s=0$.

By the root of an equation is meant any number which when substituted for the unknown quantity satisfies or fulfils the conditions of the equation, and by the solution is meant the determination of all the roots.

The coefficients of the terms of an equation are supposed to be positive or negative numbers not excluding zero.

2. Equations have been classed and named according to the highest power of the unknown quantity involved in them.

Simple equations have been named equations of the first degree as involving only the first power of the unknown quantity, and quadratic equations have been designated equations of the second degree as involving the second power, with or without the first power of the unknown quantity. In like manner cubic and biquadratic equations have been named equations of the third and fourth degree respectively, as involving the third and fourth powers, with or without lower powers of the unknown quantity. It will be seen that equations of the first and second degree will be found to be auxiliary in the solution of equations of the third and fourth degree.*

It has been already shewn that every simple equation is reducible to the form $x-a=0$, and every complete quadratic equation to the form $x^2-p_1x+p_2=0$; and if a, b be the two roots or values of x, the unknown quantity in this equation, then $x=a$, $x=b$ and $x-a=0$,

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* In 1823, Mr. Buck, of Queen's College, published a method of solving algebraical equations by reducing them to the ordinary forms of quadratics. His method of solution for equations of the third and fourth degree will be understood from examples.

1. An example of an equation of the third degree, p. 10 :—

Let $x^3-6x^2-4x-21=0$, $x^2-6x-4=\frac{21}{x}$, let $w=\frac{21}{x}$, then $wx=21$.

and $x^2-6x+9=w+13$, $\therefore$ $x=3\pm(w+13)^{\frac{1}{2}}$. If $w=3$, then $(w+13)^{\frac{1}{2}}=4$, and $x=3\pm4=7$ and -1; but $wx=21$, and $\therefore$ 7 is a value of x which satisfies the equation.

2. An example of an equation of the fourth degree, p. 33 :—

Let $x^4-7x^3+9x^2+8x+16=0$, $x^2-7x+9=-\left(\frac{8}{x}+\frac{16}{x^2}\right)$, let $\frac{8}{x}+\frac{16}{x^2}=w$,

$$\text{then } x^2-7x+9=-w, \text{ and } x=\frac{7}{2}\pm\frac{(13-4w)^{\frac{1}{2}}}{2}. \quad (1)$$

$$\text{But } w=\frac{8}{x}+\frac{16}{x^2}, \;\therefore\; wx^2-8x=16, \text{ and } x=\frac{4}{w}\pm\frac{4(w+1)^{\frac{1}{2}}}{w}. \quad (2)$$

$$\text{If } w=3, \text{ then from (1) } x=\frac{7}{2}\pm\frac{1}{2}=4 \text{ and } 3,$$

$$\text{from (2) } x=\frac{4}{3}\pm\frac{8}{3}=4 \text{ and } \frac{-4}{3}.$$

The value $x=4$ satisfies the equation.

It will be seen that the value of w is to be found *by trial.*

$x-b=0$, and therefore $(x-a)(x-b)=0$, or $x^2-(a+b)x+ab=0$ is an equation identical with the equation $x^2-p_1x+p_2=0$,

$$\text{and } x^2-p_1x+p_2=(x-a)(x-b)=x^2-(a+b)x+ab,$$

$$\text{and } \therefore\ -p_1=-(a+b),\ p_2=ab.$$

In the same manner a cubic equation may be formed of three simple equations. If a, b, c be the roots of the cubic equation $x^3-p_1x^2+p_2x-p_3=0$, $x=a$, $x=b$, $x=c$, and $x-a=0$, $x-b=0$, $x-c=0$,

$$\text{and therefore } (x-a)(x-b)(x-c)=0$$

$$\text{or } x^3-(a+b+c)x^2+(ab+ac+bc)x-abc=0$$

is an equation identical with the equation $x^3-p_1x^2+p_2x-p_3=0$,

$$\text{and } \therefore\ -p_1=-(a+b+c),\ p_2=ab+ac+bc,\ -p_3=-abc.$$

In a similar manner, if a, b, c, d denote the roots of the biquadratic equation $x^4-p_1x^3+p_2x^2-p_3x+p_4=0$, then $(x-a)(x-b)(x-c)(x-d)=0$,

$$\text{or } x^4-(a+b+c+d)x^3+(ab+ac+ad+bc+bd+cd)x^2$$
$$-(abc+abd+acd+bcd)x+abcd=0$$

is an equation identical with $x^4-p_1x^3+p_2x^2-p_3x+p_4=0$,

$$\therefore\ -p_1=-(a+b+c+d),$$
$$+p_2=ab+ac+ad+bc+bd+cd=ab+(a+b)c+(a+b+c)d,$$
$$-p_3=-(abc+abd+acd+bcd)=-\{(a+b)cd+(c+d)ab\}$$
$$=-\{abc+(ab+ac+bc)d\},$$
$$+p_4=abcd.$$

Hence the following relations exist between the roots and the coefficients of an equation, when the coefficient of the highest power of the unknown quantity is unity.

The coefficient of the second term of an equation with its proper sign is equal to the algebraical sum of the roots with their signs changed.

The coefficient of the third term is equal to the algebraical sum of the products of every two roots with their signs changed.

The coefficient of the fourth term is equal to the algebraical sum of the product of every three roots with their signs changed, &c. And the last term is equal to the product of all the roots with their signs changed.

Hence also it appears that if the roots of an equation be all positive, the signs of the coefficients of the terms are alternately positive and negative; and if the roots be all negative, the signs of the coefficients may be shewn to be all positive. Hence it is evident that if the signs of the alternate terms of an equation be changed, beginning with the second, all the signs of the roots are changed.*

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* If $+1$, $+2$, $+3$ be the roots of an equation, then $x=1$, $x=2$, $x=3$, and $x-1=0$, $x-2=0$, $x-3=0$, $\therefore$ $(x-1)(x-2)(x-3)=0$, or $x^3-6x^2+11x-6=0$, an equation whose coefficients are alternately *positive* and *negative*.

If -1, -2, -3 be roots of an equation, then $x=-1$, $x=-2$, $x=-3$, and $x+1=0$, $x+2=0$, $x+3=0$, and $\therefore$ $(x+1)(x+2)(x+3)=x^3+6x^2+11x+6=0$, an equation which has all the signs of its coefficients *positive*.

And obviously if the signs of the coefficients of the alternate terms of

3. A cubic equation $x^3-p_1x^2+p_2x-p_3=0$, whose roots are a, b, c, may be considered to be composed of a simple and a quadratic equation; for if a, b be the roots of $x^2-p_1x+p_2=0$, then $(x-a)(x-b)=0$ is an equation identical with $x^2-p_1x+p_2=0$, and if $x=c$, $x-c=0$, and therefore $(x-a)(x-b)(x-c)=0$ is an equation identical with $(x^2-p_1x+p_2)(x-c)=0$. Hence $(x^2-p_1x+p_2)(x-c)=0$ is an equation identical with $x^3-p_1x^2+p_2x-p_3=0$.

A biquadratic equation may consist of a cubic equation and a simple equation; for if a, b, c, d be the roots of the biquadratic equation,

$x^4-p_1x^3+p_2x^2-p_3x+p_4=0$, then $(x-a)(x-b)(x-c)(x-d)=0$.

But it has been shewn that if a, b, c be the roots of

$x^3-p_1x^2+p_2x-p_3=0$, then $x^3-p_1x^2+p_2x-p_3=0$, and $(x-a)(x-b)(x-c)=0$ are identical equations.

If $x=d$, $x-d=0$, then $(x-d)(x^3-p_1x^2+p_2x-p_3)=0$,

and $(x-a)(x-b)(x-c)(x-d)=0$ are identical equations.

Hence it follows that $(x-d)(x^3-p_1x^2+p_2x-p_3)=0$,

and $x^4-p_1x^3+p_2x^2-p_3x+p_4=0$ are identical equations.

A biquadratic equation may also consist of two quadratic equations; for if a, b denote the roots of $x^2-p_1x+p_2=0$, and c, d the roots of $x^2-q_1x+q_2=0$, then $(x^2-p_1x+p_2)(x^2-q_1x+q_2)=0$ is an equation identical with $(x-a)(x-b)(x-c)(x-d)=0$, and if a, b, c, d be the roots of $x^4-p_1x^3+p_2x^2-p_3x+p_4=0$; it follows that $(x^2-p_1x+p_2)(x^2-q_1x+q_2)=0$ and $x^4-p_1x^3+p_2x^2-p_3x+p_4=0$ are identical equations.

4. As an equation can be formed when the roots are given, so also an equation can be formed when the successive coefficients of the descending powers of the unknown quantity are given.

Let 1, $-p_1$, $+p_2$, $-p_3$ be the successive coefficients of the terms of a cubic equation.

First, add the second coefficient $-p_1$ with its proper sign to x, and multiply the algebraic sum by x, then $x(x-p_1)=x^2-p_1x$.

Next, add the third coefficient $+p_2$ to this product, and multiply the sum by x, then $x(x^2-p_1x+p_2)=x^3-p_1x^2+p_2x$.

Thirdly, add the fourth coefficient $-p_3$ to this last product, and make the aggregate equal to zero, and $x^3-p_1x^2+p_2x-p_3=0$, is a cubic equation, which has its coefficients p_1, p_2, p_3 numerical.

In this equation it is not known whether the coefficients p_1, p_2, p_3 are or are not directly subject to the same conditions as when the roots are

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$x^3-6x^2+11x-6=0$, be changed, beginning with the second, the equation becomes $x^3+6x^2+11x+6=0$ an equation whose roots are *negative*.

And in a similar manner, if 1, 2, 3, 4 be the positive roots of a biquadratic equation, the equation is $x^4-10x^3+35x^2-50x+24=0$, and if the alternate signs of the terms be changed, beginning with the second, the roots of the biquadratic equation $x^4+10x^3+35x^2+50x+24=0$ are all positive.

If zero be a root of an equation, it will assume the form $x^4-p_1x^3+p_2x^2-p_3x=0$.

known to be definite integral numbers, or in other words, whether the roots be rational, irrational, or impossible.

And the biquadratic equation $x^4-p_1x^3+p_2x^2-p_3x+p_4=0$ may be formed in the same manner, in which the coefficients 1, $-p_1$, $+p_2$, $-p_3$, $+p^4$ are any numbers whatever, subject or not subject to the relations which subsist between the coefficients and roots of the equation, when the roots are rational.

Hence in general it appears that the direct proposition, "To form an equation either with given roots or with given coefficients," is always possible; but the reverse process, "To determine the roots of a given equation in terms of the coefficients," is not so simple a process, but involves difficulties of such a nature as have not yet been fully overcome.*

A general formula has been found for the solution of quadratic equations, and of such as can be reduced to the form of quadratics; but no general formula has yet been discovered for the solution of cubic and biquadratic equations, or equations of a higher degree. The formulæ which have been devised for the solution of equations of the third and fourth degree are both limited and restricted in their application.

The relations between the roots and the coefficients of an equation are true, not only for equations of the second, third, and fourth degree, but also for equations of higher degrees. In cases where the roots are so related that terms are wanting in an equation, for every term so wanting there arises one additional relation between the roots and coefficients of the equation. These relations may be such as to afford facilities for the solution of the equation. As for instance, if the roots of the biquadratic equation $x^4-p_1x^3+p_2x^2-p_3x+p_4=0$ be so related that p_1, p_3, the coefficients of the second and fourth terms of the equation, are zero, the equation becomes $x^4+p_2x^2+p_4=0$, which can be solved as a quadratic, and its four roots determined.

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* "The solutions of cubic and biquadratic equations have been known for nearly three centuries; and all the attempts which have hitherto been made to proceed beyond them, at least in equations in which there exists no relation of dependence amongst the several coefficients, and no presumed or presumable relation amongst the roots, have altogether failed of success; and if we consider that this great problem has been subjected to the most scrutinising and laborious examination by nearly all the greatest analysts who have lived in that period, we may be justified in concluding that this failure is rather to be attributed to the essential impossibility of the problem itself, than to the want of skill or perseverance on the part of those who have made the attempt. But in the absence of any complete and uncontrovertible proof of this impossibility, the question cannot be considered as concluded, and will still remain open to speculations upon the part of those with whom extensive and well-matured knowledge, and a deep conviction founded upon it, have not altogether extinguished hope."—*Report of the British Association for the Advancement of Science*, 1833, *pp.* 306, 307.

There are also some forms of equations higher than the fourth degree, which can be reduced so as to admit of solution by means of equations not higher than the fourth degree.

As for instance, every complete equation of the fifth degree, like every equation of the third degree, must have one root either integral or irrational; and when this root has been found exactly or approximately, the equation can be reduced to one of the fourth degree.

An equation of the third degree may have three integral or three irrational roots; one integral and two irrational; or one integral or irrational and two impossible. In cases where the equation contains one integral root, this root can always be found, and the equation depressed to a quadratic which will give the other two roots.

An equation of the fourth degree may have its four roots all integral, all irrational, or all impossible; or it may have two roots integral and two roots irrational; or two roots integral or irrational and two roots impossible.

When the equation contains two integral roots, these can be found, and the equation depressed to a quadratic, which will contain the other two roots, which will be both irrational or both impossible.

When the equation contains monomial roots, irrational or impossible, these can be detected, and the equation depressed.

If one root of a cubic equation be known, or can in any way be determined, the equation may be depressed to a quadratic, from which the remaining two roots can be found.

In the same manner, if one root of a biquadratic equation be known, the biquadratic can be reduced to a cubic equation, and if two roots be known, it can be reduced to a quadratic equation.

In any given equation, all or some of the roots may be rational or irrational, and positive or negative, or impossible.

If the integral root or roots can by any process be determined, the given equation can be depressed to one of lower dimensions, and this depressed equation can involve only irrational or impossible roots, or some irrational and some impossible. If the limits of the irrational roots can be determined, then approximations to the values of these roots can be found to any required degree of accuracy; and the equation, when depressed, will contain only impossible roots: or if the impossible roots can be detected, the equation, when depressed, will contain only irrational roots.

5. *To explain the method of finding the integral roots of a cubic and of a biquadratic equation.*

Let $x^3-p_1x^2+p_2x-p_3=0$ be an equation of the third degree.

If this equation contain one or more integral roots, they must be factors of the last term.

Suppose a is an integral root of this equation,

Then $a^3-p_1a^2+p_2a-p_3=0$,

$$\therefore\ \frac{p_3}{a} = a^2 - p_1 a + p_2, \text{ let } \frac{p_3}{a} = Q_1$$

$$\therefore\ Q_1 - p_2 = a^2 - p_1 a,$$

$$\frac{Q_1 - p_2}{a} = a - p_1, \text{ let } \frac{Q_1 - p_2}{a} = Q_2,$$

$$\therefore\ Q_2 + p_1 = +a,$$

$$\text{and } \frac{Q_2 + p_1}{a} = +1.$$

From this process it will appear, in order that a may be an integral root of the cubic equation $x^3 - p_1 x^2 + p_2 x - p_3 = 0$, the final result will be $+1$.*

The labour of making trial of all the divisors of the last term of the equation may be considerably lessened, by depressing the equation when $+1$ or -1 are among the roots, and by excluding all divisors of the last term, which are greater than the greatest positive root, and less than the least negative root, and trying only the divisors which are included within these limits.

The same process with these limitations is applicable for finding the integral roots of equations of the fourth degree.

6. ***When the coefficients of equations are rational, impossible roots enter these equations in conjugate pairs.***

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* Determine if the cubic equation $x^3 - x^2 - 20x - 24 = 0$ has integral roots.

If a be a root of this equation, it must be a divisor of the last term, and may be ± 2, ± 3, ± 4, ± 6, ± 12; 1 is obviously not a root.

Then $a^3 - a^2 - 20a - 24 = 0$

$$\frac{24}{a} = a^2 - a - 20, \text{ but } \frac{24}{a} = Q_1$$

$$\frac{Q_1 + 20}{a} = a - 1, \text{ but } \frac{Q_1 + 20}{a} = Q_2$$

$$\frac{Q_2 + 1}{a} = +1.$$

Try $\pm a = \pm$	2	± 3	± 4	± 6	± 12
then $\frac{24}{a} =$	± 12	± 8	± 6	± 4	± 2
$Q_1 + 20 =$	32 or 8	28 or 12	26 or 14	24 or 16	22 or 18
$\frac{Q_1 + 20}{a} =$	16 or -4	,, -4	,,	4 ,,	,,
$Q_2 + 1 =$	17 or -3	,, -3	,,	5 ,,	,,
$\frac{Q_2 + 1}{a} =$	,,	,, $+1$	,,	,, ,,	,,

Here -3 makes the final result of $\frac{Q_2 + 1}{a}$ equal to $+1$, therefore -3 is the only integral root of the equation, and by substituting -3 for x the equation is satisfied.

To find the other two roots, $x = -3$, and $x + 3 = 0$,

$\frac{x^3 - x^2 - 20x - 24}{x + 3} = 0$, or $x^2 - 4x - 8 = 0$, which gives $x = 2 \pm 2\sqrt{3}$, the other two roots of the equation.

If $a+b\sqrt{-1}$ be a root of the cubic equation $x^3-p_1x^2+p_2x-p_3=0$; then $a-b\sqrt{-1}$ shall also be a root.

Since $x=a+b\sqrt{-1}$, then substituting this value of x in the equation, it becomes $(a+b\sqrt{-1})^3-p_1(a+b\sqrt{-1})^2+p_2(a+b\sqrt{-1})-p_3=0$,

or $(a^3-p_1a^2+p_2a-p_3-3ab^2+p_1b^2)+(3a^2b-2p_1ab+p_2b-b^3)\sqrt{-1}=0$, an equation consisting partly of possible, and partly of impossible quantities.

These parts are each equal to zero,

$$\therefore\ a^3-p_1a^2+p_2a-p_3-3ab^2+p_1b^2=0,$$

$$\text{and } (3a^2b-2p_1ab+p_2b-b^3)\sqrt{-1}=0;$$

$\therefore$ By subtraction,

$$(a^3-p_1a^2+p_2a-p_3-3ab^2+p_1b^2)-(3a^2b-2p_1ab+p_2b-b^3)\sqrt{-1}=0,$$

$$\text{or } (a-b\sqrt{-1})^3-p_1(a-b\sqrt{-1})^2+p_2(a-b\sqrt{-1})-p_3=0,$$

$\therefore$ $a-b\sqrt{-1}$ satisfies the equation, and therefore is also a root.

By a similar process it may be shewn that surd roots enter cubic equations in conjugate pairs: and in the same manner an equation of the fourth degree may be shewn to have one pair or two pairs of impossible or surd roots.*

If an equation be known to have one impossible or surd root, a second impossible or surd is also known, and if the given equation be a cubic, it can be depressed to a simple equation; and if it be a biquadratic equation it can be depressed to a quadratic.

A cubic equation can have only one pair of conjugate roots, and one real root positive or negative. A cubic equation also may contain three real roots, all positive or all negative; or it may contain two positive and one negative, or one positive and two negative.

A biquadratic equation may contain two pairs of impossible roots, or one pair of impossible roots and two real roots both positive or both negative, or one positive and the other negative. It may have its four roots real, all positive or all negative, or two positive and two negative, or three positive and one negative, or three negative and one positive.

If all the four roots of a biquadratic equation be impossible, then whatever numbers, positive or negative, be substituted for the unknown quantity, the results will always be affected with the same sign; for if the results gave different signs, one root at least would lie between the two results, one negative and the other positive.

As impossible roots always enter equations in conjugate pairs of the forms $a\pm b\sqrt{-1}$ or $\pm b\sqrt{-1}$, as also surd roots of the forms $a\pm\sqrt{b}$ or $\pm\sqrt{b}$; the product of a pair of conjugate roots of the forms $a\pm b\sqrt{-1}$ or $\pm b\sqrt{-1}$ is always positive; the product of a pair of roots of the form $a \pm \sqrt{b}$ may be positive or negative, and the product of a pair of the form $\pm\sqrt{b}$ is always negative.

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* The determination of the pairs of impossible roots of the form $a\pm b\sqrt{-1}$ in any equation is a problem of considerable difficulty. The theorem of Sturm for determining the number and situation of the real roots of an equation, may be deferred, as also the researches of Budan and Fourier.

If a cubic or biquadratic equation contain a pair of roots of the form $\pm b\sqrt{-1}$ they enter as two positive or two negative roots; and when such roots exist in an equation they can be detected, as also a pair of roots of the form $\pm\sqrt{b}$, and the equation depressed to one of lower dimensions. But the method of detecting pairs of roots of the forms $\pm b\sqrt{-1}$ and $\pm\sqrt{b}$ does not apply for detecting pairs of roots of the form $a \pm b\sqrt{-1}$ or $a \pm \sqrt{b}$.*

If, however, an equation involving a pair of conjugate roots of the form $a \pm b\sqrt{-1}$ or $a \pm \sqrt{b}$, can be transformed into one whose pairs of roots are of the form $\pm b\sqrt{-1}$ or $\pm\sqrt{b}$, such roots can then be detected.†

7. *To transform an equation into one whose roots shall be the reciprocals of the roots of the given equation.*

Let $x^3 - p_1x^2 + p_2x - p_3 = 0$ be a cubic equation whose roots are a, b, c; it is required to transform it to an equation whose roots shall be $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$.

In the given equation for x substitute $\frac{1}{x}$, then the given equation

becomes $\frac{1}{x^3} - \frac{p_1}{x^3} + \frac{p^2}{x} - p_3 = 0$, or $1 - p_1x + p_2x^2 - p_3x^3 = 0$,

or $x^3 - \frac{p_2x^2}{p_3} + \frac{p_1x}{p_3} - \frac{1}{p_3} = 0$, an equation whose roots are $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$.

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* If the equation $x^4 - 7x^3 + 10x^2 + 14x - 24 = 0$ contains two roots of the form $\pm\sqrt{b}$,
then $\underline{x^4 + 7x^3 + 10x^2 - 14x - 24 = 0}$ „ „ $\mp\sqrt{b}$,
$-14x^3 + 28x = 0$, and $-x^2 + 2 = 0$, $x^2 = 2$, $x = \pm\sqrt{2}$, are the two conjugate roots, and $\frac{x^4 - 7x^3 + 10x^2 + 14x - 24}{x^2 - 2} = 0$, or $x^2 - 7x + 12 = 0$ contains the other two roots, $x = 3$, $x = 4$. The truth of this can be verified.

† For example, the roots of the equation $x^4 - x^3 - 9x^2 - 5x + 2 = 0$ are -1, -2, $2 \pm \sqrt{3}$.

Since $x^4 - x^3 - 9x^2 - 5x + 2 = 0$ has roots -1, -2, $2 \pm \sqrt{3}$
$\underline{x^4 + x^3 - 9x^2 + 5x + 2 = 0 \quad „ \quad 1, \quad 2, \quad -2 \mp \sqrt{3}}$
$-2x^3 - 10x = 0$, and $x^2 + 5 = 0$, $x = \pm\sqrt{-5}$ which fails to give $2 \pm \sqrt{8}$, the two correct roots.

The equation $x^4 + x^3 + 11x^2 + 9x + 18 = 0$ involves two roots of the form $\pm b\sqrt{-1}$,
then $\underline{x^4 - x^3 + 11x^2 - 9x + 18 = 0}$ „ „ $\mp b\sqrt{-1}$,
$2x^3 + 9x = 0$, and $x^2 + 9 = 0$, $x = \pm 3\sqrt{-1}$,
$\therefore \pm 3\sqrt{-1}$ are two roots of the equation, and this may be verified,
$x = +3\sqrt{-1}$, $x = -3\sqrt{-1}$, then $x - 3\sqrt{-1} = 0$, $x + 3\sqrt{-1} = 0$,
$\therefore (x - 3\sqrt{-1})(x + 3\sqrt{-1}) = 0$, or $x^2 + 9 = 0$,
$\therefore \frac{x^4 + x^3 + 11x^2 + 9x + 18}{x^2 + 9} = 0$, or $x^2 + x + 2 = 0$,
and $x = \pm\frac{1}{2}(-1 \pm \sqrt{-7})$ are the other two roots.

By the same process a biquadratic equation may be transformed.[*] If a, b, c, d denote the roots of the equation

$$x^4-p_1x^3+p_2x^2-p_3x+p_4=0,$$

$$\text{then } x^4-\frac{p_1x^3}{p_4}+\frac{p_2x^2}{p_4}-\frac{p_3x}{p_4}+\frac{1}{p_4}=0 \text{ is the equation}$$

$$\text{whose roots are } \frac{1}{a},\ \frac{1}{b},\ \frac{1}{c},\ \frac{1}{d}.$$

If the given equation be of the form $x^3-px^2+px-1=0$, let $\frac{1}{x}$ be

substituted for x, then $\frac{1}{x^3}-\frac{p}{x^2}+\frac{p}{x}-1=0$, or $1-px+px^2-x^3$,

$$\text{or } x^3-px^2+px^2-1=0.$$

The transformed equation is identical with the given equation, and the roots of the transformed equation are the same as the roots of the given equation.

Also the biquadratic equation $x^4-px^3+qx^2-px+1=0$, when transformed into one whose roots are the reciprocals of the roots, will be of the same form as the given equation.

When the signs and coefficients of an equation taken in a direct and inverted order are the same, the equation is named a reciprocal or recurring equation.

8. *To transform an equation into another whose roots shall be respectively any multiples or submultiples of the roots of the given equation.*

Let $x^3-p_1x^2+p_2x-p_3=0$ be a cubic equation, whose roots are a, b, c, it is required to transform it into an equation whose roots shall be ma, mb, mc.

Assume $y=mx$, then $x=\frac{y}{m}$, substitute this value of x in the given equation, $\therefore \frac{y^3}{m^3}-\frac{p_1y^2}{m^2}+\frac{p_2y}{m}-p_3=0$, or $y^3-p_1my^2+p_2m^2y-m^3p_3=0$, is the equation whose roots are ma, mb, mc.

Next, let $y=\frac{x}{m}$, then $x=my$. Substitute in the given equation, $\therefore m^3y^3-p_1m^2y^2+p_2my-p_3=0$, or $y^3-\frac{p_1y^2}{m}+\frac{p_2y}{m^2}-\frac{p_3}{m^3}=0$ is the equation whose roots are $\frac{a}{m}$, $\frac{b}{m}$, $\frac{c}{m}$.[†]

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[*] By means of this transformation an equation can be found of which the greatest and least roots are respectively the least and greatest roots of the given equation.

[†] For instance, the equation $x^3-6x^2+11x-6=0$, roots 1, 2, 3, is transformed into $x^3-18x^2+99x-162=0$, roots 3, 6, 9, by *multiplying* the second, third, and fourth terms respectively by 3, 3^2, 3^3. And conversely, the equation $x^3-18x^2+99x-162=0$, roots 3, 6, 9, is transformed to $x^3-6x^2+11x-6=0$, whose

By the same process a biquadratic equation may be transformed.

If a, b, c, d denote the roots of the equation

$$x^4-p_1x^3+p_2x^2-p_3x+p_4=0,$$

then $y^4-p_1my^3+p_2m^2y^2-p_3m^3y+m^4p_4=0$ is the equation whose roots are ma, mb, mc, md.

And $y^4-\frac{p_1y^3}{m}+\frac{p_2y^2}{m^2}-\frac{p_3y}{m^3}+\frac{p_4}{m^4}=0$ is the equation whose roots are $\frac{a}{m}$, $\frac{b}{m}$, $\frac{c}{m}$, $\frac{d}{m}$.

9. *To transform equations of the third and fourth degree into others whose roots shall be* (1) *the squares,* (2) *the cubes of the roots of the given equations.*

(1) Let a, b, c be the roots of the equation $x^3-p_1x^2+p_2x-p_3=0$, then $x^3+p_2x=p_1x^2+p_3$,

squaring each side, $x^6+2p_2x^4+p_2^2x^2=p_1^2x^4+2p_1p_3x^2+p_3^2$

and $x^6-(p_1^2-2p_2)x^4+(p_2^2-2p_1p_3)x^2-p_3^2=0$.

Let $y=x^2$, then $y^3-(p_1^2-2p_2)y^2+(p_2^2-2p_1p_3)y-p_3^2=0$, is the equation whose roots are a^2, b^2, c^2.[*]

(2) Let $y=x^3$, then $x=y^{\frac{1}{3}}$, substituting this value of x in the given equation, it becomes $y-p_1y^{\frac{2}{3}}+p_2y^{\frac{1}{3}}-p_3=0$, and $y-p_3=p_1y^{\frac{2}{3}}-p_2y^{\frac{1}{3}}$, raising each side of this equation to the third power,

$$(y-p_3)^3=p_1^3y^2-p_2^3y-3p_1p_2y(p_1y^{\frac{2}{3}}-p_2y^{\frac{1}{3}})=p_1^3y^2-p_2^3y-3p_1p_2y(y-p_3).$$

Hence $y^3-(p_1^3-3p_1p_2+3p_3)y^2+(p_2^3-3p_1p_2p_3+3p_3^2)y-p_3^3=0$ is the equation whose roots are a^3, b^3, c^3.

(1) Next, let a, b, c, d be the roots of the equation $x^4-p_1x^3+p_2x^2-p_3x+p_4=0$, then $x^4+p_2x^2+p_4=p_1x^3+p_3x$. Square each side of this equation and arrange the terms in order on one side, and

$$x^8-(p_1^2-2p_2)x^6+(p_2^2-2p_1p_3+2p_4)x^4-(p_3^2-2p_2p_4)x^2+p_4^2=0.$$

Let $x^2=y$,

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roots 1, 2, 3, by *dividing* the second, third, and fourth terms by 3, 3^2, 3^3. When the coefficient of the highest power of the unknown quantity in an equation is not unity, or if any of the coefficients be fractional; by means of this transformation the equation may be reduced to another in which the coefficient of the highest power of the unknown quantity shall be unity, and all the coefficients integral.

Also an equation whose coefficients are large numbers may be reduced to one whose coefficients shall be smaller numbers. As, for example, if $x=100y$, the equation $x^3-20000x-5000000=0$ is reduced to $y^3-2y-5=0$.

[*] Since the square of every quantity, positive or negative, is positive; if a complete cubic equation have its roots real, positive or negative, the transformed equation will have all its roots positive, and consequently the signs of its terms will be alternately positive and negative. If therefore the transformed equation have not the signs of the alternate terms positive and negative, the proposed equation must involve two impossible roots. The equation $x^3+x^2-22x-40=0$ has its roots -2, -4, $+5$; when transformed into an equation whose roots are $(-2)^2$, $(-4)^2$, $(+5)^2$, it becomes $y^3-43y^2+128y-160=0$. But the equation $x^3-9x^2+33x-65=0$, whose roots are, $+5$, $2+3\surd-1$, $2-3\surd-1$, when transformed into one whose roots are $(+5)^2$, $(2+3\surd-1)^2$, $(2-3\surd-1)^2$ becomes $y^3-15y^2-81y-4225=0$.

then $y^4-(p_1^2-2p_2)y^3+(p_2^2-2p_1p_3+2p_4)y^2-(p_3^2-2p_2p_4)y+p_4^2=0$ is the equation whose roots are a^2, b^2, c^2, d^2.*

10. *To transform an equation into another whose roots shall be respectively greater or less than the roots of the given equation by any given quantity.*

Let $x^3-p_1x^2+p_2x-p_3=0$, the given equation whose roots are a, b, c, be transformed into one whose roots shall be $a+e$, $b+e$, $c+e$.

Assume $y=x+e$, then $x=y-e$.

Substitute this value of x in the given equation, and it becomes $(y-e)^3-p_1(y-e)^2+p_2(y-e)-p_3=0$, the terms of which when expanded and arranged give

$$y^3-(p_1+3e)y^2+(p_2+2p_1e+3e^2)y-(e^3+p_1e^2+p_2e+p_3)=0,$$

the equation whose roots are respectively $a+e$, $b+e$, $c+e$.

By the same process the equation $x^3-p_1x^2+p_2x-p_3=0$ can be transformed into one whose roots shall be $a-e$, $b-e$, $c-e$.

Assume $y=x-e$, then $x=y+e$ and the transformed equation $y^3-(p_1-3e)y^2+(p_2-2p_1e+3e^2)y-(e^3-p_1e^2+p_2e-p_3)=0$ is the equation whose roots are respectively $a-e$, $b-e$, $c-e$.

In the latter transformed cubic may be noted—

1. The expression $e^3-p_1e^2+p_2e-p_3$ is identical with the terms of the given equation, having e in the place of x, and constitutes the last term of the transformed equation.

2. The coefficient of the last term but one $3e^2-2p_1e+p_2$ is formed from $e^3-p_1e^2+p_2e-p_3$ by multiplying each term by the index of the power of e and reducing the index by unity.

3. The coefficient of the last term but two $3e-p_1$ is formed by multiplying each term of $3e^2-2p_1e+p_2$ by the index of e, reducing the index by unity and dividing the result by 2. †

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* The transformation of a complete biquadratic equation into another whose roots shall be respectively the cubes of the roots of the proposed equation, may be more conveniently effected by determining in succession the coefficients of the required equation in terms of the coefficients of the proposed equation.

† By means of this transformation an equation may be transformed into another which shall want the second or any other term, by assuming the coefficient of this term equal to zero, and finding the value of the quantity e by which the roots of the given equation are to be respectively increased or diminished.

In the case of the cubic equation $x^3-p_1x^2+p_2x-p_3=0$, the second term disappears from the transformed equation when $p_1\pm3e=0$.

The third term disappears when $3e^2\pm2ep_1+p_2=0$, and the fourth or last term when $e^3\pm p_1e^2+p_2e\pm p_3=0$, or when e is equal to a root of the given equation.

When the second and third terms disappear by the same transformation, then $p_1\pm3e=0$, and $3e^2\pm2p_1e+p_2=0$; by eliminating e, then $p_1^2=3p_2$, which is the relation which *must* subsist between the coefficients of the second and third terms of the given cubic in order that both of these terms may disappear in the transformed equation.

By a similar process may be found the values of e, in order that the second or any other term of a transformed biquadratic equation may be made to disappear.

By a similar process it may be shewn that if a, b, c, d be the roots of the equation $x^4-p_1x^3+p_2x^2-p_3x+p_4=0$, that

then $y^4-(p_1+4e)y^3+(p_2+3p_1e+6e^2)y^2-(p_3+2p_2e+3p_1e^2+4e^3)y$
$+(p_4+p_3e+p_2e^2+p_1e^3+e^4)=0$

is the equation whose roots are $a+e$, $b+e$, $c+e$, $d+e$.

And that $y^4-(p_1-4e)y^3+(p_2-3p_1e+6e^2)y^2-(p_3-2p_2e+3p_1e^2-4e^3)y$
$+(e^4-p_1e^3+p_2e^2-p_3e+p_4)=0$

is the equation whose roots are $a-e$, $b-e$, $c-e$, $d-e$.

In the latter transformed biquadratic may also be noted,—

1. The expression $e^4-p_1e^3+p_2e^2-p_3e+p_4$ consists of similar terms to the proposed equation having e in the place of x.

2. The coefficient of the last term but one $4e^3-3p_1e^2+2p_2e-p_3$ is formed by multiplying each term of $e^4-pe^3+p_2e^2-p_3x+p_4$ by the index of e in that term, and reducing the index by unity.

3. The coefficient of the last term but two $6e^2-3p_1e+p_2$ is formed from $4e^3-3p_1e^2+2p_2e-p_3$ by multiplying every term by the index of e in that term, diminishing the index by unity and dividing the result by 2.

4. The coefficient of the last term but three $4e-p_1$ is formed from $6e^2-3p_1e+p^2$ by multiplying each term by the index of e, diminishing the index by unity and dividing the result by 3.

11. *If the signs of the terms of an equation whose roots are possible be arranged in order, there are at least as many changes of signs from + to —, and from — to +, as it has real and positive roots; and at least as many permanencies or continuations of the same sign, from + to +, and from — to —, as it has real and negative roots.*

The simple equation $x-a=0$ and $x+a=0$ have their signs + — and + + respectively.

The quadratic equation $x^2-p_1x+p_2=0$, having positive roots, has its signs + − +, which gives + —, — +, two changes.

Let the equation $x-c=0$, whose root is positive, be introduced into the quadratic equation, then

$$(x^2-p_1x+p_2)(x-c)=x^3-(p_1+c)x^2+(p_2+ap_1)x-ap_2=0,$$

or $x^3-p_1x^2+p_2x-p_3=0$, which has its signs + — + −, and gives + —, — +, + — ; ∴ when a positive root is introduced into a quadratic equation, one additional change of signs is introduced into the resulting cubic equation.

Next, let the equation $x-d=0$, whose root is positive, be introduced into the cubic equation $x^3-p_1x^2+p_2x-p_3=0$; it becomes

$$(x^3-p_1x^2+p_2x-p_3)(x-d)=x^4-(p_1+d)x^3+(dp_1+p_2)x^2-(dp_2+p_3)x$$
$$+dp_3=0,$$

or $x^4-p_1x^3+p_2x^2-p_3x+p_4=0$, whose signs + — + — + give
+ —, — +, + —, — +.

∴ When a positive root is introduced into a cubic equation, one additional change of signs is introduced into the resulting biquadratic equation.

In the same manner it may be shewn that when a negative root is introduced, it will be found that a permanence of the same sign $++$, or $--$, would be introduced into the resulting cubic or biquadratic equation.

Hence, conversely, if the signs of a cubic and biquadratic equation be arranged in order, there will be as many *positive* roots in the equation as there are changes of signs from $+$ to $-$, and from $-$ to $+$; and as many *negative* roots as there permanencies of the same sign $++$ and $--$, when all the roots are possible.

But if the equation contain impossible roots, every cubic and biquadratic equation has *at least* as many changes of signs from $+$ to $-$ and from $-$ to $+$ as it has real and positive roots, and *at least* as many continuations of the same signs from $+$ to $+$, and from $-$ to $-$, as it has real and negative roots.

But as every pair of impossible conjugate roots under this rule appear as two positive or two negative roots, the rule is not applicable for discovering pairs of conjugate roots of the form $a \pm b\sqrt{-1}$ in an equation.

If any coefficient of an equation be zero, the sign of that term may be considered positive or negative, as in the equations $x^3-p_1x^2+p_2x-p_3=0$, and $x^4-p_1x^3+p_2x^2-p_3x+p_4=0$, if the coefficients of the second terms be zero, the equations may be written $x^3\pm 0x^2+p_2x-p_3=0$, and $x^4\pm 0x^3+p_2x^2-p_3x+p_4=0$, and when the roots are possible, both signs $\pm$ exhibit the same number of changes and permanencies of signs.*

12. *To explain what is meant by the limits of the roots of an equation.*

Let the roots of the equation $x^3-p_1x^2+p_2x-p_3=0$ be a, b, c; a greater than b, and b greater than c;

Then $x^3-p_1x^2+p_2x-p_3=(x-a)(x-b)(x-c)$.

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* The truth of this may be illustrated in the following examples :—

1. The cubic equation $x^3-7x+6=0$ has its roots $+1$, $+2$, -3.

When the term whose coefficient is zero, is restored, $x^3\pm 0x^2-7x+6=0$.

If the $+$ sign be taken, the signs of the terms are $+\ +\ -\ +$, which give $++$, $+-$, $-+$, two changes and one permanence of signs, indicating two positive roots and one negative root.

If the sign $-$ be taken, the signs of the equation are $+\ -\ -\ +$, which give $+-$, $--$, $-+$, two changes and one permanence, as before.

2. The biquadratic equation $x^4-9x^2+2x+6=0$ has its roots $+1$, -3, $+(1+\sqrt{3})$, $-(\sqrt{3}-1)$, and when the term whose coefficient is zero is restored, it is $x^4\pm 0x^3-9x^2+2x+6=0$.

If the sign $+$ be taken, the signs of the terms are $+\ +\ -\ +\ +$, which give $++$, $+-$, $-+$, $++$, two changes and two permanences of signs.

If the sign $-$ be taken, the signs of the terms are $+\ -\ -\ +\ +$, which give $+-$, $--$, $-+$, $++$, two changes and two permanences of signs.

Let m_1, m_2, m_3, m_4 be four numbers such that m_1 is greater than a, m_2 less than a but greater than b, m_3 less than b but greater than c, and m_4 less than c.

And let m_1, m_2, m_3, m_4 be successively substituted for x; when $x=m_1$, $x^3-p_1x^2+p_2x-p_3$ becomes $(m_1-a)(m_1-b)(m_1-c)$, which is positive.

$x=m_2$, " " " $(m_2-a)(m_2-b)(m_2-c)$, which is negative.

$x=m_3$, " " " $(m_3-a)(m_3-b)(m_3-c)$, which is positive.

$x-m_4$, " " " $(m_4-a)(m_4-b)(m_4-c)$, which is negative.

Hence it appears if each of the four numbers between every two of which lie the three possible roots of a cubic equation be successively substituted for the unknown quantity, the four results are alternately positive and negative.

In the same manner, if a, b, c, d be the roots of a biquadratic equation, $x^4-p_1x^3+p_2x^2-p_3x+p_4=0$,

then $x^4-p_1x^3+p_2x^2-p_3x+p_4=(x-a)(x-b)(x-c)(x-d)$,

and if m_1, m_2, m_3, m_4, m_5 be numbers such that m_1 is greater than the greatest root a, and m_5 is less than the least root d, and m_2, m_3, m_4 lies respectively between the roots a, b, c, d, it may be shewn that when m_1, m_2, m_3, m_4, m_5 are successively substituted for x in the expression $x^4-p_1x^3+p_2x^2-p_3x+p_4$, the successive results will be positive and negative alternately.

Hence it appears that *numbers* which are limits to the roots of the equation (or numbers between which the roots lie), when substituted successively for the unknown quantity, *give results* alternately positive and negative.

Conversely, if two numbers m_1, m_4 when substituted for x give results affected with different signs, one root *at least* must lie between these two numbers.

And if a series of numbers m_1, m_2, m_3, m_4, taken in order of magnitude when successively substituted for x give results alternately positive and negative; these numbers must be limits to the roots of the equation, or the roots must lie between these numbers.

If every number greater than m_1 when substituted for x gives positive results, these numbers are respectively greater than the greatest root of the equation; and if every number less than m_4 when substituted for x gives negative results, these other numbers are less respectively than the least root of the equation.

If the results arising from the substitution of any two numbers for the unknown quantity be both positive or both negative, either no root, or an even number of roots of the equation, lies between the two numbers.

13. *To find a limit greater than the greatest positive root of an equation.*

Let a, b, c be the roots of the equation $x^3-p_1x^2+p_2x-p_3=0$, and when this equation is transformed into one whose roots are respectively diminished by e, it becomes

$$y^3-(p_1-3e)y^2+(p_2-2p_1e+3e^2)-(e^3-p_1e^2+p_2e-p_3)=0.$$

If such a value can be assigned to e which shall make every term of the transformed equation positive, then all the roots are negative, and that value of e is greater than the greatest positive root of the proposed equation.*

If the signs of the roots of the given equation be changed, a limit greater than the greatest root of this equation with its sign changed is less than the least root of the proposed equation.

14. *To find an equation whose roots shall be limits between the roots of a proposed equation when the roots of the latter are possible.*

It has been shewn that when $x^3-p_1x^2+p_2x-p_3=0$ is transformed into an equation whose roots are $a-e$, $b-e$, $c-e$, it becomes

$$y^3-(p_1-3e)y^2+(3e^2-2p_1e+p_2)y-(e^3-p_1e^2+p_2e-p_3)=0,$$

and the coefficient of the last term but one of this equation is the sum of the products of every two roots with their signs changed,

$$\therefore\ 3e^2-2p_1e+p_2=(e-a)(e-b)+(e-a)(e-c)+(e-b)(e-c).$$

If a, b, c be successively substituted for e, the results are respectively,

$(a-b)(a-c)$ which is positive,
$(b-a)(b-c)$,, negative,
$(c-a)(c-b)$,, positive,

therefore a, b, c are limits to the roots of the equation $3e^2-2p_1e+p_2=0$, and substituting x for e.

The roots of the equation $3x^2-2p_1x+p_2=0$ respectively lie between the roots of the equation $x^3-p_1x^2+p_2x-p_3=0$, and if a', b' be the roots of $3x^2-2p_1x+p_2=0$, then a, a', b, b', c are arranged in order of magnitude.

The case here taken is that in which the three roots of the cubic are positive; the same process applies when all the roots are negative, or when two are positive and one negative, or two negative and one positive.

By the same process it may be shewn that the roots of the equation

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* When x is written for e in $3e^2-2p_1e+p_2$ and $3e-p_1$, the coefficients of the transformed cubic equation; these expressions $3x^2-2p_1x+p_2$ and $3x-p_1$ have been named the first and second derived functions of the given cubic equation $x^3-p_1x^2+p_2x-p_3=0$.

Also when x is written for e in $4e^3-3p_1e^2+2p_2e-p_3$, $6e^2-3p_1e+p_2$, and $4e-p_1$, the coefficients of the transformed biquadratic equation; these expressions $4x^3-3p_1x^2+2p_2x-p_3$, $6x^2-3p_1x+p_2$ and $4x-p_1$ have also been named the first, second, and third derived functions of the given biquadratic

$$x^4-p_1x^3+p_2x^2-p_3x+p_4=0.$$

$4x^3-3p_1x^2+2p_2x-p_3=0$ are limits between the roots of the equation $x^4-p_1x^3+p_2x^2-p_3x+p_4=0$.

Also in the same manner an equation may be found whose roots shall be limits between the roots of the equation

$$4x^3-3p_1x^2+2p_2x-p_3=0.$$

If a cubic equation contain two equal roots, the limiting equation will contain one of them, and the proposed equation and the limiting equation will involve a common factor of the first degree, which will contain one of the equal roots.

Also, if a biquadratic equation contain three equal roots, the limiting equation will involve two of them, and the given and limiting equations will contain a common quadratic factor which will contain two of them.*

If all the four roots of a biquadratic equation are impossible, the limiting equation formed by the ordinary process, being a cubic equation, will contain at least one real root.†

15. *To investigate the solution of a cubic equation by the method of Cardan.*

Let $x^3-p_1x^2+p_2x-p_3=0$ be a complete cubic equation, and let its roots be denoted by a, b, c.

When this equation is transformed into $x^3-qx+r=0$, an equation wanting its second term; if α, β, γ denote its roots,

$$\text{then } \alpha=a+\frac{p_1}{3},\ \beta=b+\frac{p_1}{3},\ \gamma=c+\frac{p_1}{3};$$

and if the roots of the transformed equation can be found, the roots of the proposed equation are also known.

Assume $x=y+z$, then the equation $x^3-qx+r=0$ by substitution becomes $y^3+(3yz-q)(y+z)+z^3+r=0$.

And since y and z, two unknown quantities, have been substituted for one—namely, x; another condition may be assumed in the solution. Let $3yz-q=0$ be this condition, then $z=\frac{q}{3y}$ and $y=\frac{q}{3z}$, and the transformed equation is $y^6-ry^3+\frac{q^3}{27}=0$, a quadratic equation. Also $z^6-rz^3+\frac{q^3}{27}=0$, an identical equation is obtained by substituting the value of z instead of y.

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* It is obvious from this transformation, that if a cubic equation contain two equal roots with the same or different signs; and if a biquadratic contain two or three equal roots, these equal roots can be detected by inspection. As in the examples.

The cubic equation $x^3-x^2-8x+12=0$, can be put into the form $x^3-x^2-2^2x+2^2.3=0$; and since the last term and last term but one contain the factors 2^2, 2; therefore 2 is one of the equal roots of the equation.

The biquadratic $x^4+x^3-30x^2+76x-56=0$ can be put into the form $x^4+x^3-2.3.5x^2+19.2^2x-7.2^3=0$, of which the last and preceding terms contain the factors 2^3, 2^2, 2; 2 is one of the three equal roots of the equation.

† The equation $x^4-8x^3+42x^2-80x+125=0$, has the roots $1\pm2\sqrt{-1}$, $3\pm4\sqrt{-1}$.

The limiting equation is $4x^3-24x^2+84x-80=0$, or $x^3-6x^2+21x-20=0$,

when $x=1$, $x^3-6x^2+21x-20=1-6+21-20=22-26=-4$, a negative result,

$x=2$, $x^3-6x^2+21x-20=8-24+42-20=50-44=+6$, a positive result,

$\therefore$ one root of the equation $x^3-6x^2+21x-20=0$ lies between 1 and 2, and is some incommensurable number.

Hence one of the values y^3, found from the former equation, is a value of z^3 found from the latter.

And therefore $y^3=-\frac{r}{2}+\left(\frac{r^2}{4}-\frac{q^3}{27}\right)^{\frac{1}{2}}$, $z^3=-\frac{r}{2}-\left(\frac{r^2}{4}-\frac{q^3}{27}\right)^{\frac{1}{2}}$;

Consequently

$$x=y+z=\left\{-\frac{r}{2}+\left(\frac{r^2}{4}-\frac{q^3}{27}\right)^{\frac{1}{2}}\right\}^{\frac{1}{3}}+\left\{-\frac{r}{2}-\left(\frac{r^2}{4}-\frac{q^3}{27}\right)\right\}^{\frac{1}{3}}.$$

But since $y=\left\{-\frac{r}{2}+\left(\frac{r^2}{4}-\frac{q^3}{27}\right)^{\frac{1}{2}}\right\}^{\frac{1}{3}}$; y has three values, one possible and two impossible;* and similarly since $z=\left\{-\frac{r}{2}-\left(\frac{r^2}{4}-\frac{q^3}{27}\right)^{\frac{1}{2}}\right\}^{\frac{1}{3}}$, z has also three values of the same forms.

And since $x=y+z$, these values of y and z appear to give the following nine values of x:—

$$\begin{array}{lll} y+z, & y+\omega z, & y+\omega^2 z, \\ \omega y+z, & \omega^2 y+z, & \omega y+\omega^2 z, \\ \omega y+\omega z, & \omega^2 y+\omega z, & \omega^2 y+\omega^2 z. \end{array}$$

But the equation of condition $yz=\frac{q}{3}$ indicates that only those of the nine values are admissible whose products are rational numbers; that is, the three roots of the transformed equation are $y+z$, $\omega y+\omega^2 z$,

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* Every number positive or negative has three cube roots, one of which is numerical, and the other two symbolical, or impossible.

To find the cube roots of $+a^3$ and $-a^3$.

Let $x^3=a^3$, then $x=(a^3)^{\frac{1}{3}}=a$, the numerical cube root of a^3.

To find the other two roots, $x^3-a^3=0$, and $x-a=0$.

$\therefore \frac{x^3-a^3}{x-a}=x^2+ax+a^2=0$, a quadratic equation, of which the roots are $x=\frac{1}{2}(-1+\sqrt{-3})a$, $x=\frac{1}{2}(-1-\sqrt{-3})a$.

And the three cube roots of $(+a)^3$, are a, $\frac{1}{2}(-1+\sqrt{-3})a$, $\frac{1}{2}(-1-\sqrt{-3})a$.

Similarly, the three cube roots of $(-a)^3$ are $-a$, $\frac{1}{2}(1+\sqrt{-3})a$, $\frac{1}{2}(1-\sqrt{-3})a$.

If $a=1$, then the three cube roots of $+1$ are 1, $\frac{1}{2}(-1+\sqrt{-3})$, $\frac{1}{2}(-1-\sqrt{-3})$,
and the three cube roots of -1 are -1, $\frac{1}{2}(1+\sqrt{-3})$, $\frac{1}{2}(1-\sqrt{-3})$.

And since $\left\{\frac{1}{2}(-1-\sqrt{-3})\right\}^2=\frac{1}{2}(-1+\sqrt{-3})$, or the square of one of the symbolic cube roots of $+1$ is equal to the first power of the other, the three cube roots of $+1$ may be conveniently represented by 1, ω, ω^2, and similarly for the three cube roots of -1. Also $\frac{1}{2}(-1+\sqrt{-3})=\left\{\frac{1}{2}(-1-\sqrt{-3})\right\}^{-1}$,
and $\frac{1}{2}(1+\sqrt{-3})=\left\{\frac{1}{2}(1-\sqrt{-3})\right\}^{-1}$.

$\omega^2 y+\omega z$, of which the first is rational and the other two impossible, and as these are denoted by α, β, γ, the roots of the proposed equation are denoted by $a=\alpha-\frac{p_1}{3}$, $b=\beta-\frac{p_1}{3}$, $c=\gamma-\frac{p_1}{3}$.

In order that Cardan's method may be applicable, the expression $\left(\frac{r^2}{4}-\frac{q^3}{27}\right)^{\frac{1}{2}}$ must not be an impossible quantity.

If $\frac{r^2}{4}>\frac{q^3}{27}$, then $\left(\frac{r^2}{4}-\frac{q^3}{27}\right)^{\frac{1}{2}}$ is a possible quantity, and one root of the equation is possible and the other two impossible.

If $\frac{r^2}{4}=\frac{q^3}{27}$, then $\left(\frac{r^2}{4}-\frac{q^3}{27}\right)^{\frac{1}{2}}=0$, and two roots of the equation are equal.

Hence Cardan's method is applicable when two roots of the cubic are equal, as well as when two roots are impossible.*

If $\frac{r^2}{4}<\frac{q^3}{27}$, then $\left(\frac{r^2}{4}-\frac{q^3}{27}\right)^{\frac{1}{2}}$ is an impossible quantity, and Cardan's formula is inapplicable; and this will always be the case when the three roots of the cubic are possible, excepting when two of them are equal.

16. *To investigate the solution of a biquadratic equation by the method of Descartes.*

Let $x^4+qx^2+rx+s=0$ be the given equation when the second term is removed, and let its roots be denoted by a, b, c, d.

Suppose this equation to be composed of the two quadratic equations $x^2+ex+f=0$ and $x^2-ex+g=0$.

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* Solve the cubic equation $x^3+6x-2=0$ by Cardan's rule.

Generally $x=\left\{-\frac{r}{2}+\left(\frac{r^2}{3}-\frac{q^3}{27}\right)^{\frac{1}{2}}\right\}^{\frac{1}{3}}+\left\{-\frac{r}{2}-\left(\frac{r^2}{4}-\frac{q^3}{27}\right)^{\frac{1}{2}}\right\}^{\frac{1}{3}}$.

Here, comparing $x^3-qx+r=0$ with $x^3+6x-2=0$, $q=-6$, $r=-2$.

$$\therefore x=\{-1+(1+8)^{\frac{1}{2}}\}^{\frac{1}{3}}+\{-1-(1+8)^{\frac{1}{2}}\}^{\frac{1}{3}}=\{-1+3\}^{\frac{1}{3}}+\{-1-3\}^{\frac{1}{3}}=2^{\frac{1}{3}}+(-4)^{\frac{1}{3}}$$
$$=2^{\frac{1}{3}}-2^{\frac{2}{3}}=-(2^{\frac{2}{3}}-2^{\frac{1}{3}}),$$

that is $x=-(2^{\frac{2}{3}}-2^{\frac{1}{3}})$, and $x+(2^{\frac{2}{3}}-2^{\frac{1}{3}})=0$, also $x^3+6x-2=0$,

$\therefore \frac{x^3+6x-2}{x+(2^{\frac{2}{3}}-2^{\frac{1}{3}})}=0$, or $x^2-(2^{\frac{2}{3}}-2^{\frac{1}{3}})x+(2^{\frac{4}{3}}+2+2^{\frac{2}{3}})=0$, the quadratic equation which will give the other two roots, which are impossible.

Solve the cubic equation $x^3-27x+54=0$.

Here $q=27$, $r=54$, $\therefore \frac{r^2}{4}-\frac{q^3}{27}=\frac{27^2.4}{4}-\frac{27^3}{27}=27^2-27^2=0$,

therefore two roots of the equation are equal, and Cardan's method is applicable,

$\therefore x=\{-27\}^{\frac{1}{3}}+\{-27\}^{\frac{1}{3}}=-3-3=-6$ one root,

and $x+6=0$, but $x^3-27x+54=0$, $\therefore \frac{x^3-27x+54}{x+6}=0$,

or $x^2-6x+9=0$ is the equation which contains the two equal roots 3, 3,

$\therefore$ the roots are 3, 3, 6.

$\therefore\ x^4-2Ax^2-8\alpha\beta\gamma x+(A^2-4B)=0$, which is an equation identical with $x^4+qx^2+rx+s=0$.

Hence $q=-2A$, $r=-8\alpha\beta\gamma$, $s=A^2-4B$.

And $\alpha^2+\beta^2+\gamma^2=A=-\frac{q}{2}$,

$$\alpha^2\beta^2+\alpha^2\gamma^2+\beta^2\gamma^2=B=\frac{A^2-s}{4}=\frac{q^2-4s}{16},\quad \alpha^2\beta^2\gamma^2=\frac{r^2}{64}.$$

From these equations it is evident that

$-\frac{q}{2}$, $\frac{q^2-4s}{16}$, $\frac{r^2}{64}$, are the coefficients of the second, third, and fourth terms of the cubic equation $y^3-\frac{q}{2}y^2+\frac{q^2-4s}{16}y-\frac{r^2}{64}=0$, whose roots are α^2, β^2, γ^2, if $x^2=y$.

By comparing the reducing cubic in Euler's solution

$$y^3-\frac{q}{2}y^2+\frac{q^2-4s}{16}y-\frac{r^2}{64}=0,\ \text{or}\ (4y)^3-2q(4y)^2+(q^2-4s)(4y)-r^2=0,$$

$$\text{with}\ y^3-qy^2+(q^2-4s)y-r^2=0,$$

the reducing cubic of Descartes; it will be seen that the roots of the former equation are respectively quadruple of the roots of the latter.

Since $x=\alpha+\beta+\gamma$ is a root of the biquadratic, and α^2, β^2, γ^2 are the roots of the reducing cubic; it follows that $\pm\alpha$, $\pm\beta$, $\pm\gamma$ appear to furnish eight different values of x.*

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* To solve the biquadratic equation $x^4-25x^2+60x-36=0$ by Euler's method.

Generally, the reducing cubic of $x^4+qx^2+rx+s=0$, is $y^3-\frac{q}{2}.y^2+\frac{q^2-4s}{16}y-\frac{r^2}{64}=0$.

It follows that the reducing cubic of $x^4-25x^2+60x-36=0$ is

$$y^3-\frac{25}{2}.y^2+\frac{769}{16}.y-\frac{225}{4}=0.$$

Transforming this cubic into one whose roots shall be four times as great, by writing $4y$ for y, it becomes $y^3-50y^2+769y-3600=0$.

The roots of this equation have been already determined to be 9, 16, 25 (p. 21).

Hence the roots of $y^3-\frac{25}{2}y^2+\frac{769}{16}y-\frac{225}{4}=0$ are $\frac{9}{4}$, $\frac{16}{4}$, $\frac{25}{4}$, which are respectively the value of α^2, β^2, γ^2, $\therefore\ \alpha=\frac{\pm 3}{2}$, $\beta=\frac{\pm 4}{2}$, $\gamma=\frac{\pm 5}{2}$.

And since the last term $\frac{225}{4}$ of the reducing cubic is negative, all the values of $\frac{\pm 3}{2}$, $\frac{\pm 4}{2}$, $\frac{\pm 5}{2}$ must be negative, or two positive and one negative.

The four roots of the proposed biquadratic will be

$$-\alpha-\beta-\gamma=-\frac{3}{2}-\frac{4}{2}-\frac{5}{2}=-2-4=-6$$

$$-\alpha+\beta+\gamma=-\frac{3}{2}+\frac{4}{2}+\frac{5}{2}=2+1=3$$

$$\alpha-\beta+\gamma=\frac{3}{2}-\frac{4}{2}+\frac{5}{2}=-2+4=2.$$

$$\alpha+\beta-\gamma=\frac{3}{2}+\frac{4}{2}-\frac{5}{2}=+2-1=1,$$

But $\alpha^2\beta^2\gamma^2 = \frac{r^2}{64}$, $\therefore \alpha\beta\gamma = \pm\frac{r}{8}$. If $\frac{r}{8}$ be positive, the product $\alpha\beta\gamma$ must be positive, and this condition requires that α, β, γ be all positive, or one of them positive and two negative.

In this case the four roots of the biquadratic equation are

$$\alpha+\beta+\gamma, \quad \alpha-\beta-\gamma, \quad -\alpha+\beta-\gamma, \quad -\alpha-\beta+\gamma.$$

If $\frac{r}{8}$ be negative, the product $\alpha\beta\gamma$ must be negative, and this can only be the case when α, β, γ are all negative, or when two are positive and one negative.

And the four roots of the biquadratic are

$$-\alpha-\beta-\gamma, \quad -\alpha+\beta+\gamma, \quad \alpha-\beta+\gamma, \quad \alpha+\beta-\gamma.$$

18. *To solve a biquadratic equation by the method of Simpson, or of Waring, as it is more commonly known.*

This method does not, like the methods of Descartes and Euler, require the transformation of the equation to another wanting the second term. The solution is made to depend on the assumption that the given biquadratic may be made equal to the difference of the squares of two algebraical expressions.

Let $x^4+p_1x^3+p_2x^2+p_3x+p_4=0$ be the proposed equation.

Assume $x^4+p_1x^3+p_2x^2+p_3x+p_4=(x^2+\frac{1}{2}p_1+A)^2-(Bx+C)^2$.

Expanding and arranging the terms of this equation

$$x^4+p_1x^3+p_2x^2+p_3x+p_4$$
$$=x^4+p_1x^3+(2A-B^2+\tfrac{1}{4}p_1^2)x^2+(p_1A-2BC)x+A^2-C^2.$$

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or 1, 2, 3, -6 are the roots of $x^4-25x^2+60x-36=0$.

Solve the equation $x^4-8x^3+14x^2+4x-8=0$ by Euler's method.

First transform the equation into one which shall want the second term, and it becomes $x^4-10x^2-4x+8=0$, the reducing cubic of which is $y^3-5y^2+\frac{17}{4}y-\frac{1}{4}=0$, and this equation transformed into one with integral coefficients is

$$y^3-10y^2+17y-2=0.$$

The roots of this equation will be found to be 2, $4+\sqrt{15}$, $4-4\sqrt{15}$, which are double of the roots of $y^3-5y^2+\frac{17}{4}y-\frac{1}{4}=0$.

Hence the roots of the reducing cubic are $\alpha^2=1$, $\beta^2=\frac{1}{2}(4+\sqrt{15})$, $\gamma^2=\frac{1}{2}(4-\sqrt{15})$, and $\alpha=\pm1$, $\beta=\pm\frac{1}{2}(\sqrt{5}+\sqrt{3})$, $\gamma=\pm\frac{1}{2}(\sqrt{5}-\sqrt{3})$, and the product of $\alpha\beta\gamma$ is negative.

Therefore the four roots of the biquadratic $x^4-10x^2-4x+8=0$ are

$$\alpha+\beta+\gamma=1+\tfrac{1}{2}(\sqrt{5}+\sqrt{3})+\tfrac{1}{2}(\sqrt{5}-\sqrt{3})=1+\sqrt{5}.$$
$$\alpha-\beta-\gamma=1-\tfrac{1}{2}(\sqrt{5}+\sqrt{3})-\tfrac{1}{2}(\sqrt{5}-\sqrt{3})=1-\sqrt{5}.$$
$$-\alpha+\beta-\gamma=-1+\tfrac{1}{2}(\sqrt{5}+\sqrt{3})-\tfrac{1}{2}(\sqrt{5}-\sqrt{3})=-1+\sqrt{3}.$$
$$-\alpha-\beta+\gamma=-1-\tfrac{1}{2}(\sqrt{5}+\sqrt{3})+\tfrac{1}{2}(\sqrt{5}-\sqrt{3})=-1-\sqrt{3}.$$

And since the roots of $x^4-10x^2-4x+8=0$ are respectively greater by 2 than the roots of $x^4-8x^3+14x^2+4x-8=0$,

$\therefore$ $3+\sqrt{5}$, $3-\sqrt{5}$, $1+\sqrt{3}$, $1-\sqrt{3}$ are the four roots of the proposed equation.

This may be easily verified by shewing that the equation

$\{x-(3+\sqrt{5})\}.\{x-(3-\sqrt{5})\}.\{x-(1+\sqrt{3})\}.\{x-(1-\sqrt{3})\}=0$, is identical with

$$x^4-8x^3+14x^2+4x-8=0.$$

Equating the coefficients of corresponding powers of x,

$$\therefore\ 2A-B^2+\tfrac{1}{4}p_1^2=p_2,\qquad p_1A-2BC=p_3,\qquad A^2-C^2=p_4,$$

$$\text{and } B^2=2A+\tfrac{1}{4}p_1^2-p_2,\qquad 2BC=p_1A-p_3,\qquad C^2=A^2-p_4.$$

Now obviously $B^2\times C^2=\frac{1}{4}(2BC)^2$, since $B^2+2BC+C^2$ is a complete square, $\therefore\ (2A+\frac{1}{4}p_1^2-p_2)(A^2-p_4)=(p_1A-p_3)^2$,

$$\text{or } 2A^3-p_2A^2+2(p_4+p_1p_3)A-(\tfrac{1}{4}p_1^2p_4-p_2p_4-p_3^2)=0,$$

a cubic equation from which the values of A can be found in terms of

$$p_1,\ p_2,\ p_3,\ p_4.$$

And since $B^2=2A+\frac{1}{4}p_1^2-p_2$ and $C^2=A^2-p_4$,

$\therefore$ B and C are also known when A is known.

But $(x^2+\frac{1}{2}p_1+A)^2-(Bx+C)^2$

$$=\{(x^2+\tfrac{1}{2}p_1x+A)+(Bx+C)\}\{(x^2+\tfrac{1}{2}p_1x+A)-(Bx+C)\}=0,$$

$$\therefore\ x^2+\tfrac{1}{2}p_1x+A+(Bx+C)=0 \text{ and } x^2+\tfrac{1}{2}p_1x+A-(Bx+C)=0,$$

are the two quadratic equations from which the four values of x can be determined in terms of p_1, p_2, p_3, p_4.*

19. *To explain Newton's method of finding an approximate value of an incommensurable root of a cubic or biquadratic equation.*†

Let $x^3-p_1x^2+p_2x-p_3=0$ be a cubic equation having an incommensurable root.

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* Solve the equation $x^4-4x^3+8x^2-8x+3=0$ by Simpson's or Waring's method. Assume $x^4-4x^3+8x^2-8x+3=(x^2-2x+A)^2-(Bx+C)^2$

$$=x^4-4x^3+(2A+4-B^2)x^2-(4A+2BC)x+A^2-C^2.$$

Equating coefficients of corresponding powers of x
$2A+4-B^2=8$, $4A+2BC=8$, $A^2-C^2=3$. Eliminating B and C, the cubic equation $A^3-4A^2+5A-2=0$ is obtained, of which the roots or values of A are 1, 1, 2, and taking $A=2$, $B=0$, and $C=1$.

$$\therefore\ (x^2-2x+A)^2-(Bx+C)=(x^2-2x+2)^2-1=0,$$

$$\text{and } (x^2-2x+3)(x^2-2x+1)=0, \text{ and } x^2-2x+3=0,\ x^2-2x+1=0$$

are the two quadratics from which the four values of x can be determined.

It is possible that there are some forms of biquadratic equations, the solution of which may be more readily effected by making the solution to depend on the assumption that the given equation may be transformed into the sum instead of the difference of the squares of two algebraical expressions.

† Of the various methods of approximating to the incommensurable roots of numerical equations, that of Newton is the least tedious and the most simple and direct in its application. Both Lagrange and Fourier have pointed out the defects of Newton's method, and great caution is necessary in its application, especially when two roots falling between two consecutive integers are nearly equal. But, notwithstanding the objections, it is capable of a more direct and ready application than other methods. By substituting in the given equation the successive approximate values obtained, the results will show the correctness or incorrectness of any of these approximate values.

Lagrange's method of approximation is effected by means of continued fractions.

Mr. Horner's method of approximation is a process requiring successive transformations, and a series of operations by which the figures of the root are found in succession. The theorem of Sturm, combined with the method of Mr. Horner, renders the theory and solution of numerical equations complete. It appears, however, that the study of these methods may be advantageously deferred until the general theory of equations is considered.

Let a, b be two consecutive numbers, a less than b, which when substituted for the unknown quantity give, one a positive and the other a negative result.

Then one real root at least lies between a and b.

Let e denote some number less than unity, and when $a+e$ is substituted for x in the given equation, it becomes

$$(a+e)^3-p_1(a+e)^2+p_2(a+e)-p_3=0.$$

This equation when reduced and arranged according to powers of e is

$$(a^3-p_1a^2+p_2a-p_3)+(3a^2-2p_1a+p_2)e+(3a-p_1)e^2+e^3=0;$$

but since e is less than unity, e^2 is less than unity, and e^3 less than e^2.

If the term involving e^3 be rejected, then

$$(a^3-p_1a^2+p_2a-p_3)+(3a^2-2p_1a+p_2)e+(3a-p_1)e^2=0.$$

If also e^2 be rejected, $(a^3-p_1a^2+p_2a-p_3)+(3a^2-2p_1a+p_2)e=0$. If values of e be obtained from these equations, the value of e when added to a will give $a+e$, a nearer value of x than a.

If the value of $a+e$ when substituted for x in the given equation give a result having the same sign as when a was substituted for x; then $a+e$ is nearer the true value of x than a.

A still nearer approximation may be made by repeating the process, and the approximate value may be made to approach the exact value of x to any required degree of accuracy.

By a similar process an approximation can be obtained by *diminishing by e*, the greater limit b, in the same manner as has been done by *increasing by e*, the smaller limit a.*

20. In the solution of cubic and biquadratic equations, the most important object is to determine the commensurable and incommensur-

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* Find an approximate value of one of the roots of the equation $x^3-3x+1=0$.

If $x=1$, $1-3+1=-1$; if $x=2$, $8-6+1=+2$; ∴ one root lies between 1 and 2.

But if $x=1{\cdot}5$, $(1{\cdot}5)^3-3(1{\cdot}5+1=-{\cdot}125$, and ∴ one root lies between 1·5 and 2.

Let $x=1{\cdot}5+e$, substituting this value of x in the given equation,

$$(1{\cdot}5+e)^3-3(1{\cdot}5+e)+1=0, \text{ or } -{\cdot}125+3{\cdot}75e+4{\cdot}5e^2+e^3=0.$$

Since e is less than unity, let $e^3+4{\cdot}5e^2$ be rejected, then $3{\cdot}75e-{\cdot}125=0$, and $e={\cdot}033$ nearly, ∴ $x=1{\cdot}5+e=1{\cdot}5+{\cdot}033=1{\cdot}533$ nearly, an approximate value of one value of x.

To obtain a nearer approximation.

Let $x=1{\cdot}533+e'$, and by a similar process it will be found that

$$4{\cdot}05026e'+{\cdot}00368=0, \text{ and } e'=-{\cdot}00091 \text{ nearly.}$$

∴ $x=1{\cdot}533+e'=1{\cdot}533-{\cdot}00091=1{\cdot}53209$ nearly, a nearer approximation than the first. A still nearer approximation may be found by repeating the process.

But if only e^3 be rejected, then $4{\cdot}5e^2+3{\cdot}75e-{\cdot}125=0$ is a quadratic equation whose approximate roots are $e={\cdot}0321$ and $e=-{\cdot}8354$.

Taking the positive value of e,

$x=1{\cdot}5+e=1{\cdot}5+{\cdot}0321=1{\cdot}5321$ nearly, which is obviously an approximate value of x nearly equal to the second approximate value obtained by the first process.

The process is exactly the same for finding an approximate value of an incommensurable root of a biquadratic equation. When one root is found, the equation can always be depressed, if a cubic, to a quadratic; if a biquadratic, to a cubic.

able roots, as these only are capable of an intelligible interpretation. Impossible roots, when such exist in an equation, do not admit of any such interpretation.

As the equations which most frequently occur in philosophical questions and investigations are such as involve numerical coefficients, the real roots of these equations can in general be obtained either exactly or approximately.

The most direct process for the solution of equations consists in first finding the integral roots. If an equation involve roots of the form ± 1, these can be found by inspection, and other integral roots can always be found by the method of divisors.

When the integral roots are determined the equation can be depressed to one of lower dimensions, which will contain only incommensurable roots or impossible roots.

If every complete biquadratic equation could be reduced to two quadratic factors, the four roots could be directly determined by the solution of a quadratic equation. But as the ordinary process for such a reduction involves the solution of an equation of the fourth degree, it therefore fails in its general application. Each of the processes by which a complete biquadratic equation can be reduced to the product of two quadratic factors, involves the solution of a cubic equation.

Methods of approximation can be applied *directly* for finding any incommensurable root of an equation, to any required degree of accuracy, when the two consecutive integral numbers are known, between which the incommensurable root lies.

EXERCISES.

I.

1. Explain the principle on which depends the ordinary solution of quadratic equations. Why cannot a similar method be applied to solve cubic and biquadratic equations?

2. Show that every cubic equation has at least one real positive or negative root, according as the last term is negative or positive, and that a cubic equation cannot have more than three roots.

3. Explain how any equation can be formed having given roots; and why the converse process of determining the roots of a given equation cannot be effected by any general rule for cubic and biquadratic equations.

4. What is meant by a *real* and by an *imaginary* or *impossible* root of an equation? Write the general form of a complete biquadratic equation, and shew whether the general assertion that "an equation has as many roots as it has dimensions," ought to be considered as an axiomatic or a demonstrable truth.

5. State and exhibit the relations which subsist between the coefficients and the roots of an equation, (1) of the third degree, (2) of the fourth degree.

6. If any coefficient of an equation be changed in its value or sign, the values of the roots are changed.

7. If the signs of the terms of an equation whose roots are real be alternately positive and negative, the roots are positive, and if all the signs be positive, the roots are negative.

8. If a cubic or a biquadratic equation have only one change of sign, it can have only one positive root.

9. Every equation of four dimensions having its last term negative and all the coefficients rational, has at least two rational roots, one positive and the other negative.

II.

Form the cubic equations, each of which shall have respectively the roots,

1. $3, 4, 5$. 2. $-3, 4, 5$. 3. $-3, -4, 5$. 4. $-3, -4, -5$.
5. $3, \pm\sqrt{2}$. 6. $5, 2\pm\sqrt{3}$. 7. $4, 2\pm\sqrt{-3}$. 8. $-1, 1\pm\sqrt{-1}$.

III.

Form the equations of the fourth degree which shall respectively have the following roots:—

1. $2, 3, 4, 5$. 2. $-2, -3, -4, -5$. 3. $-2, 3, 4, -5$.
4. $9, 11, \pm\sqrt{5}$. 5. $3, -3, \pm\sqrt{3}$. 6. $1, -5, \pm\sqrt{-3}$.

7. $\pm\sqrt{3}$, $\pm\sqrt{-5}$. 8. $1, 2, 1\pm\sqrt{2}$. 9. $-1, -3, 1\pm\sqrt{-3}$
10. $2\pm\sqrt{3}, 3\pm\sqrt{2}$. 11. $2\pm\sqrt{3}, 2\pm\sqrt{-3}$. 12. $0, -1, 1\pm\sqrt{-1}$.
13. $\frac{1}{2}\{1+\sqrt{3}\pm(2.3^{\frac{1}{2}})^{\frac{1}{2}}\}$, $\frac{1}{2}\{1-\sqrt{3}\pm(-2.3^{\frac{1}{2}})\}$.
14. $3\pm2\sqrt{-1}$, $\pm\sqrt{-5}$.

IV.

Solve the following equations:—

1. $x^3-3x-110=0$, one root 5.
2. $x^3-11x^2+36x-36=0$, one root 6.
3. $6x^3-x^2-75x+100=0$, one root -4.
4. $x^3+8x^2-3x-24=0$, one root $\sqrt{3}$.
5. $x^3-6x^2+6x+8=0$, one root $1-\sqrt{3}$.
6. $x^3-2x^2-7x+2=0$, one root $2+\sqrt{3}$.
7. $x^3-9x^2+19x-35=0$, one root $1-4\sqrt{-1}$.
8. $x^4+4x^3+5x^2+2x-6=0$, one root $-1-\sqrt{3}$.
9. $x^4+x^3-8x^2-16x-8=0$, one root $1+\sqrt{5}$.
10. $x^4+2x^3-4x^2-4x+4=0$, one root $\sqrt{2}$.
11. $x^4-3x^3+5x^2-x-10=0$, one root $1+2\sqrt{-1}$.
12. $x^4-3x^3-5x^2+13x+6=0$, one root $1+\sqrt{2}$.
13. $x^4-3x^3-14x^2-6x+4=0$, one root $3+\sqrt{7}$.
14. $x^4+3x^3-10x^2-54x-72=0$, one root $-2-\sqrt{-2}$.
15. $x^4-5x^3+8x^2-10x+12=0$, one root $-\sqrt{-2}$.

V.

Find by inspection one of the roots of each of the following equations, depress the equations, and find the remaining roots:—

1. $x^3-3x^2-x+3=0$. 2. $x^3-3x+2=0$.
3. $2x^3-x-1=0$. 4. $x^3-7x-6=0$.
5. $x^3-2x+4=0$. 6. $x^3-6x-9=0$.
7. $x^3+3x^2-6x-8=0$. 8. $4x^3-3a^2x-a^3=0$.
9. $2x^3+3x^2+5=0$. 10. $x^3-3a^2x+2a^3=0$.
11. $(x-5)(x^2-5x+5)+x=5$. 12. $x(x+1)(x+2)=9\times8\times7$.
13. $3x^4+13x^3-117x-243=0$. 14. $x^4-5x^3-x+5=0$.
15. $x^3-x^2-2x-2=0$. 16. $x^4+4x^3+5x^2+2x-6=0$.
17. $x^4-4x^3+8x^2-16=0$. 18. $x^4-18x^3+99x^2-162x-280=0$.
19. $(x-3)(x-4)(x-5)(x-6)=1.2.3.4$.

VI.

Determine whether the following pairs of equations have roots in common; and if so, find them, and complete the solution of the equations:—

1. $x^3-3x^2+11x-9=0$ and $x^3-5x^2+11x-7=0$.
2. $x^3-9x^2+26x-24=0$ and $x^3-5x^2+11x-7=0$.
3. $x^3-10x^2+27x-18=0$ and $x^4-11x^3+41x^2-61x+30=0$.
4. $x^4-4x^3+8x^2-16x+16=0$ and $x^4-6x^3+13x^2-12x+4=0$.

5. $x^4-2x^3-7x^2+26x-20=0$ and $x^4+4x^3-2x^2-12x+8=0$.
6. $x^4-4x^3+16x-16=0$ and $2x^4-17x^3+53x^2-72x+36=0$.
7. $x^4+x^3-5x^2-7x+10=0$ and $2x^5-3x^4-10x^3+28x^2-31x+14=0$.

VII.

How may it be discovered when an equation contains equal roots? Ascertain if the following equations contain equal roots :—

1. $x^3-7x^2+16x-12=0$. 2. $x^3+11x^2+24x-36=0$.
3. $x^3-18x^2+96x-128=0$. 4. $x^3-3x^2-45x+175=0$.
5. $x^3-x^2-8x+12=0$. 6. $9261x^3-13230x^2-4000=0$.
7. $x^4-6x^2+8x-3=0$. 8. $4x^4-24x^3+45x^2-29x+6=0$.
9. $x^4-33x^2+100x-84=0$. 10. $x^4-6x^3+8x^2-2x-1=0$.
11. $x^4+2x^3-12x^2-40x-32=0$. 12. $x^4-24x^2-64x-48=0$.

VIII.

Explain what is meant by a recurring equation. Solve the following recurring equations, and verify the fact that each pair of conjugate roots is of the form $a, \frac{1}{a}$.

1. $x^3\pm1=0$. 2. $x^3\pm x^2+x\pm1=0$. 3. $x^3-5x^2+5x-1=0$.
4. $3x^3+7x^2+7x+3=0$. 5. $x^4+ax^3+a^2x^2+a^3x+a^4=0$.
6. $x^4-x^3-4x^2-x+1=0$. 7. $2x^4-5x^3+4x^2-5x+2=0$.
8. $12x^4-91x^3+194x^2-91x+12=0$. 9. $x^5+2x^4+2x+1=0$.
10. $x^5+9x^3+9x^2+1=0$. 11. $x^5+2x^4+3x^3+3x^2+2x+1=0$.
12. $x^5-11x^4+17x^3+17x^2-11x+1=0$.
13. $2x^5-15x^4+37x^3+37x^2+15x-2=0$.
14. $5x^5+26x^4+5x^3-5x^2-26x-5=0$.
15. $6x^5-11x^4-33x^3+33x+11x-6=0$.
16. $x^6-3x^5+6x^4-6x^2+3x-1=0$.

IX.

Shew that if the sum of the coefficients of an equation with their proper signs be equal to zero, one of the roots is $+1$; and if the sum of the coefficients of the odd terms be equal to the sum of the coefficients of the even terms, one of the roots is -1.

Solve the following equations :—

1. $x^3+3x^2+9x-13=0$. 2. $x^3-9x^2+27x-19=0$.
3. $x^3-6x^2+11x-6=0$. 4. $27x^3-135x^2+225x-117=0$.
5. $x^3-9x^2+23x-15=0$. 6. $7x^3+5x^2+x+3=0$.
7. $3x^3-x^2+18x-20=0$. 8. $20x^3+3x^2+x+18=0$.
9. $x^4-5x^3+4x^2-7x+7=0$. 10. $x^4+3x^3-5x^2+8x-7=0$.
11. $x^4+4x^3-13x^2-4x+12=0$. 12. $x^4-11x^2+18x-8=0$.
13. $x^4+13x^3+33x^2+31x+10=0$. 14. $x^4-5x^3-5x^2+45x-36=0$.
15. $x^5+15x^4-9x^3+53x^2+8x-48=0$. 16. $x^4+2x^3-2x-1=0$.
17. $x^5+2x^4-10x^3-20x^2+9x+18=0$. 18. $x^4-2x^3+2x-1=0$.

X.

Solve the following equations :—

(1) When the roots are in arithmetical progression.

1. $x^3-15x^2+66x-80=0$. 2. $x^3-9x^2+23x-15=0$.
3. $x^3-15x^2+71x-105=0$. 4. $x^3-20x^2+117x-162=0$.
5. $x^4-10x^3+35x^2-50x+24=0$. 6. $x^4-8x^3+14x^2+8x-15=0$.
7. $x^4+22x^3+159x^2+418x+280=0$. 8. $x^4+x^3-11x^2+9x+18=0$.

(2) When the roots are in harmonical progression.

1. $x^3-11x^2+36x-36=0$. 2. $8x^3-6x^2-3x+1=0$.
3. $x^3-22x^2+144x-288=0$. 4. $105x^3-71x^2+15x-1=0$.
5. $x^3-23x^2+135x-225=0$. 6. $6x^3-15x^2+6x-1=0$.

(3) When the roots are in geometrical progression.

1. $x^3-7x^2+14x-8=0$. 2. $x^3-13x^2+39x-27=0$.
3. $x^3-14x^2+56x-64=0$. 4. $x^3-26x^2+156x-216=0$.
5. $x^4+2x^3+4x^2+8x+16=0$. 6. $x^4-30x^3+28x^2-960x+1024=0$.

XI.

1. Shew how to increase or diminish the roots of an equation by a given quantity, and state some of the uses of this transformation.

2. Transform the equations $x^3+3x^2-8x+10=0$ and $x^4-4x^3-9x+20x+32=0$ into equations which shall want their second terms.

3. Transform $3x^3+15x^2+25x-3=0$, $x^4-8x^3+24x^2+60x-168=0$ into equations in which the coefficient of the second term shall be zero; and shew that the value of e which causes the second terms to vanish, will also cause the third term to vanish in the first equation, and also the third and fifth terms to vanish in the second equation.

4. Transform $x^3-5x^2+7x-12=0$ and $x^4-12x^3+30x^2-16x+48=0$ into equations wanting the third term.

5. Transform the equation $x^3-9x^2-27x-8=0$ into another in which the second and third terms disappear.

6. Determine the resulting equation, when the second and fourth terms are made to vanish from the equation $x^4+4x^3+3x^2-2x-6=0$.

7. Transform the equation $2x^3-2x^2+3x+6=0$ into one which shall have all its signs alternately positive and negative.

8. Transform the equation $x^3-x^2-\frac{3x}{2}+3=0$ into one which shall have all its terms positive.

9. Transform the equation $5x^3-7x^2+2x-1=0$ into one in which the highest power of the unknown quantity shall be unity.

10. Transform the equation $x^4-40x+39=0$ into one whose roots shall be (1) the sum, (2) the difference respectively of every two roots of the original equation.

11. Transform $216x^3-504x^2+24x+11=0$
and $8x^4-4x^3+12x^2+2x-1=0$
into equations whose coefficients shall be integers, and the coefficients of the highest power of x in each equation unity.

12. Transform $x^3-10x+4=0$ into an equation whose roots shall be the reciprocals of the products of every two roots of the given equation.

13. Transform the equation $x^3-6x^2+11x-6=0$ into one whose roots shall be (1) greater than, (2) less than the roots of the given equation by 4, 5, 6 respectively.

14. Find the two equations which shall respectively have their roots, (1) the products of every two, (2) the products of every three roots of the equation $x^4+4x^3-13x^2+5x+3=0$, and shew that the resulting equations will be of six and of four dimensions.

15. Transform the equation $x^3-12x^2+41x-29=0$, (1) into an equation whose roots shall be the squares of the differences of the roots, and (2) into one whose roots shall be the differences of the squares of the roots of the given equation.

XII.

Explain the method of divisors for finding the integral roots of an equation, and shew how the number of substitutions may be lessened.

Find the roots of the following equations:

1. $x^3-4x^2-6x+12=0$. 2. $x^3-9x^2+26x-24=0$.
3. $x^3-4x^2-7x+10=0$. 4. $x^3-11x^2+38x-40=0$.
5. $4x^3+x^2-13x-12=0$. 6. $8x^3-45x^2+73x-30=0$.
7. $x^4+2x^3-14x^2+2x-15=0$. 8. $x^4-5x^3+3x^2+2x+8=0$.
9. $x^4-6x^3-27x^2-4x+36=0$. 10. $x^4-x^3-13x^2+16x-48=0$.
11. $2x^4-11x^3+19x^2+20x-60=0$. 12. $x^4-6x^3+10x^2+4x-24=0$.
13. $3x^4-19x^3+34x^2-45x-24=0$. 14. $x^4-40x+39=0$.

XIII.

If a, b, c be the roots of the equation $x^3-px^2+qx-r=0$; find in terms of the coefficients of the equations the following expressions:—

1. $a^2+b^2+c^2$. 2. $a^3+b^3+c^3$. 3. $a^4+b^4+c^4$.
4. $a^{-2}+b^{-2}+c^{-2}$. 5. $a^{-3}+b^{-3}+c^{-3}$. 6. $a^{-4}+b^{-4}+c^{-4}$.
7. $(a+a^{-1})+(b+b^{-1})+(c+c^{-1})$. 8. $(a^2+a^{-2})+(b^2+b^{-2})+(c^2+c^{-2})$.
9. $(a+b)(b+c)(c+a)$. 10. $a^3b^3+a^3c^3+b^3c^3$. 11. $a^5+b^5+c^5$.
12. $(ab+ac+bc).\left\{\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right\}$. 13. $\frac{a}{b}+\frac{b}{a}+\frac{a}{c}+\frac{c}{a}+\frac{b}{c}+\frac{c}{b}$.
14. $(a+b+c).\left\{\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right\}+(ab+ac+bc).\left\{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right\}+abc+\frac{1}{abc}$.
15. $\left\{\frac{a-b}{a+b}\right\}^2+\left\{\frac{b-c}{b+c}\right\}^2+\left\{\frac{c-a}{c+a}\right\}^2$.
16. $(a^{\frac{2}{3}}-b^{\frac{1}{3}}c^{\frac{1}{3}})(b^{\frac{2}{3}}-a^{\frac{1}{3}}c^{\frac{1}{3}})(c^{\frac{2}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}})$.

XIV.

If a, b, c, d be the roots of the equation $x^4-px^3+qx^2-rx+s=0$. find the values of the following expressions in terms of the coefficients of the equation :—

1. $a^2+b^2+c^2+d^2$. 2. $a^3+b^3+c^3+d^3$. 3. $a^4+b^4+c^4+d^4$.

4. $a^{-2}+b^{-2}+c^{-2}+d^{-2}$. 5. $a^{-3}+b^{-3}+c^{-3}+d^{-3}$.

6. $b^{-4}+b^{-4}+c^{-4}+d^{-4}$. 7. $a^2bcd+b^2acd+c^2abd+d^2abc$.

8. $(b+c-a-d)(c+a-b-d)(a+b-c-d)$.

9. $\frac{a}{b}+\frac{b}{a}+\frac{a}{c}+\frac{c}{a}+\frac{a}{d}+\frac{d}{a}+\frac{b}{c}+\frac{c}{b}+\frac{b}{d}+\frac{d}{b}+\frac{c}{d}+\frac{d}{c}$.

10. $\frac{a^2}{bcd}+\frac{b^2}{acd}+\frac{c^2}{bcd}+\frac{d^2}{abc}$. 11. $\frac{abc}{d}+\frac{abd}{c}+\frac{acd}{b}+\frac{bcd}{a}$.

12. $a^2b^2+a^2c^2+a^2d^2+b^2c^2+b^2d^2+c^2d^2$.

13. $a^2b^2c^2+a^2b^2d^2+a^2c^2d^2+b^2c^2d^2$. 14. $a^5+b^5+c^5+d^5$.

15. Given $x^4-5x^3+5x^2+5x-6=0$, whose roots are a, b, c, $-d$; required the sum of the four functions of the roots of the form $a+2b+3c+4d$, in terms of the coefficients of the equation.

XV.

Find the roots of the following equations :—

1. $x^3-13x^2+44x-32=0$, one root double of another.

2. $12x^3+13x^2-20x+4=0$, sum of two roots one third of -4.

3. $x^3-15x^2+66x-80=0$, sum of two roots 13.

4. $x^3+2x^2-21x+18=0$, quotient of two roots -2.

5. $x^3-12x^2+47x-60=0$, difference of two roots 2.

6. $x^3-12x-16=0$, one root the square of another.

7. $8x^3+14x^2-5x-2=0$, one root the sum of the reciprocals of the other two.

8. $x^3+4x^2-9x-36=0$, two roots of the form $\pm a$.

9. $x^3-4x^2-3x+12=0$, two roots of the form $\pm\sqrt{a}$.

10. $2x^3-11x^2+17x-6=0$, two roots of the form a, a^{-1}.

11. $x^3-10x^2+27x-18=0$, a, b, c, roots such that $a=2b$, $b=3c$.

12. $x^3-17x^2+94x-168=0$, two roots in the ratio of 2 to 3.

13. $x^3-16x^2+68x-80=0$, roots proportional to 1, 2, 3.

14. $x^3-4x^2+x+c=0$, one root 3, find value of c.

15. $x^3-31x^2+300x-900=0$, roots successive triangular numbers.

XVI.

Find the roots of the following biquadratic equations :—

1. $x^4-4x^3+5x^2-16x+4=0$, two roots of the form a, $\frac{1}{a}$.

2. $x^4-11x^3+41x^2-61x+30=0$, sum of two roots equal to the greatest root.

3. $x^4-45x^2-40x+84=0$, difference of two roots 3.

4. $x^4-7x^3+11x^2+7x-12=0$, 3 is the product of two roots.

5. $x^4+x^3-62x^2-80x+1200=0$, product of two roots 30.

6. $x^4-47x^2-126x-80=0$, product of two roots equal to that of the other two.

7. $x^4-10x^3+35x^2-50x+24=0$, roots of form $a\pm 1$, $b\pm 1$.

8. $6x^4-43x^3+107x^2-108x+36=0$, roots of form a, b, $\frac{a}{b}$, $\frac{b}{a}$.

9. $x^4-4x^3+11x^2-14x+10=0$, roots of the form $a+b\sqrt{-1}$, $a+2b\sqrt{-1}$.

10. $x^4-2a^2x^2+4ab^2x+a^4-b^4=0$, one root of form $a+b\sqrt{-1}$.

11. $x^4-2ax^3+bx^2+cx+f=0$, a is the sum of two roots.

12. $x^4-3x^3-7x^2+27x-18=0$, two roots of the form $\pm a$.

XVII.

Transform the equation $x^3-px^2+qx-r=0$, whose roots are a, b, c, into equations whose roots shall be respectively—

1. $a+b-c$, $a+c-b$, $b+c-a$.
2. $a(b+c)$, $b(a+c)$, $c(a+b)$.
3. a^2+a, b^2+b, c^2+c.
4. a^2+bc, b^2+ac, c^2+ab.
5. $a^3\pm b^2$, $b^2\pm c^2$, $c^2\pm a^2$.
6. $a^2(b+c)$, $b^2(a+c)$, $c^2(a+b)$.
7. $ab(a+b)$, $ac(a+c)$, $bc(b+c)$.
8. $a+b+ab$, $a+c+ac$, $b+c+bc$.
9. $(a+b)(a+c)$, $(b+c)(b+a)$, $(c+a)(c+b)$.
10. a^3+b^3, a^3+c^3, b^3+c^3.
11. $a+a^2+a^3$, $b+b^2+b^3$, $c+c^2+c^3$.
12. $\frac{1}{a+b}$, $\frac{1}{b+c}$, $\frac{1}{c+a}$.
13. $\frac{ab}{c}$, $\frac{ac}{b}$, $\frac{bc}{a}$.
14. $\frac{a}{b}+\frac{b}{a}$, $\frac{a}{c}+\frac{c}{a}$, $\frac{b}{c}+\frac{c}{b}$.
15. $\frac{1}{a^2}+\frac{1}{b^2}$, $\frac{1}{b^2}+\frac{1}{c^2}$, $\frac{1}{c^2}+\frac{1}{a^2}$.
16. $a+\frac{1}{bc}$, $b+\frac{1}{ac}$, $c+\frac{1}{ab}$.
17. $a^2+b^2+\frac{1}{ab}$, $b^2+c^2+\frac{1}{bc}$, $a^2+c^2+\frac{1}{ac}$.
18. $\frac{c}{a+b-c}$, $\frac{b}{a+c-b}$, $\frac{a}{b+c-a}$.
19. $\frac{abc}{ab+bc}$, $\frac{abc}{ac+bc}$, $\frac{abc}{ab+ac}$.
20. $\left\{\frac{b+c}{a}\right\}^2$, $\left\{\frac{a+b}{c}\right\}^2$, $\left\{\frac{a+c}{b}\right\}^2$.

XVIII.

1. Determine the relation between the coefficients of the equation $x^4-px^3+qx^2-rx+s=0$, in order (1) that the second and third terms may vanish by one transformation, also (2) the third and fourth terms.

2. Transform $x^4+qx^2+rx+s=0$ into an equation whose roots shall be the squares of the reciprocals of the roots.

3. Shew that the third term of the equation $x^3-px^2+qx-r=0$ cannot be removed by transforming the equation if $3q$ be greater than p^2.

4. Form the cubic equation whose roots shall be $a+b^{\frac{1}{3}}$.

5. Shew that the roots of the equation $x^3+2qx^2+qx-r^2=0$ are respectively the squares of the roots of the equation $x^3+qx+r=0$.

6. The roots of $x^2+mx+n=0$ are a_1, b_1; the roots of $x^2+px+q=0$ are a_2, b_2; form the equation whose roots are $\frac{a_1}{a_2}$, $\frac{a_1}{b_2}$, $\frac{b_1}{a_2}$, $\frac{b_1}{b_2}$.

7. If a, b, c be roots of $x^3+px^2+qx+r=0$; find the equation whose roots shall be $a+b$, $a+c$, $b+c$, and apply the resulting equation to the equation whose roots shall be the sum of every two of the roots of the equation $x^3-8x^2+23x-22=0$.

8. If a, b, c be the roots of the equation $x^3+qx-r=0$; find the equation whose roots shall be a^2-ab+b^2, a^2-ac+c^2, b^2-bc+c^2.

9. Transform the equation $x^4-px^3+qx^2-rx+s=0$, whose roots are a, b, c, d, into one whose roots are $ab+cd$, $ac+bd$, $ad+bc$.

10. Transform the equations $x^4-x^2y^2+y^4=85$ and $xy(x^2-y^2)=18$ into two other equations in which the new values z, w shall be $z=x^2-y^2$, $w=xy$, and shew that the resulting equation involving only z is $z^4-85z^2+324=0$.

11. Eliminate x and y between the equations

$$x+y+z=6,\quad x^2+y^2+z^2=14,\quad x^2+2y^2+3z^2=20,$$

and solve the resulting biquadratic equation.

12. Transform the equation $x^3-p_1x^2+p_2x-p_3=0$ into an equation whose roots shall be the squares of the differences of the roots of the given equation; and shew that the differences of the roots of the given equation are respectively equal to the differences of the roots of the transformed equation.

XIX.

1. Prove that impossible roots of the forms $a\pm b\sqrt{-1}$ and $\pm b\sqrt{-1}$ enter equations with integral coefficients in conjugate pairs; as also surd roots of the forms $a\pm\sqrt{b}$ and $\pm\sqrt{b}$.

2. Prove that a cubic equation whose coefficients are whole numbers has no fractional root, and shew why it does not apply to the equation $x^3-x^2+x=0$.

3. If $\frac{1}{2}a+\sqrt{b}$ be a root of $x^3+px^2+qx+r=0$, then a^2+pa+q is a factor of r; b not being a perfect square, and p, q, r integers.

4. If $x^3+qx+r=0$ have a root of the form $\frac{1}{2}(a+\sqrt{b})$ where $\sqrt{b}$ is irrational, shew that a is a root of the equation $x^3+qx-r=0$.

5. If $a+b\sqrt{-1}$ be a root of the equation $x^3+qx+r=0$, then a is a root of $8x^3+2qx-r=0$, and $3a^2-b^2=-q$; but if $a+b\sqrt{-1}$ be a root of $x^3-px^2-r=0$, then a is a root of $8x^3-8px^2+2p^2x+r=0$.

6. Find a condition that $x^4+2qx^2+4rx-s=0$ may have two imaginary roots.

7. Transform $x^4-4x^3+10x^2-12x+9=0$ into an equation whose roots shall respectively be less by unity, and from the result shew that the roots of the proposed equation are all impossible.

8. If $a+b\sqrt{-1}$ be a rout of $x^4+px^4+qx^3+rx+s=0$; then two of the roots will be $\frac{1}{2}\left\{-(p+2a)\pm\left(p^2-2q-2a^2+2b^2-\frac{2s}{a^2+b^2}\right)^{\frac{1}{2}}\right\}$.

9. A biquadratic equation cannot have all its roots real, if the coefficient of any term be less than a mean proportional between those of the two adjacent terms.

XX.

1. Shew that $x^3-7x^2+16x-10$ can never be negative for any value of x greater than unity.

2. Find between which of the roots of $x^3-9x^2+23x-15=0$ the number 2 lies.

3. Find a number greater than the greatest positive root of the equations $x^3-6x+23=0$ and $x^4-8x^3+12x^2+16x-39=0$ respectively.

4. Find a number greater than the greatest negative root of the equation $x^3+4x^2-31x-70=0$.

5. Determine a limit less than the least positive root of the equations $x^3+8x^2-8x-64=0$ and $x^4-3x^3-5x^2+2x+3=0$.

6. Find a number less than the least negative root of the equation $x^3-4x^2-x+20=0$.

7. Find the superior and inferior limits of the roots of the equation $x^3-3x^2-11x+21=0$.

8. Find a number less than the difference of any two roots of the equation $x^3-7x+7=0$.

9. Find the number and the situation in order of the real roots of the equation $x^3-5x^2+7x-36=0$.

10. Required the number and situation of the real roots of the equation $x^4-2x^3-7x^2+10x+10=0$.

11. Find two limits, one of which is greater than the greatest, and the other less than the least root of $x^4-19x^3+117x^2-261x+162=0$.

12. Find the number of real roots of $x^4-4x^3+11=0$ and the consecutive integers between which they are placed.

13. Find the first figure of one of the roots of the cubic equation $x^3-17x^2+54x-350=0$.

14. Find the first figure of one of the roots of the biquadratic equation $x^4+3x^3+2x^2+6x-148=0$.

XXI.

1. If the successive substitution of a' and b' in any equation for the unknown quantity, give results affected with different signs, then one at least or an odd number of roots lies between a' and b'; but if the results are affected with the same sign, an even number of roots or none at all lies in the above-named interval.

2. Find the limits of a and c in order that the equation

$$x^4-4ax^3-2a^2x^2+12a^3x+c=0 \text{ may have three real roots equal.}$$

3. If the real roots of a biquadratic equation be substituted in the limiting equation, the results will be alternatively positive and negative. Ex. $x^4-10x^3+35x^2-50x+24=0$.

4. In a complete cubic equation $x^3-p_1x^2+p_2x-p_3=0$, if the square of the coefficient of the second term be less than three times the coefficient of the third term, the proposed equation has a conjugate pair of impossible roots.

5. Shew that every number which written for x in the equation $x^4-px^3+qx^2+rx+s=0$ makes it and all the derived functions positive is a superior limit of the positive roots of the equation.

Ex. 1. $x^4-2x^3+10x-3=0$. Ex. 2. $x^4+x^3-10x^2-x+15=0$.

6. If a, b, c, d be the roots of $x^4-p_1x^3+p_2x^2-p_3x+p_4=0$, and $4x^3-3p_1x^2+2p_2x-p_3=0$ be the limiting equation;

$$\text{then } \frac{4x^3-3p_1x^2+2p_2x-p_3}{x^4-p_1x^3+p_2x^2-p_3x+p_4}=\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}+\frac{1}{x-d}.$$

7. Given the equation $4x^3+9x^2-12x+2=0$; employ the roots of its first derived equation to discover the number and position of its real roots.

XXII.

Investigate Cardan's method of solving a cubic equation, and solve the following equations, first shewing when the method is applicable by applying the criterion:—

1. $x^3+3x+4=0$.
2. $x^3-6x+6=0$.
3. $x^3+6x-32=0$.
4. $x^3-3x-18=0$.
5. $x^3+6x+20=0$.
6. $x^3-63x-370=0$.
7. $x^3-3x^2+15x+50=0$.
8. $x^3+2x^2+2x+4=0$.
9. $x^3-6x^2+10x-8=0$.
10. $2x^3+8x^2+16x-1=(x+3)^3$.
11. $x^3-4ax^2+5a^2x-2a^3=0$.
12. $x^3-x^2-5x+21=0$.

XXIII.

Solve the following biquadratic equations:—

By the method of Descartes.

1. $x^4-6x^3+8x^2+6x-9=0$
2. $x^4-5x^2-4x+30=0$.
3. $x^4-3x^2-42x-40=0$.
4. $x^4+4x^2-x+6=0$.
5. $x^4+8x^3+x^2-x-10=0$.
6. $x^4-5x^3+8x^2-10x+12=0$.

By the method of Euler.

1. $x^4+3x^2+2x+3=0$.
2. $x^4-2x^2+16x-15=0$.
3. $x^4-25x^2+60x-36=0$.
4. $x^4-5x^2-4x+30=0$.

By the method of Simpson or Waring.

1. $x^4-6x^3+12x^2-12x+4=0$.
2. $x^4-6x^3-14x^2-16x+8=0$.

3. $x^4-12x-5=0.$ 4. $x^4+2x^3+5x^2+2x+4=0.$
5. $x^4-12x^2+12x-3=0.$ 6. $x^4-38x^3+210x^2+538x+289=0.$

XXIV.

Explain Newton's method of approximation to the incommensurable roots of an equation, and determine the conditions for a successful application of his method.

Find an approximate value of an incommensurable root in each of the following equations:—

1. $x^3-2x-5=0.$ 2. $2x^3-x^2+x-3000=0.$
3. $x^3-7x-1=0.$ 4. $x^3+9x-20=0.$
5. $x^3-4x^2-7x+24=0.$ 6. $x^4-5x^3+3x^2+35x-70=0.$
7. $x^4-2x^3+2x^2-6x-3=0.$ 8. $x^4-5x^3+7x^2-5x-6=0.$
9. $x^4-6x^2+8x-5=0.$ 10. $x^4-5x^3-2x^2+15x-3=0.$

XXV.

1. The equation $x^3-px^2+qx-r=0$ has two equal roots, shew that one of them is $\frac{9r-pq}{6q-2p^2}$. Is the converse necessarily true? Find the third root.

2. The root common to the equations $x^3+px^2+qx+r=0$ and $x^3+p'x^2+q'x+r'=0$ is real, if $q-q'=2\{(p-p')(r-r')\}^{\frac{1}{2}}$.

3. If the roots of $x^3+ax^2+bx+c=0$ be of the form α, $\frac{1}{\alpha}$, β; shew that $c(a-c)=b-1$.

4. The product of the two unequal roots of $ax^3-bx^2+cx-d=0$ is equal to unity; prove that the third root will be equal to $\frac{c-a}{b-d}$.

5. If two roots of the equation $ax^3+3bx^2+3cx+a=0$ be equal, then shall $4(ac-b^2)(bd-c^2)-(ad-bc)^2=0$.

6. If $x^3+qx+r=0$ have two equal roots, the product of the roots of its limiting equation will be $-\left(\frac{r}{2}\right)^{\frac{2}{3}}$.

7. If the equations $x^3+px^2+qx+r=0$ and $3x^2+2px+q=0$ have a common root, each equation involves the same factor of the first degree, and $27r^2+2(2p^2-9q)pr+q^2(4q-p^2)=0$.

8. Shew that if $ax^3+bx^2+cx+d=0$ and $a'x^3+b'x^2+c'x+d'=0$ have a common root, the coefficients must satisfy the relations

$$\frac{ad'-a'd}{a'b-ab'}=\frac{bd'-b'd}{a'c-ac'}=\frac{cd'-c'd}{a'd-ad'}.$$

9. Shew that the equations $ax^3+bx^2+cx+d=0$

and $dx^3+cx^2+bx+a=0$

will have two roots in common when the following conditions are fulfilled:—

$$\frac{abc-a^2b-b^2d-acd}{ac-bd}=\frac{ac^2-bcd-a^3+abd}{ab-cd}=\frac{d(ac-bd)}{a^2-d^2}.$$

10. If the equations $x^2-p_1x+q_1=0$, $x^2-p_2x+q_2=0$, $x^2-p_3x+q_3=0$ have roots b, c; c, a; a, b respectively, the equation

$$x^3-\tfrac{1}{2}(p_1+p_2+p_3)x^2+(q_1+q_2+q_3)x-(q_1q_2q_3)^{\frac{1}{2}}=0$$

has its roots a, b, c.

11. If $x^3+px^2+qx+r=0$, $x^3+qx^2+rx+p=0$, $x^3+rx^2+qx+p=0$ have a common root; shew that $q^2+r^2=2pq$.

12. If each of two pairs of the equations $ax^2+bx+c=0$, $bx^2+cx+a=0$, $cx^2+ax+b=0$ have a common root; prove that a common root exists for the third pair, and that if they all have a common root, $a^3+b^3+c^3=3abc$.

13. The roots of the equation $x^3+3px^2+3qx+r=0$ cannot be all real if q be greater than p^2 or pr greater than q^2.

14. If a, a, c be the roots of the equation $x^3+3px^2+qx+r=0$, and c, c, a be the roots of $x^3+3p_1x^2+q_1x+r_1=0$, then shall

$$3(p-p_1)=\frac{q-q_1}{p+p_1}=\frac{(p+p_1)(r-r_1)}{r+r_1}\text{, and }\frac{qq_1}{9pp_1}=(rr_1)^{\frac{1}{3}}.$$

15. The roots of the equation $x^3-px^2+qx-r=0$ are a, b, c, and the roots of $x^3-p'x^2+q'x-r'=0$ are a, b, c'. Determine c and c' in terms of the coefficients of the equations.

XXVI.

1. Shew that $\frac{10}{3+x}+\frac{11}{5+x}+\frac{12}{7+x}=1$, and $\frac{3x-1}{x+3}+\frac{2x-1}{x+5}+\frac{x-5}{x+7}=5$ are identical equations with real roots.

2. Solve the equations $x^3-7x^2+36=0$ and $x^3-3x^2-10x+24=0$, the former of which has one root equal to 3 times one of the roots of the latter.

3. Having given

$$(x-1)(x-2)(x-3)+a(x-1)(x-2)+b(x-1)+c=0,$$

determine the values of a, b, c, so that 4, 5, 6 may be the three roots of the equation.

4. If $x^3+px^2+qx+r=0$, of which the roots a, b, c are real, be satisfied by $x=3+\sqrt{8}$, it will be satisfied by $x=-r$.

5. The roots of the equation $ax^2+bx+c=0$, are two of the roots of the cubic equation $a^2x^3+(b^2-ac)bx+c^3=0$; find the third root.

6. If a, b, c be the roots of the equation $x^3-3x+7=0$, determine the numerical values of $a^4+b^4+c^4$ and $a^2b^2+a^2c^2+b^2c^2$.

XXVII.

1. If s be the sum and p the product of the roots of the equation $x^4(ax+b)-(a+bx)=0$, then $as+bp=0$.

2. Given the equation $x^3-px^2+qx-r=0$ whose roots are a, b, c; express $(a-b)^2$ in terms of c.

3. If $pq=r$, what is the nature and value of the roots in the equation $x^3-px^2+qx-r=0$? Ex. $x^3-3x^2+4x-12=0$.

4. Solve the equation $x^3+px^2+qx+r=0$, when $3pr=q^2$, and apply it to find the roots of $x^3-4x^2+6x-3=0$.

5. If a, b, c, arranged in order of magnitude, be the roots of $x^3-3px^2+3qx-r=0$; shew that the equation $rx^2-2q^2x+rq=0$ contains the greatest and least of them.

6. Determine the ratio of x to y in the equation

$$ax^3+bx^2y+cxy^2+dy^3=0.$$

7. Solve the equations $ax^3+bx^2y+cxy^2+dy^3=0$, and

$$dx^3+cx^2y+bxy^2+ay^3=0.$$

8. Given $x+y=z$, $x^2+y^2=a^2$, $x^3+y^3=b^3$; shew that $z^3-3a^2z+2b^3=0$.

9. Prove that the product $(x^2+ax+m)(x^2+bx+m)(x^2+cx+m)$ will contain no powers of x whose indices are not multiples of 3, if $a+b+c=0$ and $a^2+b^2+c^2=6m$; and by means of the resulting expression determine the roots of $x^6-20x^3+343=0$.

10. The equation $\dfrac{a_1}{x+b_1}+\dfrac{a_2}{x+b_2}+\dfrac{a_3}{x+b_3}+\dfrac{a_4}{x+b_4}=0$ will be satisfied by only one finite value of x, if $a_1+a_2+a_3+a_4=0$,

and $a_1b_1+a_2b_2+a_3b_3+a_4b_4=0$.

11. Find the values of x, y, z from the equations $xy+xz+yz=26$,

$$xy(x+y)+xz(x+z)+yz(y+z)=168,$$

and $xy(x^2+y^2)+xz(x^2+z^2)+yz(y^2+z^2)=538$.

12. The surface of a rectangular parallelopiped is 22 square inches, the length of its diagonal is $\sqrt{14}$ linear inches, and its solid content is 6 cubic inches; find its edges.

13. Shew that impossible roots in an equation lie under the form of two positive or two negative roots in equations in which they occur. Determine whether the equation $x^3-(a^2+b^2+c^2)x-2abc=0$ has impossible roots.

XXVIII.

1. If $x^4+px^3+qx^2+rx+1=0$ and $x^4+rx^3+qx^2+px+1=0$, have a common root, then $p+r=q+2$.

2. If $x^4+px+q=0$, and $x^4+qx^3+p=0$, have two roots in common; find the relation between p and q.

3. The equation $x^4+rx+s=0$ has equal roots when $\left(\dfrac{r}{4}\right)^4=\left(\dfrac{s}{3}\right)^3$; and no real root when $\left(\dfrac{r}{4}\right)^4$ is less than $\left(\dfrac{s}{3}\right)^3$.

4. The equation $x^4-px^3+qx^2-rx+s=0$ has two pairs of equal roots: find their value in terms of the coefficients.

5. If $x^4+4px^3+6qx^2+4rx+s=0$ have three equal roots, prove that $qs+2pqr-r^2-p^2s-q^3=0$, $s-4pr+3q^2=0$.

6. Shew how to find the equal roots of an equation. Does the method apply to the case where the same imaginary root occurs more than once in an equation? Apply the method to find the equal roots of the equation

$$x^4+13x^3+33x^2+31x+10=0.$$

XXIX.

1. Shew that the value of the expression

$$x^4-28x^3+277x^2-1134x+792,$$

if x be between 6 and 8, cannot exceed 8.

2. Resolve x^5-5x^3+4x into its simple factors.

3. Shew that $x^4+2x^3+qx^2+rx+s=0$ becomes a quadratic, if $q=r+1$. Solve the equation $x^4+2x^3+9x^2+8x-9=0$.

4. Solve the equation

$$(x+1)(x+2)(x+3)(x+4)=(x+1)^2+(x+2)^2+(x+3)^2+(x+4)^2.$$

5. Find the roots of the following equations, if the equation $x^4-10x^3+35x^2-50x+24=0$ contain all the different roots of the equations $x^3-7x^2+14x-8=0$ and $x^3-6x^2+11x-6=0$.

XXX.

1. In the equation $x^4-px^3+qx^2-rx+s=0$, determine the conditions (1) that the sum of two of the roots shall be equal to the sum of the other two, and (2) that the product of two of the roots shall be equal to the product of the other two.

2. Solve the equations $x^4-2ax^3+(a^2-2b^2)x^2+2ab^2x-a^2b^2=0$, and

$$2x(x^2+3x-7)(x^2+x-3)=21(x^2+3x-3)-9(x^2+3x-7).$$

3. The equation $x^4-2(p+q)x^3+2(p-q)x-1$ has four real roots when $27p^2q^2-(q^2-p^2-1)^3$ is negative.

4. If -1, $-a$, $-b$, $-c$ be the roots of the biquadratic equation

$$x^4+p_1x^3+p_2x^2+p_3x+p_4=0,$$

then shall $p_1-p_2+p_3-p_4=1$.

5. If a, b, c be the roots of the cubic equation $x^3-p_1x^2+p_2x-p_3=0$,

$$\text{then } (1-p_2)^2+(p_1-p_3)^2=(1+a^2)(1+b^2)(1+c^2).$$

And if a, b, c, d be the roots of $x^4-p_1x^3+p_2x^2-p_3x+p_4=0$,

$$\text{then also } (1-p_2+p_4)^2+(p_1-p_3)^2=(1+a^2)(1+b^2)(1+c^2)(1+d^2).$$

6. Given $x^4+ax+b=0$ and $x^2+cx+y=0$; shew that the final equation determining y will be a biquadratic wanting the second and fourth terms if $c^6+4bc^2-a^2=0$.

7. Required the conditions to be satisfied that in the division of x^4-qx^2-rx-s by x^2+ax+b the remainder may be zero independent of the value of x after three terms of the quotient have been obtained; and when these conditions are satisfied, obtain an equation for finding b from q, r, s given, and shew how to solve it.

XXXI.

1. If the coefficients of a reciprocal equation be whole numbers, shew that it can have no commensurable roots except $+1$ and -1.

2. A recurring equation of five dimensions can be reduced to one of four dimensions, and the roots can be found by the solution of an equation of two dimensions, and a recurring equation of six dimensions may be solved by means of an equation of three dimensions.

3. A recurring equation of four dimensions has its roots of the form a, $\frac{1}{a}$; b, $\frac{1}{b}$; and an equation having all its coefficients whole numbers cannot have a fractional root. Shew that these two propositions are not inconsistent.

4. In the transformation of a given equation into one whose roots shall be the reciprocals of the roots of the given equation; find under what conditions the transformed equation will be identical to the given equation.

5. Transform the equation $x^4+x^3+x^2+x+1=0$ into one whose roots shall be the squares of the roots of the given equation; and shew from the roots themselves that the transformation is correct.

6. Prove that the equation whose roots are a, b, $\frac{1}{a}$, $\frac{1}{b}$ is

$$q(x^4+1)+(q^2+p^2+1)x^2=px(q+1)(x^2+1).$$

7. Transform $x^4+4x^3-8x-4=0$ into a reciprocal equation by increasing each of the roots by the same quantity; and then find its roots.

8. Determine the roots of $x^4-4.2^{\frac{1}{4}}x^3+6.4^{\frac{1}{4}}.x^2-4.8^{\frac{1}{4}}.x+2=0$.

9. Shew that the sum of the squares of the roots of the recurring equation $x^4+px^3+qx^2+px+1=0$ is less than the sum of the squares of the roots of the reducing quadratic by 4.

10. In the recurring equation $x^4+px^3+qx^2+px+1=0$, whose roots are a, b, c, d, prove that

$$\frac{a^2}{b^2}+\frac{b^2}{a^2}+\frac{a^2}{c^2}+\frac{c^2}{a^2}+\&c.=(p^2+2q+2)(p^2-2q-2).$$

XXXII.

1. Determine the relations which subsist between p, q, r, in the equation $x^3-px^2+qx-r=0$, (1) when the roots are in arithmetical progression, (2) when they are in harmonical progression, and (3) when they are in geometrical progression.

2. Form the equation whose roots shall be the arithmetic, harmonic, and geometric means between the roots of the equation $x^2-px+q=0$.

3. Transform $x^3-px^2+qx-r=0$, whose roots are a, b, c, into an

equation whose roots shall be mean proportionals between every two roots.

4. Solve the equation $(x+a_1)(x+a_2)(x+a_3)(x+a_4)=d$; a_1, a_2, a_3, a_4 being in arithmetical progression.

5. If $x^3-px^2+qx-r=0$ have its roots in harmonical progression, shew that the equation $q^2x^2-(pq^2-3qr)x+(q^3-3pqr+9r^2)=0$ contains the greatest and the least roots.

6. If the roots of $x^3-px^2+qx-r=0$ be in harmonical progression, and p, q be integral; then r is the square of the greatest root.

7. The harmonic mean of two roots of the equation

$$6x^3-x^2-75x+100=0 \text{ is } 2;$$

find the roots.

8. Find in terms of m and a the roots of the equation

$$x^4+pax^3+(m^2+m)a^2x^2+rax+a^4=0,$$

which are in geometric progression.

9. If the roots of the equation $ax^4+bx^2+cx+d=0$ be in harmonical progression, prove that $c^2=4bd$, and $c^4+400ad^3=0$.

10. Shew that if the two roots of a biquadratic equation be an arithmetic and harmonic mean respectively between the remaining two, the product of the greatest and least roots, or of the two others, shall be equal to the square root of the last term of the equation.

Hence solve the equation $x^4-30x^3+320x^2-1440x+2304=0$.

XXXIII.

1. Shew the advantage which arises in the solution of a cubic equation by Cardan's rule, by first transforming the complete equation $x^3+p_1x^2+p_2x+p_3=0$ into the form $x^3+qx+r=0$.

2. Point out from the inconsistency of the assumptions made in Cardan's solution of a cubic equation, that the method fails when the three roots are real, except in one case.

3. If $x=(\pm a^3)^{\frac{1}{3}}+(\pm b^3)^{\frac{1}{3}}$, determine all the values of x for which the values of $(\pm a^3)^{\frac{1}{3}}.(\pm b^3)^{\frac{1}{3}}$ are possible.

4. If 1, α, β, be the roots of $x^3+1=0$, shew that $\alpha^3+\beta^3=-2$.

5. If the equation $x^3-qx+r=0$ have two of the roots $a\pm b\sqrt{-3}$; then shall $-\frac{r}{2}+\left\{\frac{r^2}{4}-\frac{q^3}{27}\right\}=(b-a)^3$.

6. If a be the possible root of the cubic $x^3-qx+r=0$, which can be solved by Cardan's method; then must a be greater than $2\sqrt{\frac{q}{3}}$.

7. In the solution of a cubic equation by Cardan's method, shew that the expression $-\frac{1}{2}\{(y+z)\pm(y-z)\sqrt{-3}\}$ contains the two impossible roots.

8. The formula $x=\{r+\sqrt{(r^2-q^3)}\}^{\frac{1}{3}}+\{r-\sqrt{(r^2-q^3)}\}^{\frac{1}{3}}$, expresses the complete solution of the equation $(x^3-2r)^3-27q^3x^3=0$. In what

manner can the formula be modified, so as to express the roots of the equation $x^3-3qx-2r=0$ only,

XXXIV.

1. If e be the sum of two roots of the equation $x^4+qx^2+rx+s=0$, then (without consideration of the reducing cubic) shew that e^2 has only three different values.

2. Express the four values of the unknown quantity in the biquadratic equation in terms of the roots of reducing cubic equation.

3. If a biquadratic equation, when deprived of its second term, has the coefficient of x^2 positive, two of the roots at least are imaginary.

4. Prove that Descartes' solution of a biquadratic equation succeeds when all the roots are possible, and when two of them are equal.

5. The sum of the roots of Descartes' reducing cubic is equal to the sum of the squares of the roots of the biquadratic.

6. If a be one of the roots of Descartes' reducing cubic, the four roots of the equation $x^4+qx^2+rx+s=0$, are

$$-\frac{\sqrt{a}}{2}\pm\left\{-\frac{q}{2}-\frac{a}{4}+\frac{r}{2\sqrt{a}}\right\}^{\frac{1}{2}}, \text{ and } +\frac{\sqrt{a}}{2}+\left\{-\frac{q}{2}-\frac{a}{4}-\frac{r}{2\sqrt{a}}\right\}^{\frac{1}{2}}.$$

7. Prove that the roots of Descartes' reducing cubic are respectively quadruple of the roots of Euler's reducing cubic.

8. Express the roots of the cubic equation employed in the process of solving a biquadratic equation by Euler's method, in terms of the roots of the biquadratic.

9. Explain why the general root of Euler's solution of a biquadratic is assumed to be entirely surd, and on what the index of the surd depends.

10. Find the roots of the equation

$$x^4-2(a^2+b^2+c^2)x^2-8abcx+2(a^4+b^4+c^4)-(a^2+b^2+c^2)^2=0$$

by Euler's method of solving a biquadratic equation.

11. Shew that the reducing cubic in Euler's solution of a biquadratic equation will have its roots real, except when the original equation has two possible and two impossible roots.

12. In the solution of the biquadratic equation by the methods of Descartes and Waring, shew that the product of the roots of the reducing cubics will be the same in each case if $7r^2+4qs=0$.

13. Shew that if $x^4+4bx^3+6cx^2+4dx+e$ can be expressed as the difference of two squares $(x^2+2bx+m)^2-(px+q)^2$,

$$\text{then } m^3-3cm^2+(4bd-e)m+3ec-2b^2e-2d^2=0.$$

14. Shew that the solution of $x^4+px^2+4rx+s=0$ may be made to depend on that of the cubic $4y^3+(4p-s)y-(p^2s-r^2)=0$; and if a, b, c, d be the roots of the given biquadratic, the roots of the auxiliary cubic are $\frac{1}{4}(ab+cd)$, $\frac{1}{4}(ac+bd)$, $\frac{1}{4}(ad+bc)$.

XXXV.

1. If a, b, c, d be the four roots of the equation $x^4-px^3-qx^2-rx+s=0$, determine the equation whose roots shall be respectively the products of every three roots of the proposed equation.

2. If α, β be two roots of $x^4+px^2+qx+r=0$; shew that the other two roots will be $-\frac{\alpha+\beta}{2}\pm\left\{\left(\frac{\alpha-\beta}{2}\right)^2+\frac{q}{\alpha+\beta}\right\}^{\frac{1}{2}}$.

3. If the equation $x^3+px^2+r=0$ and $x^3+qx+s=0$ have a common root; determine the equation which connects the coefficients p, q, r, s.

4. The equation $x^4-px^3+qx-1=0$ has four real roots if

$$\left\{\frac{p+q}{4}\right\}^{\frac{2}{3}}-\left\{\frac{p-q}{4}\right\}^{\frac{2}{3}}$$

be greater than unity, and only two in the contrary case.

5. Determine the four roots of the biquadratic $x^4+4a^3x-a^4=0$ by means of a quadratic equation.

6. Determine the four roots of the biquadratic equation

$$\frac{a^4(x-b)(x-c)}{(a-b)(a-c)}+\frac{b^4(x-c)(x-a)}{(b-c)(b-a)}+\frac{c^4(x-a)(x-b)}{(c-a)(c-b)}=x^4.$$

7. Depress to a cubic the equation

$$x^6-px^5+qx^4-rx^3+qsx^2-ps^2x+s^3=0.$$

8. Reduce $x^9-px^8+qx^7-rx^6+sx^5-sx^4+rx^3-qx^2+px-1=0$ to an equation of four dimensions.

9. Reduce the equation

$$x^9+x^8-9x^7+3x^6-8x^5-8x^4+3x^3-9x^2+x+1=0$$

to simple and quadratic equations, and find all the nine roots.

10. Shew that the equation

$$v^4-\{(b^2+c^2)l^2+(c^2+a^2)m^2+(a^2+b^2)n^2\}v^2+b^2c^2l^2+c^2a^2m^2+a^2b^2n^2=0,$$

when $l^2+m^2+n^2=1$, can be put into the form

$$\frac{l^2}{v^2-a^2}+\frac{m^2}{v^2-b^2}+\frac{n^2}{v^2-c^2}=0.$$

HINTS, RESULTS, ETC., FOR THE EXERCISES ON EQUATIONS OF THE THIRD AND FOURTH DEGREE.

I.

1. See Section VIII. pp. 1, 2. The solution of a quadratic equation is effected by making the terms which involve the unknown quantity a complete second power. If the terms involving the unknown quantity of a cubic or biquadratic equation be arranged on one side, and the known quantity placed on the other, there is no general method known by which the terms can be increased or diminished so as to become a complete third power in one case, or a complete fourth power in the other.

3. See p. 3. 5. See p. 3. 7. See p. 3.

II.

1. Let $x=3$, $x=4$, $x=5$; then $x-3=0$, $x-4=0$, $x-5=0$,
and $(x-3)(x-4)(x-5)=0$, or $x^3-12x^2+47x-60=0$.

2. $x^3-6x^2-7x+60=0$.
3. $x^3+2x^2-23x-60=0$.
4. $x^3+12x^2+47x+60=0$.
5. $x^3-3x^2-2x+6=0$.
6. $x^3-9x^2+21x-5=0$.
7. $x^3-8x^2+23x-28=0$.
8. $x^3-x^2+2=0$.

III.

1. $x^4-14x^3+71x^2-154x+120=0$.
2. $x^4+14x^3+71x^2+154x+120=0$.
3. $x^4-27x^2+14x+120=0$.
4. $x^4-20x^3+94x^2+100x-495=0$.
5. $x^4-12x^2+27=0$.
6. $x^4+4x^3-2x^2+12x-15=0$.
7. $x^4+2x^2-15=0$.
8. $x^4-8x^2+7x^2-x-2=0$.
9. $x^4+2x^3-3x^2+2x+6=0$.
10. $x^4-10x^3+32x^2-34x+7=0$.
11. $x^4-8x^3+24x^2-32x+7=0$.
12. $x^4-x^3+2x=0$.
13. $x^4-2x^3+4x^2-2x+1=0$.
14. $x^4-6x^3+18x^2-30x+65=0$.

IV.

1. Since 5 is one of the roots, $x=5$ and $x-5=0$, and $\dfrac{x^3-3x-110}{x-5}=0$, or $x^2+5x+22=0$, a quadratic which gives the other two roots, $x=\frac{1}{2}(-5\pm\sqrt{-63})$.

2. $x=2$, $x=3$ the other two roots. 3. $x=2\frac{1}{2}$, $x=3\frac{1}{3}$ the other two roots.

4. Since $+\sqrt{3}$ is one root, $-\sqrt{3}$ is another, then $x=+\sqrt{3}$, $x=-\sqrt{3}$; $x-\sqrt{3}=0$, $x+\sqrt{3}=0$, $(x-\sqrt{3})(x+\sqrt{3})=0$, or $x^2-3=0$, then $\dfrac{x^3+8x-3x-24}{x^2-3}=0$, or $x+8=0$, $\therefore x=-8$, the third root.

5. $x=4$.
6. $x=-2$.
7. $x=7$.
8. $x=1\pm\sqrt{-2}$.
9. $x=-1$, $x=-2$.
10. $x=-1\pm\sqrt{3}$.
11. $x=2$, $x=-1$.
12. $x=3$, $x=-2$.
13. $x=-1$, $x=-2$.
14. $x=4$, $x=-3$.
15. $x=3$, $x=2$.

V.

1. The equation $x^3-3x^2-x+3=0$ is the same as $x^2(x-3)-(x-3)=0$, or $(x-3)(x^2-1)=0$, $\therefore x-3=0$ and $x^2-1=0$, $\therefore x=3$ and $x=\pm 1$.

2. $x^3-3x+2=0$, or $x^3-x-2x+2=0$, or $x(x^2-1)-2(x-1)=0$,
or $(x-1)(x^2+x+2)=0$, $\therefore x-1=0$ and $x^2+x+2=0$.

8. $2x^3-x-1=0$, or $x^3-x+x^3-1=0$, or $x(x^2-1)+(x-1)(x^2+x+1)=0$,
or $(x-1)(2x^2+2x+1)=0$, $\therefore\ x-1=0$, $2x^2+2x+1=0$.

11. $(x-5)(x^2-5x+5)+x=5$, $(x-5)(x^2-5x+5)+(x-5)=0$,
or $(x-5)(x-5x+6)=0$.

12. $x(x+1)(x+2)=9\times 8\times 7=7(7+1)(7+2)$, an identity, $\therefore\ x=7$, one of the roots.

19. The equation $(x-3)(x-4)(x-5)(x-6)=1.2.3.4$
$=(7-6)(7-5)(7-4)(7-3)$, an identical expression:
$\therefore$ 7 is one of the roots. Or thus,

Since the addition of unity to the product of four consecutive numbers makes the m a complete square, $\therefore\ (x-3)(x-4)(x-5)(x-6)+1=1.2.3.4+1$,
expanding and extracting the square root of each side, $x^2-9x+19=\pm 5$, which gives the four roots of the equation.

VI.

1. When two or more equations have roots in common, whether real or imaginary, the equations involve each a common factor which will contain the equal root or roots.

If $x^3-3x^2+11x-9=0$ and $x^3-5x^2+11x-7=0$ contain a common root; $x^3-3x^2+11x-9$ and $x^3-5x^2+11x-7$ have a common factor, which will be found to be $x-1$, $\therefore\ x-1=0$ and $x=1$, the root common to the two equations.

If the two expressions have no common factor, there is no root common to the two equations.

VII.

When an equation contains two or three equal roots, the limiting equation will always contain one or two of these roots respectively, and the given and limiting equations involve a common factor.

1. The limiting equation of $x^3-7x^2+16x-12=0$ is $3x^2-14x+16=0$, and the expressions $3x^2-14x+16$ and $x^3-7x^2+16x-12$ have a common factor which by the process described, Section V. pp. 1, 2, will be found to be $x-2$, and $x-2=0$, $\therefore\ x=2$, one of the equal roots.

7. The limiting equation of the biquadratic $x^4-6x^2+8x-3=0$ is
$4x^3-12x+8=0$, or $x^3-3x+2=0$, the expressions x^3-3x+2 and x^4-6x^2+8x-3 have a common factor, x^2-2x+1, and $x^2-2x+1=0$, or $(x-1)^2=0$ and $x=1$.

The given equation has three roots each equal to 1.

VIII.

1. The three roots $x^3+1=0$ are 1, $\frac{1}{2}(-1+\sqrt{-3})$, $\frac{1}{2}(-1-\sqrt{-3})$: and the three roots of $x^3-1=0$ are -1, $\frac{1}{2}(1+\sqrt{-3})$, $\frac{1}{2}(1-\sqrt{-3})$.

Also if $a=\dfrac{-1+\sqrt{-3}}{2}$, then $\dfrac{-1-\sqrt{-3}}{2}=\dfrac{-1-\sqrt{-3}}{2}\times\dfrac{-1+\sqrt{-3}}{-1+\sqrt{-3}}$

$$=\frac{1+3}{2(-1+\sqrt{-3})}=\frac{2}{-1+\sqrt{-3}}=\frac{1}{a}.$$

2. The roots of $x^3+x^2+x+1=0$ are -1, $\pm\sqrt{-1}$: and of $x^3-x^2+x-1=0$ are 1, $\pm\sqrt{-1}$.

3. $x=1$, $x=2\pm\sqrt{3}$. 4. -1, $\frac{1}{2}(2\pm\sqrt{-5})$. 5. $x=\frac{1}{2}(\mp\sqrt{5}\pm 1)a$.

6. $x^2-3x+1=0$, $x^2+2x+1=0$ are the two quadratics which give the four roots.

7. $2x^4-5x^3+4x^2-5x+2=0$, $x^4-\dfrac{5}{2}x^3+2x^2-\dfrac{5}{2}x+1=0$,

$(x^4+2x^2+1)-\dfrac{5}{2}x(x^2+1)=0$, $(x^2+1)^2-\dfrac{5}{2}x(x^2+1)=0$, $(x^2+1)\left(x^2+1-\dfrac{5x}{2}\right)=0$,

$\therefore\ x^2+1=0$, $x^2-\dfrac{5x}{2}+1=0$, are the two quadratics for finding the four roots.

9. $x^5+2x^4+2x+1=0$, or $(x^5+1)+2x(x^3+1)=0$, or $(x+1)(x^4+x^3-x^2+x+1)=0$; ∴ $x+1=0$, and $x^4+x^3-x^2+x+1=0$, or

$$(x^2+1)^2+x(x^2+1)=3x^2, \text{ a quadratic, from which } x^2+1+\frac{x}{2}=\frac{\pm x\sqrt{13}}{2}.$$

11. $x^5+2x^4+3x^3+3x^2+2x+1=0$, or $(x^5+1)+2x(x^3+1)+3x^2(x+1)=0$, or $(x+1)(x^4+x^3+2x^2+x+1)=0$, ∴ $x+1=0$ and $x=-1$, and $x^4+x^3+2x^2+x+1=0$,

$$\therefore \left(x^2+2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)+1=0,\quad \left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)+1=0.$$

$$\text{Let } x+\frac{1}{x}=z, \text{ then } z^2+z+1=0.$$

12. The five roots are -1, $\frac{1}{2}(3\pm\sqrt{5})$, $\frac{1}{2}(19\pm\sqrt{357})$.

16. $x^6-3x^5+6x^4-6x^2+3x-1=0$, or $(x^2-1)(x^4-3x^3+6x^2-3x+1)=0$,

$$\therefore x^2-1=0 \text{ and } x^4-3x^3+6x^2-3x+1=0.$$

IX.

Of the equations under this number each contains at least one root $+1$ or -1, which is rendered possible by certain relations between the coefficients of the terms of the equations. See Section IV. pp. 23, 24.

1. The equation $x^3+3x^2+9x-13=0$ has the algebraical sum of its coefficients with their proper signs equal to zero. In this case $x^3+3x^2+9x-13$ is divisible by $x-1$, and if $x-1=0$, $\frac{x^3+3x^2+9x-13}{x-1}=0$, or $x^2+4x+13=0$ is the quadratic which contains the other two roots.

5. $x^3-9x^2+23x-15=0$. Here $1-9+23-15=0$, and $x-1$ is a divisor of $x^3-9x^2+23x-15$, then $\frac{x^3-9x^2+23x-15}{x-1}=0$, or $x^2-8x+15=0$.

6. $7x^3+5x^2+x+3=0$, in which the sum of the first and third coefficients is equal to the sum of the second and fourth, or $7+1=5+3$, and $x+1$ is a divisor; if $x+1=0$, $\frac{7x^3+5x^2+x+3}{x+1}=0$, or $7x^2-2x+3=0$, a quadratic which contains the other two roots.

13. $x^4+13x^3+33x^2+31x+10=0$. Here $1+33+10=13+31$, and $x+1$ is a divisor.

$$\text{If } x+1=0, \ \frac{x^4+13x^3+33x^2+31x+10}{x+1}=0, \text{ or } x^3+12x^2+21x+10=0.$$

And since $1+21=12+10$ in the equation $x^3+12x^2+21x+10=0$, $x+1$ is a divisor, and if $x+1=0$, $\frac{x^3+12x^2+21x+10}{x+1}=0$, or $x^2+11x+10=0$.

And again since $1+10=11$ in the equation $x^2+11x+10=0$, $x+1$ is a divisor,

$$\text{and if } x+1=0, \ \frac{x^2+11x+10}{x+1}=0, \text{ or } x+10=0.$$

The equation is $(x+1)^3(x+10)=0$, which has one root -10, and three roots each -1.

14. $x^4-5x^3-5x^2+45x-36=0$. Here $1-5-5+45-36=0$, $x-1$ is a divisor; if $x-1=0$, then $\frac{x^4-5x^3-5x^2+45x-36}{x-1}=0$, or $x^3-4x^2-9x+36=0$, a cubic which by inspection can be put into the form $x^2(x-4)-9(x-4)=0$, or $(x-4)(x^2-9)=0$, ∴ $x-4=0$, and $x^2-9=0$, ∴ $x=4$, $x=\pm 3$, and $x=1$.

X.

(1) When the roots are in arithmetical progression.

1. $x^3 - 15x^2 + 66x - 80 = 0$. Let $a-b$, a, $a+b$ denote the three roots, then $-(a-b) - a - (a+b) = -15$, and $-(a-b) \times -a \times -(a+b) = -80$;

$\therefore\ 3a = 15$ and $a = 5$, also $5(25 - b^2) = 80$; $\therefore\ b^2 = 9$ and $b = \pm 3$;

$\therefore\ a - b = 5 - 3 = 2$, $a = 5$, $a + b = 8$, or 2, 5, 8 are the three roots of the equation.

(2) When the roots are in harmonical progression.

1. $x^3 - 11x^2 + 36x - 36 = 0$. Let a, b denote the first and third roots,

then $\frac{2ab}{a+b}$ is the second, and a, $\frac{2ab}{a+b}$, b, will denote the three roots in order,

and $a + \frac{2ab}{a+b} + b = 11$, $\frac{2a^2b^2}{a+b} = 36$. Find a and b from these equations.

Otherwise the given equation may be transformed into one whose roots shall be the reciprocals of the roots of the given equation, and then the roots of the transformed equation will be in arithmetical progression.

(3) When the roots are in geometrical progression.

1. $x^3 - 7x + 14x - 8 = 0$. Let $\frac{a}{b}$, a, ab denote the roots,

then $\frac{a}{b} + a + ab = 7$, $\frac{a}{b} \times a \times ab = 8$ or $a^3 = 8$; $\therefore\ a = 2$, and $b = 2$ and $\frac{1}{2}$;

therefore 1, 2, 4 are the roots.

XI.

1. See Art. 10, p. 12.
2. See Art. 10, p. 12. The required equations are $y^3 - 11y + 20 = 0$ and $y^4 - 15y^2 - 6y + 40 = 0$.
3. The resulting equations are $27y^3 + 98 = 0$, and $y^4 + 92y = 0$.
4. Let $x = y + e$, then the equation $x^3 - 5x^2 + 7x - 12 = 0$ becomes

$$y^3 + (3e - 5)y^2 + (3e^2 - 10e + 7)y + (e^3 - 5e^2 + 7e - 12) = 0,$$

and it is required that $3e^2 - 10e + 7 = 0$, whence $e = 1$ and $\frac{7}{3}$.

If $x = 1$, then $y^3 - 2y^2 + 2 = 0$. If $e = \frac{7}{3}$, then $27y^3 + 54y^2 - 275 = 0$.

(2) Let $x = y + e$, then the equation $x^4 - 12x^3 + 30x^2 - 16x + 48 = 0$ becomes

$$y^4 + (4e - 12)y^3 + (6e^2 - 36e + 30)y^2 + (4e^3 - 36e^2 + 60e - 16)y$$
$$+ (e^4 - 12e^3 + 30e^2 - 16e + 48) = 0.$$

Here $6e^2 - 36e + 30 = 0$, and $e = 5$ and 1.

If $e = 1$, then $y^4 - 8y^3 + 12y + 51 = 0$. If $e = 5$, $y^4 + 8y^3 - 116y - 152 = 0$.

13. First find the roots of the given equation, next form the equations whose roots shall be increased and diminished by the given numbers.

XII.

These exercises are subject to the general method, of which an example is given in the note, p. 7. When the coefficient of the highest power of the unknown quantity is greater than unity, let the equation be transformed into another having unity for the coefficient of the highest power of the unknown quantity. The number of substitutions may be restricted by finding the superior and inferior limits of the roots of the given equation.

XIII.

1. p^2-2q. 2. $p^3-3pq+3r$. 3. $p^4-4p^2q-4pr+2q^2$.

4. $\dfrac{q^2-2pr}{r^2}$. 5. $\dfrac{q^3-3pqr+3r^2}{r^3}$. 6. $\dfrac{q^4-4pq^2r+3p^2r^2+2pr^3}{r^4}$.

7. $p+\dfrac{q}{r}$. 8. $\left(p^2+\dfrac{q^2}{r^2}\right)-2\left(q+\dfrac{p}{r}\right)$. 9. $pq-r$.

10. $q^3-3pqr+3r^2$. 11. $p^5-5p^3q+4p^2r+3pq^2+2p^2q-5qr$. 12. $\dfrac{pq}{r}$.

13. $\dfrac{pq-3r}{r}$. 14. $\dfrac{p^2+q^2+r^2+1}{r}$.

15. $\left\{\dfrac{a-b}{a+b}\right\}^2+\left\{\dfrac{b-c}{b+c}\right\}^2+\left\{\dfrac{c-a}{c+a}\right\}^2$

$$=\frac{(a-b)^2(b+c)^2(c+a)^2+(b-c)^2(a+b)^2(c+a)^2+(c-a)^2(a+b)^2(b+c)^2}{(a+b)^2(b+c)^2(c+a)^2}$$

$$=\frac{3(a^2+b^2)(b^2+c^2)(c^2+a^2)+2\{a^3b^3+a^3c^3+b^3c^3+abc(a^3+b^3+c^3)-2abc[(a+b+c)(ab+ac+bc)-3abc]\}-24a^2b^2c^2}{(a+b)^2(b+c)^2(c+a)^2}$$

$$=\frac{3(a^2+b^2+c^2)(a^2b^2+a^2c^2+b^2c^2)+6a^2b^2c^2+2\{a^3b^3+a^3c^3+b^3c^3+abc(a^3+b^3+c^3)-4abc(a+b+c)(ab+ac+bc)+12a^2b^2c^2-24a^2b^2c^2}{(a+b)^2(b+c)^2(c+a)^2}$$

$$=\frac{3(a^2+b^2+c^2)(a^2b^2+a^2c^2+b^2c^2)+2\{a^3b^3+a^3c^3+b^3c^3+abc(a^3+b^3+c^3)\}-4abc(a+b+c)(ab+ac+bc)-6a^2b^2c^2}{(a+b)^2(b+c)^2(c+a)^2}.$$

But since $a+b+c=p$, $ab+ac+bc=q$, $abc=r$, $\therefore a^2+b^2+c^2=p^2-2q$,
$a^2b^2+a^2c^2+b^2c^2=q^2-2pr$, $a^3+b^3+c^3=p^3-3pq+3r$.
Also since $(a+b+c)^3=a^3+b^3+c^3+3(a+b)(b+c)(c+a)$,
$\therefore p^3=p^3-3pq+3r+3(a+b)(b+c)(c+a)$,
and $(a+b)(b+c)(c+a)=pq-r$, also $a^3b^3+a^3c^3+b^3c^3=q^3-3r(pq-r)$,

$\therefore \left\{\dfrac{a-b}{a+b}\right\}^2+\left\{\dfrac{b-c}{b+c}\right\}^2+\left\{\dfrac{c-a}{c+a}\right\}^2$

$$=\frac{3(p^2-2q)(q^2-2pr)+2\{q^3-3pqr+3r^2+r(p^3-3pq+3r)\}-4pqr-6r^2}{(pq-r)^2}$$

$$=\frac{3p^2q^2-4(p^3r+pqr+q^3)+6r^2}{(pq-r)^2}$$

16. $(a^{\frac{1}{3}}+b^{\frac{1}{3}}c^{\frac{1}{3}})(b^{\frac{1}{3}}+a^{\frac{1}{3}}c^{\frac{1}{3}})(c^{\frac{1}{3}}-a^{\frac{1}{3}}b^{\frac{1}{3}})=(a^2+b^2+c^2)a^{\frac{2}{3}}b^{\frac{2}{3}}c^{\frac{2}{3}}-(a^2b^2+a^2c^2+b^2c^2)$
$=(p^2-2q)r^{\frac{2}{3}}-(q^2-2pr)$.

XIV.

The exercises under this number require no remarks, as they may be effected in the same manner as those under the preceding number.

1. p^2-2q. 2. $p^3-3pq+3r$. 3. $p^4-4p^2q+4pr+2q^2-4s$.

12, 13 are the coefficients, $q^2-2pr+2s$ and $2qs-r^2$, of the third and fourth terms of the equation which contains as its roots the squares of the roots of the given equation.

14. $p^5-5p^3q+4p^2r+3pq^2+2p^2q-5qr-5ps$.

15. The sum of the total number is $60(a+b+c+d)$. The roots of
$x^4-5x^3+5x^2+5x-6=0$ are ± 1, 2, 3.

XV.

1. $x^3-13x^2+44x-32=0$. Let a, $2a$, b denote the roots,
then $3a+b=13$, $2a^2b=32$, $a^2b=16$, $b=13-3a$, $\therefore\ 3a^3-13a^2+16=0$,
or $3a^3+3-13a^2+13=0$, or $3(a^3+1)-13(a^2-1)=0$,
$\therefore\ (a+1)\{3(a^2-a+1)-13(a-1)\}=0$, or $(a+1)(3a^2-16a+16)=0$; $\therefore\ a+1=0$ and $3a^2-16a+16=0$, $\therefore\ a$ is known, and $b=13-3a$. The roots are -1, -2, $+16$.

2. $12x^3+13x^2-20x+4=0$, $\therefore\ x^3+\frac{13}{12}x^2-\frac{5}{3}x+\frac{1}{3}=0$. Let a, $-b$, c denote the roots, then the algebraical sum of two roots $a-b=-\frac{4}{3}$, and $-a+b-c=\frac{13}{12}$, $=\frac{1}{3}$; $\therefore\ c=-(a-b)-\frac{13}{12}=\frac{4}{3}-\frac{13}{12}=\frac{3}{12}=\frac{1}{4}$, and $ab=\frac{1}{3c}=\frac{4}{3}$.
From the equations $a-b=-\frac{4}{3}$ and $ab=\frac{4}{3}$; $a=\frac{2}{3}$ and -2, and $b=2$ and $-\frac{2}{3}$.
The three roots are $\frac{2}{3}$, -2, $\frac{1}{4}$.

3. Let a, b, c be the roots, then $a+b=13$, but $a+b+c=-15$, $\therefore\ c=-2$ one of the roots, and $abc=80$, $\therefore\ bc=40$ and $a+b=13$. Hence a and b can be found.

4. The three roots are -6, 3, 1.

5. The three roots are 3, 4, 5.

6. The roots are -2, -2, 4.

7. Let a, b, c denote the roots, then $a=\frac{1}{b}+\frac{1}{c}$ or $abc=b+c$.

8. If $x^3+4x^2-9x-36=0$ have two roots $\pm a$, then $x^3-4x^4-9x+36=0$ has two roots $\mp a$; $\therefore\ x^3+4x^2-9x-36$ and $x^3-4x^2-9x+36$ have a common quadratic factor.

10. The third root is obviously $+3$.

11. Here $a+b+c=10$, and $a=2b$, $b=3c$; $\therefore\ 2b+b+\frac{1}{3}b=10$, and $b=3$.

12. Let $2a$, $3a$, b denote the three roots.

13. Let a, $2a$, $3a$ denote the three roots.

14. $x=3$, $x-3=0$, and $\frac{x^3-4x^2+x+c}{x-3}=x^2-x-2$, the quotient with a remainder $c-6$.

15. See note pp. 1, 2, Section X. on polygonal numbers.

XVI.

1. Let b, c be the other two roots; then since a, $\frac{1}{a}$, b, c are the four roots.

Let $x^2+px+1=0$, the quadratic which contains the roots a, $\frac{1}{a}$,
and $(x-b)(x-c)=0$ or $x^2-(b+c)x+bc=0$ is the quadratic containing the roots b, c;
$\therefore\ (x^2+px+1)\{x^2-(b+c)x+bc\}=0$ is an equation identical with the proposed equation.
By equating coefficients of corresponding powers of x, it will be found that $p=-4$;
$\therefore\ x^2-4x+1=0$ contains two of the roots, and $\frac{x^4-4x^3+5x^2-16x+4}{x^2-4x+1}=0$,
or $x^2+4=0$ contains the other two roots.

4. Since 3 is the product of two of the roots, -4 is the product of the other two.
Let $x^2+mx-3=0$, $x^2+nx+4=0$ be the two quadratics which contain the same four roots, then
$(x^2+mx-3)(x^2+nx+4)=0$, or $x^4+(m+n)x^3+(mn+1)x^2+(4m-3n)x-12=0$
is an equation identical with $x^4-7x^3+11x^2+7x-12=0$.

7. Since the four roots are of the forms $a\pm1$, $b\pm1$. The equation formed with these roots is
$$x^4-(a+b)x^3+\{(a+b)^2+2(ab-1)\}x^2-2(a+b)(ab-1)x+(a^2-1)(b^2-1)=0,$$
which is identical with $x^4-10x^3+35x^2-50x+24=0$.

8. Since the roots are of the forms a, b, $\frac{a}{b}$, $\frac{b}{a}$; if the equation be transformed into another whose roots shall be the reciprocals of the roots of the given equation, the roots of the transformed equation will be $\frac{1}{a}$, $\frac{1}{b}$, $\frac{b}{a}$, $\frac{a}{b}$. Hence the two equations will each contain two roots in common.

9. Since $a+b\sqrt{-1}$ and $a+2b\sqrt{-1}$ are roots of the equation, $a-b\sqrt{-1}$ and $a-2b\sqrt{-1}$ are also the corresponding conjugate roots, and

$$x^4-4ax^3+(6a^2+5b^2)x^2-4(a^3+10ab^2)x+(a^4+5a^2b^2+4b^4)=0$$

is an equation identical with $x^4-4x^3+11x^2-14x+10=0$.

10. $x=a+b\sqrt{-1}$, $x=a-b\sqrt{-1}$; $\therefore$ $(x-a)-b\sqrt{-1}=0$, $(x-a)+b\sqrt{-1}=0$;

$$\therefore (x-a-b\sqrt{-1})(x-a+b\sqrt{-1})=0, \text{ or } x^2-2ax+(a^2+b^2)=0,$$

$$\text{and } \frac{x^4-2a^2x^2+4ab^2x+(a^4-b^4)}{x^2-2ax+(a^2+b^2)}=0, \text{ or } x^2+2ax+(a^2-b^2)=0$$

is the equation which contains the other two roots.

11. Since a is the sum of two of the roots, from the coefficient of the second term it may be inferred that a is also the sum of the other two roots.

12. If the equation $x^4-3x^3-7x^2+27x-18=0$ have two roots of the form $\pm a$, then the equation $x^4+3x^3-7x^2-27x-18=0$ has two roots of the form $\mp a$.

XVII.

These transformations offer some good practice for acquiring skill in the process of elimination.

In some cases, instead of the method of elimination, it may be found more convenient to determine the successive coefficients in terms of the roots of the equation.

1. Since $a+b+c=p$, then $a+b-c=p-2c$, $a+c-b=p-2b$, $b+c-a=p-2a$. Assume $y=p-2x$, then $x=\frac{p-y}{2}$. Substitute this value of x in the given equation,

and $\frac{(p-y)^3}{8}-\frac{p(p-y)^2}{4}+\frac{q(p-y)}{2}-r=0$, which when reduced becomes

$y^3-py^2+(4q-p^2)y+(p^3+4pq+r)=0$, the equation whose roots are

$a+b-c$, $a+c-b$, $b+c-a$.

2. Here $a+b+c=p$, $b+c=p-a$, and $a(b+c)=ap-a^2$. Assume $y=px-x^2$ and eliminate x between this and the given equation.

3. Let $y=x^2+x$. Eliminate x^3 and x^2 between $x^3-px^2+qx-r=0$ and and $y=x^2+x$, then $x=\frac{(p+1)y+r}{y+p+q+1}$. Substitute this value of x in the given equation, and reduce it to the ordinary form.

4. $a^2+bc=a^2+\frac{abc}{a}=a^2+\frac{r}{x}$, assume $y=x^2+\frac{r}{x}$ and eliminate x between this and the given equation.

5. First transform the given equation into one whose roots shall be a^2, b^2, c^2, next into an equation whose roots shall be, first the sums, secondly the difference of the roots of the transformed equation.

6. Here $a+b+c=p$, $b+c=p-a$, and $a^2(b+c)=a^2(p-a)$. Assume $y=px^2-x^3$, and eliminate x between this equation and the given equation.

7. Here $a+b+c=p$, $abc=r$, $a+b=p-c$, $ab=\frac{r}{c}$, $\therefore ab(a+b)=\frac{(p-c)r}{c}$.

$$\text{Assume } y=\frac{(p-x)r}{x}, \text{ then } x=\frac{pr}{y+r}.$$

8. Here $a+b+ab=a+b+c+ab-c=a+b+c+\frac{abc}{c}-c=p+\frac{r}{c}+c$.

Assume $y=p+\frac{r}{x}-x$, and eliminate.

9. Here $a+b+c=p$, $\therefore\ a+b=p-c$, $a+c=p-b$,

$\therefore\ (a+b)(a+c)=p^2-(b+c)p+bc=p^2-(p-a)p+\frac{abc}{a}=pa+\frac{r}{a}$. Assume $y=px+\frac{r}{x}$, and eliminate x between this and the given equation.

10. Here $a^3+b^3=(a+b)(a^2+b^2-ab)$

$$=\left\{(a+b+c)-c\right\}.\left\{(a^2+b^2+c^2)-c^2-\frac{abc}{c}\right\}$$

$$=\left(p-c\right)\left(p^2-2q-c^2-\frac{r}{c}\right).$$

Assume $y=\left(p-x\right)\left(p^2-2q-x^2-\frac{r}{x}\right)$, and eliminate x.

11, Assume $y=x^3+x^2+x$, and eliminate x^2 and x^3 between the equations $x^3-px^2+qx-r=0$ and $y=x^3+x^2+x$; then find the value of x in terms of y, and substitute this value of x in the given equation, and the resulting equation in y will be the equation required.

12. First transform the given equation into one whose roots shall be the sums of every two roots; next transform this equation into one whose roots shall be reciprocals of the roots of this equation.

13. Since $\frac{ab}{c}=\frac{abc}{c^2}=\frac{r}{c^2}$, assume $y=\frac{r}{x^2}$ and eliminate x between this and the given equation.

14. Here $\frac{a}{b}+\frac{b}{a}=\frac{a^2+b^2}{ab}=\frac{(a^2+b^2+c^2)-c^2}{\frac{abc}{c}}=\frac{(p^2-2q)c-c^3}{r}$.

Assume $y=\frac{(p^2-2q)x-x^3}{r}$, and eliminate x.

15. Transform the equation into one whose roots are a^2, b^2, c^2; next transform this equation into one whose roots shall be the reciprocals of the roots; thirdly, transform the last equation into one whose roots shall be the sums of every two roots.

Otherwise, $\frac{1}{a^2}+\frac{1}{b^2}=\frac{a^2+b^2}{a^2b^2}=\frac{a^2+b^2+c^2-c^2}{\frac{a^2b^2c^2}{c^2}}=\frac{p^2-2q-c^2}{\frac{r^2}{x^2}}$.

Let $y=\frac{p^2-2q-x^2}{\frac{r^2}{x^2}}=\frac{(p^2-2q)x^2-c^2x^4}{r^2}$ and substitute.

16. Since $a+\frac{1}{bc}=a+\frac{a}{abc}$; assume $y=x+\frac{x}{r}$, then $x=\frac{ry}{r+1}$. Substitute this value of x in the given equation.

17. Here $a^2+b^2+\frac{1}{ab}=a^2+b^2+c^2+\frac{c}{abc}-c^2=p^2-2q+\frac{c}{r}-c^2$.

Assume $y=p^2-2q+\frac{x}{r}-x^4$, and eliminate.

18. Here $a+b+c=p$, $\therefore\ a+b-c=p-2c$, and $\frac{c}{a+b-c}=\frac{c}{p-2c}$.

Assume $y=\frac{x}{p-2x}$, then $x=\frac{py}{2y+1}$; next substitute this value of x in the given equation.

19. Since $a+b+c=p$, $a+c=p-b$, and $ac=\frac{abc}{b}$; then $\frac{abc}{ab+bc}=\frac{r}{pb-b^2}$.

Assume $y=\frac{r}{px-x^2}$, and eliminate x.

20. Here $\left(\frac{b+c}{a}\right)^2=\frac{b^2+c^2+2bc}{a^2}=\frac{b^2+c^2+a^2+2bc-a^2}{a^2}=\frac{(p^2-2q)+\frac{2r}{a}-a^2}{a^2}$
$=\frac{(p^2-2q)a+2r-a^3}{a^3}$. Assume $y=\frac{(p^2-2q)x+2r-x^3}{x^3}$, and eliminate x.

XVIII.

1. The second and third terms vanish when $3p^2-8q=0$, and the second and fourth when $8r=p(4q-p^2)$.

4. $x^3-3ax^2+3a^2x-(a^3+b)=0$.

7. $y^3+2py+(p^2+q)y+pq-r=0$: and $y^3-16x^2+87x-162=0$.

8. The equation is $y^3+5qy^2+7q^2y+3q^3+8r^2=0$.

9. Let $y^3-Py^2+Qy-R=0$ denote the required equation. Find the values of P, Q, R in terms of p, q, r, s;

$P=ab+cd+ac+bd+ad+bc=q$.

$Q=(ab+cd)(ac+bd)+(ab+cd)(ad+bc)+(ac+bd)(ad+bc)$
$=(a+b+c+d)(abc+abd+acd+bcd)-4abcd=pr-4s$.

$R=(ab+cd)(ac+ba)(ad+bc)=(p^2-4q)s+r^2$.

The equation required is $y^3-qy^2+(pr-4s)y-(p^2-4q)s-r^2=0$.

11. $x+y+z=6$, $x^2+y^2+z^2=14$, $x^2+2y^2+3z^2=20$, from the second and third equations $x=\sqrt{(8+z^2)}$, $y=\sqrt{(6-2z^2)}$, substitute in the first equation, and $\sqrt{(8+z^2)}+\sqrt{(6-2z^2)}+z=6$, which reduced becomes the biquadratic equation $3z^4-12z^3+68z^2-132z+73=0$.

And since the sums of the positive and negative coefficients are equal, unity is one of the roots of this equation. (See Exercises under IX., p. 29.)

12. Let the given equation $x^3-p_1x^2+p_2x-p_3=0$ be transformed into the equation $x^3+px+q=0$, which wants the second term,

then if a, b, c be the roots of the given equation,

$a-\frac{1}{3}p$, $b-\frac{1}{3}p$, $c-\frac{1}{3}p$ are the roots of the transformed equation,

and $a-b$, $b-c$, $c-a$ are respectively equal to $(a-\frac{1}{3}p)-(b-\frac{1}{3}p)$, $(b-\frac{1}{3}p)-(c-\frac{1}{3}p)$, $(c-\frac{1}{3}p)-(a-\frac{1}{3}p)$,

or the differences of the roots of the transformed equation are equal to the differences of the roots of the given equation, and if the equation $x^3+px+q=0$ be transformed into an equation whose roots are $a-b$, $b-c$, $c-a$, it will be the equation required.

Here $(a-b)^2=a^2+b^2-2ab=a^2+b^2+c^2-\frac{2abc}{c}-c^2=-2q+\frac{2r}{c}-c^2$.

Similarly $(b-c)^2=-2q+\frac{2r}{b}-b^2$ and $(c-a)^2=-2q+\frac{2r}{a}-a^2$ are the other roots.

Let $y=-2q+\frac{2r}{x}-x^2$. Since $x^3+qx+r=0$, $-x^2=q+\frac{r}{x}$,

$\therefore y=-q+\frac{3r}{x}$, and $x=\frac{3r}{y+q}$. Substitute this value of x in the given equation,

and $\left\{\frac{3r}{y+q}\right\}^3+q\left\{\frac{3r}{y+q}\right\}+r=0$, which when reduced becomes

$y^3+6qy^2+9q^2y+27r^2+4q^3=0$, the equation required.

The equation $x^4+qx^2+rx+s=0$, when transformed into an equation whose roots shall be the squares of the differences of the roots of the proposed equation, will be of the sixth degree, consisting of seven terms, and whose coefficients involve no less than twenty-three distinct quantities. And it may be noted that Waring, in page 85 of his "Meditationes Algebraicæ," has given the equation which results from transforming an equation of the fifth degree into one whose roots are the squares of the differences of the roots of the proposed equation. The resulting equation is of the $\frac{5.4}{1.2}$ or the tenth degree, and consists of eleven terms. The coefficients of x^9 and succeeding powers of the unknown quantity consist of 1, 2, 3, 5, 7, 10, 13, 16, 18, 19 terms respectively, making a total of only 94.

XIX.

1. See Art. 6, pp. 7–9.

2. $x^3 - x^2 + x = 0$, or $x(x^2 - x + 1) = 0$, $\therefore x = 0$, and $x^2 - x + 1 = 0$, from which $x = \frac{1 \pm \sqrt{-3}}{2}$ and the three roots of the equation are 0, $\frac{1+\sqrt{-3}}{2}$, $\frac{1-\sqrt{-3}}{2}$. The expression "fractional root" is restricted to mean *a rational fraction*, not a fraction involving an impossible quantity.

3. If $\frac{1}{2}a + \sqrt{b}$ be substituted for x in the equation, the resulting equation will consist partly of rational and partly of irrational quantities. The sum of the rational parts and the sum of the irrational parts are respectively equal to zero. Eliminate b between these two equations, and the resulting equation is reduced to $(a+p)(a^2 + ap + q) = r$.

4. Since $\frac{1}{2}(a + \sqrt{b})$ is a root of the equation, this value of x satisfies the equation, which becomes on substitution $\frac{1}{8}(a^3 + 3a^2 b^{\frac{1}{2}} + 3ab + bb^{\frac{1}{2}}) + \frac{aq}{2} + \frac{b^{\frac{1}{2}}q}{2} + r = 0$. The rational and irrational parts are respectively equal, $\therefore (3a^2 + b + 4q)b^{\frac{1}{2}} = 0$, and $a^3 + 3ab + 4qa + 8r = 0$, whence $a^3 + aq - r = 0$, or since $x = a$, $\therefore x^3 + qx - r = 0$.

5. To shew that $3a^2 - b^2 = -q$, substitute $a + b\sqrt{-1}$ for x in the equation, and equate the rational and impossible parts each to zero.

6. The limiting equation of $x^4 + 2qx^2 + 4rx - s = 0$ is $4x^3 + 4qx + 4r = 0$, or $x^3 + qx + r = 0$. If this limiting equation have two impossible roots, the given equation has also two impossible roots. See Art. 15, p. 19.

XX.

See note, Section IV., p. 6, on the meaning of the words *greater than*, and *less than*, when applied to negative numbers.

1. For x, substitute successively 1, 2, 3, &c., in the given expression.

2. The limiting equation of $x^3 - 9x^2 + 23x - 15 = 0$, is $3x^2 - 18x + 23 = 0$, whose roots are $\frac{1}{3}(9 \pm 2\sqrt{3})$ or 4·153 and 1·846.

Hence 1·846 lies between the second and third roots of the given equation when arranged in decreasing order of magnitude.

3. Transform $x^3 - 6x + 23 = 0$ into an equation whose roots shall be diminished by e. Assume $y = x - e$, then $x = y + e$,

$$\text{and } (y+e)^3 - 6(y+e) + 23 = 0, \text{ or } y^3 + 3cy^2 + (3e^2 - 6)y + (e^3 - 6e + 23) = 0,$$

is the transformed equation, and every value of e, found by trial, which makes all the coefficients of this equation positive, will be greater than the greatest value of x in the proposed equation.

If $e = 2$, then $y^3 + 6y^2 + 6y + 19 = 0$, every coefficient of which is positive, and 2 is greater than the greatest root of $x^3 - 6x^2 + 23 = 0$.

4. If the alternate signs of the equation beginning with the second be changed,

the signs of all the roots are changed, and the greatest *negative root* becomes the *least positive root*, and if this equation be transformed into one whose roots are the reciprocals of its roots, the least positive root becomes the greatest positive root of this last equation.

5. If the equation be transformed into another whose roots shall be the reciprocals of the roots of the proposed equation, the least positive root of the proposed equation becomes the greatest positive root of the transformed equation.

6. If the signs of the alternate terms of this equation be changed, beginning with the second, the signs of the roots are changed, and the *least negative root* in the given equation becomes the *greatest positive root* in the transformed equation.

12. If $x=1$, $x^4-4x^3+11=8$
$x=2$, ,, ,, $=-5$
$x=3$, ,, ,, $=-6$
$x=4$, ,, ,, $=+11$.

One root lies between 1 and 2; and another between 3 and 4.

13. When $x=10$, then $10^3-17.10^2+54.10-350=+1540-2050=-510$.
$x=20$, $20^3-17.20^2+54.20-350=+9080-7150=+1930$.

One root lies between 10 and 20, and the first figure of that root is $+1$.

14. When $x=2$, then $2^4+3.2^3+2.2^2+6.2-148=60-148=-88$.
$x=3$, $3^4+3.3^3+2.3^2+6.3-148=198-148=+50$.

∴ One root lies between 2 and 3, and its first figure is 2.

XXI.

2. The equation $4x^3-12ax^2-4a^2x+12a^3=0$ contains two of the three equal roots, and $6x^2-12ax-2a^2=0$ contains one of them.

3. The roots of the biquadratic found by the method of divisors are 1, 2, 3, 4. The limiting equation is $2x^3-15x^2+35x-25=0$.

4. The limiting equation of $x^3-p_1x^2+p_2x-p_3=0$ is $3x^2-2p_1x+p_2=0$, which gives the roots $x=\frac{1}{3}\{p_1\pm(p_1^2-3p_2)^{\frac{1}{2}}\}$; if p_2 be positive and p_1^2 be less than $3p_2$, the roots are both impossible.

Hence the cubic equation $x^3-p_1x^2+p_2x-p_3=0$ will have a conjugate pair of impossible roots, when that relation exists.

5. The derived functions of the equation
$x^4-2x^3+10x-3=0$; if $x=2$, then $x^4-2x^3+10x-3=+17$,
are $2x^3-3x^2+5$, $2x^3-3x^2+5=+9$,
$3x^2-3x$, $3x^2-3x=+6$,
$2x-1$, $2x-1=+3$.

Since $x=2$ renders the three derived functions and the given equation positive, 2 is greater than the greatest root.

XXII.

See the examples by the method of Cardan in the Note p. 19. In the examples which contain the second power of the unknown quantity, the term involving this power must first be removed.

XXIII.

See the examples worked out by those methods in the Notes pp. 21, 22, 23.

XXIV.

See the example worked out in the Note p. 25.

1. One root between 2 and 3. 2. One root between 10 and 11.

5. One root between 2 and 3. 6. One root between 2 and 3.
7. One root between 0 and 1. 9. Two real roots, find the greater.
10. One root lies between 4 and 5.

XXV.

1. $3x^2-2px+q=0$ is the limiting equation of $x^3-px^2+qx-r=0$.

The third root is $\dfrac{4pq-p^3-9r}{pr-3q}$.

2. Subtract the second from the first equation and there results $(p-p')x^2+(q-q')x+(r-r')=0$, a quadratic equation. See Art. 1, Section VIII.

4. Let a, $\frac{1}{a}$, m denote the three roots of the equation,

$$\text{then } a+\frac{1}{a}+m=\frac{b}{a},\quad 1+m\left(a+\frac{1}{a}\right)=\frac{c}{a},\quad a.\frac{1}{a}m.=\frac{d}{a}.$$

$$\text{Eliminate } a+\frac{1}{a}, \text{ and } m=\frac{c-a}{b-d}.$$

10. The equation whose roots are a, b, c

$$\text{is } x^3-(a+b+c)x^2+(ab+ac+bc)x-abc=0.$$

From the equation $x^2-p_1x+q_1=0$, $p_1=b+c$, $q_1=bc$,

$$x^2-p_2x+q_2=0,\quad p_2=a+c,\quad q_2=ac,$$
$$x^2-p_3x+q_3=0,\quad p_3=a+b,\quad q_3=ab.$$

Hence $2(a+b+c)=p_1+p_2+p_3$, $\therefore a+b+c=\frac{1}{2}(p_1+p_2+p_3)$,

$$ab+ac+bc=q_1+q_2+q_3,$$
$$a^2b^2c^2=q_1q_2q_3,\quad \therefore abc=(q_1q_2q_3)^{\frac{1}{2}}.$$

$\therefore$ Making the substitutions

$$x^3-\tfrac{1}{2}(p_1+p_2+p_3)x^2+(q_1+q_2+q_3)x-(q_1q_2q_3)^{\frac{1}{2}}=0.$$

11. Eliminating x^3 between the second and third equations, $(q-r)x^2+(r-q)x=0$, from which $q=r$.

Eliminating x^3 between the first and second equation,

$(p-q)x^2+(q-r)x+(r-p)=0$, but $q=r$, $\therefore (p-q)x^2+(q-p)=0$, and $p=q$, $\therefore p$, q, r are equal to one another, and $p^2+r^2=2pq$. If a be the given root, the equation $x^3+px^2+px+p=0$ can be depressed, and the other two roots determined.

12. $ax^2+bx+c=0$, $bx^2+cx+a=0$, $cx^2+ax+b=0$, adding the equations

$$(a+b+c)x^2+(a+b+c)x+(a+b+c)=0,\quad \therefore (a+b+c)(x^2+x+1)=0,$$
$$\text{and } \therefore a+b+c=0, \text{ also } a+b=-c,$$
$$\therefore -c^3=(a+b)^3=a^3+b^3+3ab(a+b)=a^3+b^3-3abc,\quad \therefore a^3+b^3+c^3=3abc.$$

From the first and second equation, $x=\dfrac{bc-a^2}{ac-b^2}$;

from the second and third, $x=\dfrac{ac-b^2}{ab-c^2}$; and from the first and third, $x=\dfrac{ab-c^2}{bc-a^2}$;

Also a, b, c are so related that $a+b+c=0$, whence it may be shown that the three values of x are equal to each other.

13. The limiting equation of $x^3+3px^2+3qx+r=0$ is $3x^2+6px+3q=0$, or $x^2+2px+q=0$, and $x=-p\pm(p^2-q)^{\frac{1}{2}}$, the values of x in the limiting equation, which will be impossible if q be greater than p^2. Hence in this case the proposed equation has a pair of impossible roots.

14. Since a, a, c are roots of $x^3+3px^2+qx+r=0$, $2a+c=3p$, $a^2+2ac=q$, $a^2c=r$, and a, c, c are roots of $x^2+3p_1x^2+q_1x+r_1=0$, $2c+a=3p_1$, $c^2+2ac=q_1$, $ac^2=r_1$,

Then $a^3c^3=rr_1$, and $ac=(rr_1)^{\frac{1}{3}}$, $q=a(a+2c)=3pa$, $q_1=c(c+2a)=3pc$,

$$\therefore 9pp_1ac=qq_1,\ \text{and } ac=\frac{qq_1}{9pp_1},\ \therefore \frac{qq_1}{9pp_1}=(rr_1)^{\frac{1}{3}}.$$

Subtracting the second equation from the first, $3(p-p_1)x^2+(q-q_1)x+(r-r_1)=0$, a quadratic equation.

XXVI.

1. Each equation is reducible to $x^3-18x^2-225x-656=0$.
3. $a=-9,\ b=36,\ c=-60$.
4. Since $x=3+\sqrt{8}$, $\therefore$ also $x=3-\sqrt{8}$, and $x^2-6x+1=0$,

$$\text{but } x^3+px^2+qx+r=0,\ \therefore\ \frac{x^3+px^2+qr+r}{x^2-6x+1}=0,$$

$\therefore\ x+(p+6)=0,\ (6p+36+q-1)x+(r-p-6)=0$, from which $x=-r$.

5. The third root is $\frac{b}{a}$, if $b^3+c^3-a^2c+ab^2=0$.

XXVII.

3. Here $pq=r$, this equation becomes $x^3-px^2+qx-pq=0$,

$$\text{or } x^2(x-p)+q(x-p)=0,\ \text{ or } (x-p)(x^2+q).$$

If a, b, c denote the roots of $x^3-px^2+qx-r=0$, it may be noted that since $pq=r$, $(a+b+c)(ab+ac+b)=abc$, from which may be deduced $(a+b)(b+c)(c+a)=0$.

4. One of the roots of $x^3-4x^2+6x-3=0$ is $+1$. See Exercises IX. p. 29. The other two roots are $\frac{1}{2}(3\pm\sqrt{-3})$.

6. Let $\frac{x}{y}=z$, then the equation becomes $z^3+pz^2+qz+r=0$, if p, q, r be put for $\frac{b}{a}$, $\frac{c}{a}$, $\frac{d}{a}$.

9. Here $(x^2+ax+m)(x^2+bx+m)(x^2+cx+m)$

$$=x^6+(a+b+c)x^5+(3m+ab+ac+bc)x^4+\left\{2(a+b+c)m+abc\right\}x^3$$
$$+\left\{ab+ac+bc+3m\right\}mx^2+(a+b+c)m^2x+m^3.$$

$a+b+c=0,\ \therefore\ a^2+b^2+c^2=-2(ab+ac+bc)=6m,\ \therefore\ ab+ac+bc+3m=0.$

Hence $x^6+abcx^3+m^3$. To solve this equation, $x^6-20x^3+343=0$.

Let $x^3=y$, then the equation is reduced to a cubic $y^2-20y+343=0$.

10. By reducing the equation and arranging the terms according to the descending powers of x, $a_1+a_2+a_3+a_4$ is the coefficient of x^3, and since $a_1+a_2+a_3+a_4=0$, $a_1+a_2=-(a_3+a_4)$, and by means of this relation the coefficient of x^2 can be reduced to $a_1b_1+a_2b_2+a_3b_3+a_4b_4=0$, and the equation is thereby reduced to one of the first degree, which admits of only one value of x.

11. Since $xy(x+y)+xz(x+z)+yz(y+z)=(x+y+z)(xy+xz+yz)-3xyz$, and $xy(x^2+y^2)+xz(x^2+z^2)+yz(y^2+z^2)=(x^2+y^2+z^2)(xy+xz+yz)-xyz(x+y+z)$.

The second and third equations by substitution become

$$(x+y+z)(xy+xz+yz)-3xyz=168,$$
$$\text{and } (x^2+y^2+z^2)(xy+xz+yz)-xyz(x+y+z)=538.$$

Then by substitution and reduction $\frac{1}{3}(x+y+z)^3+\frac{54}{36}(x+y+z)=52+\frac{362}{26}$, whence $x+y+z$ is known, also xyz and $xy+xz+yz$. Hence the solution is reduced to a well-known form by putting $x+y+z=p$, $xy+xz+yz=q$, $xyz=r$, and then eliminating two of the unknown quantities y, z, the resulting equation is a cubic, $x^3-px^2+qx-r=0$.

12. Let x, y, z denote the edges, then $2(xy+xz+yz)=22$, $(x^2+y^2+z^2)^{\frac{1}{2}}=\sqrt{14}$, $xyz=6$. Eliminating y and z, $z^3-6z^2+11x-6=0$, from which z will be found to be 1, 2, 3 lineal inches respectively. Also x and y can be found when the value of z is known.

13. The criterion required for Cardan's rule to be applicable is, that $\frac{r^2}{4}$ be greater

than $\frac{q^3}{27}$; in this case, that $\frac{(2abc)^2}{4}$ be greater than $\frac{(a^2+b^2+c^2)^3}{27}$; but $\frac{(2abc)^2}{4}$ is less than $\frac{(a^2+b^2+c^2)^3}{27}$ obviously, and consequently, the three roots of the equation $x^3-(a^2+b^2+c^2)x-2abc=0$ are possible, two of which are negative and one positive. If a, b, c be supposed to be 1, 2, 3 respectively, the equation becomes $x^3-14x-12=0$, which has all its roots real, and they lie between 0 and -1, -3 and -4, 4 and 5; and will be found to be $-·911$, $-3·201$, $4·113$ respectively.

This problem occurs in Newton's Universal Arithmetic, Cap. 1, Sect. 4, of which he has given several solutions. The following one is taken as an illustration. Let AB, BC, CD denote three consecutive arcs of a semicircle, of which AD is the diameter. Join AC, BD. Then $ABCD$ is a quadrilateral inscribed in a circle, and AC, BD are the diagonals.

Then by Euc. VI. D. $AC\times BD=AB\times CD+AD\times BC$. Let $AB=a$, $BC=b$, $CD=c$, $AD=x$.

$$\therefore \sqrt{(x^2-c^2)}\times\sqrt{(x^2-a^2)}=ac+bx,$$
$$\text{and } x^4-a^2x^2-c^2x^2+a^2c^2=a^2c^2+2abcx+b^2x^2,$$
$$\text{also } x^4-(a^2+b^2+c^2)x^2-2abcx=0,\ \therefore\ x^3-(a^2+b^2+c^2)x-2abc=0.$$

XXVIII.

2. Since the equations have two roots in common, each involves the same quadratic factor.

3. The limiting equation of $x^4+rx+s=0$ is $4x^3+r=0$. The expressions x^4+rx+s and $4x^3+r$ have a common factor, and the remainder is equal to zero.

4. Let a, b denote the equal roots, then $(x-a)^2(x-b)^2=0$, or $x^4-2(a+b)x^3+(a^2+4ab+b^2)x^2-2ab(a+b)x+a^2b^2=0$, is an equation identical with $x^4-px^3+qx-rx+s=0$. The values of a and b may be found in terms of any two of the four coefficients p, q, r, s.

6. If there be equal roots in the equation, its limiting equation will contain one or more of them, and it will be found that $x^4+13x^3+33x^2+31x+10$ and $4x^3+39x^2+66x+31$ have a common quadratic factor x^2+2x+1, which indicates that the given equation contains three equal roots, each equal to -1, the fourth root being -10.

It may be remarked that since the sum of the coefficients of the odd terms is equal to the sum of the coefficients of the even terms, the equation may be otherwise solved.

XXIX.

2. Obviously, x is one of the factors. Removing this factor, and let $x^4-5x^2+4=0$. This is a quadratic equation of which the roots are ±1 and ±2, then $x=0$, $x-1=0$, $x+1=0$, $x-2=0$, $x+2=0$,

$$\therefore x(x-1)(x+1)(x-2)(x+2)=x^5-5x^3+4x$$
$$\therefore x,\ x-1,\ x+1,\ x-2,\ x+2 \text{ are the factors of } x^5-5x^3+4x.$$

3. If $q=r+1$, $q-1=r$, and the equation can be put into the form

$$x^4+2x^3+x^2+(q-1)x^2+rx+s=0, \text{ or}$$
$$(x^2+x)^2+r(x^2+x)+s=0, \text{ a quadratic equation.}$$

4. Add 1 to each side, then

$$(x+1)(x+2)(x+3)(x+4)+1=(x+1)^2+(x+2)^2+(x+3)^2+(x+4)^2+1,$$
$$\text{or } (x^2+5x+5)^2=4(x^2+5x+5)+11.$$

5. First find the roots common to the first and second equation, next those which are common to the first and third; and it will be found that 1, 2, 3, 4 are the roots of the biquadratic.

XXX.

1. When the sum of two of the roots is equal to the sum of the other two, then $p^3 - 4pq + 8r = 0$, and when the product of two roots is equal to the product of the other two, $p^2 s = r^2$. In both cases the biquadratic can be solved by a quadratic.

2. The first equation can be reduced to $(x^2 - ax)^2 - 2b^2(x^2 - ax) = a^2b^2$, and the second equation to $2x^5 + 8x^4 - 14x^3 - 44x^2 + 6x = 0$.

7. When $x^4 - qx^2 - rx - s$ is divided by $x^2 + ax + b$, the quotient is
$x^2 - ax + a^2 - b - q$, and the remainder is $-(a^3 - 2ab - qa + r)x - b(a^2 - b - q) - s$.

If $x^2 + ax + b$ be one quadratic factor of $x^4 - qx^2 - rx - s$, then must $x^2 - ax + a^2 - b - q$ be the other; and the two terms of which the remainder is composed must be respectively equal to zero.

And then $a^3 - (2b + q)a + r = 0$, and $b(a^2 - b - q) - s = 0$. When a is eliminated, the resulting equation becomes a cubic involving b with q, r, s.

XXXI.

1. When a reciprocal equation is of an odd number of dimensions, it has at least one root, $+1$ or -1.

2. (1) Let $x^5 - px^4 + qx^3 - qx^2 + px - 1 = 0$ be a recurring equation of five dimensions.

It may be put into the form $(x^5 - 1) - px(x^3 - 1) + qx^2(x - 1) = 0$, $\therefore\ x - 1 = 0$, and $x = 1$, one of the roots, and the equation when depressed becomes

$$x^4 - (p-1)x^3 + (q - p + 1)x^2 - (p-1)x + 1 = 0;\ \text{divide each term by } x^2,$$

$$\therefore\ x^2 - (p-1)x + (q - p + 1) - (p-1).\frac{1}{x} + \frac{1}{x^2} = 0,$$

$$\text{and } \left(x^2 + 2 + \frac{1}{x^2}\right) - (p-1)\left(x + \frac{1}{x}\right) = 2 - q + p - 1 = p - q + 1.$$

Let $x + \frac{1}{x} = z$, then $z^2 - (p-1)z = p - q + 1$, a quadratic by means of which the other roots can be determined.

(2) Let $x^6 - px^5 + qx^4 - rx^3 + qx^2 - px + 1 = 0$, a recurring equation of six dimensions.

Divide each term by x^3, then $x^3 - px^2 + qx - r + q.\frac{1}{x} - p.\frac{1}{x^2} + \frac{1}{x^3} = 0$,

$$\text{or } \left(x^3 + \frac{1}{x^3}\right) - p\left(x^2 + \frac{1}{x^2}\right) + q\left(x + \frac{1}{x}\right) - r = 0.$$

Let $x + \frac{1}{x} = z$, then $x^2 + \frac{1}{x^2} = z^2 - 2$, $x^3 + \frac{1}{x^3} = z^3 - 3z$; make the substitutions,

$$\therefore\ z^3 - pz^2 + (q - 3)z - (2p + r) = 0,\ \text{an equation of three dimensions.}$$

5. Here $x^4 + x^2 + 1 = -(x^3 + x)$, and $(x^4 + x^2 + 1)^2 = (x^3 + x)^2$, whence $x^8 + x^6 + x^4 + x^2 + 1 = 0$, let $x^2 = y$, then $y^4 + y^3 + y^2 + y + 1 = 0$ is the equation whose roots are the squares of the roots of the given equation. It is obvious that the two equations are identical, and that the four roots of the transformed equation may be shewn to be equal to the squares of the four roots respectively of the given equation.

6. Form the equation whose roots shall be a, $\frac{1}{a}$, b, $\frac{1}{b}$, and shew that it is identical with the given equation when the latter is reduced to the form

$$x^4 - \frac{p(q+1)}{q}.x^3 - \frac{(q^2 + p^2 + 1)}{q}.x^2 - \frac{p(q+1)}{q}.x + 1 = 0.$$

7. Let the roots be increased by e; assume $y = x + e$, $x = y - e$, then $y^4 - 4(e-1)y^3 + 6e(e-2)y^2 - 8(e^3 - 1)y + (e^4 - 4e^3 + 8e - 4) = 0$ is the transformed equation. And in order that this equation may become a reciprocal equation, the

coefficients of the first and last terms, as also the coefficients of the second and last term but one, must be equal; that is $4(e-1)=8(e^2-1)$, and $e^4-4e^3+8e-4=1$.

From $4(e-1)=8(e^2-1)$, $2c^2-e-1=0$, and $e=1$, $e=-\frac{1}{2}$, the former value satisfies the equation $e^4-4e^3+8e-4=0$. Lastly, substituting $e=1$ in the transformed equation, then $y^4-6y^2+1=0$ is the required reciprocal equation, whose roots can be found by a quadratic, also the roots of the given equation.

8. The equation can be reduced to the form $x^4-4mx^3+6m^2x^2-4m^3x+m^4=0$.

9. The sum of the squares of the roots of the equation $x^4+px^3+qx^2+px+1=0$ is p^2-2q. Next divide the equation by x^2,

$$\text{then } x^2+px+q+p.\frac{1}{x}+\frac{1}{x^2}=0, \text{ and } x^2+2+\frac{1}{x^2}+p\left(x+\frac{1}{x}\right)+q-2=0.$$

$$\text{If } x+\frac{1}{x}=z, \text{ then } z^2+pz=2-q \text{ is the reducing quadratic.}$$

Let z_1, z_2 denote the roots of this equation.

$$\text{Then } z_1^2+z_2^2=\left\{-\frac{p}{2}+\sqrt{(\tfrac{1}{4}p^2+2-q)}\right\}^2+\left\{-\frac{p}{2}-\sqrt{(\tfrac{1}{4}p^2+2-q)}\right\}^2$$

$$=\frac{p^2}{4}-p\sqrt{(\tfrac{1}{4}p^2+2-q)}+\frac{p^2}{4}+2-q+\frac{p^2}{4}+p\sqrt{(\tfrac{1}{4}p^2+2-q)}+\frac{p^2}{4}+2-q$$

$$=p^2-2q+4$$

$$\therefore (p^2-2q+4)-(p^2-2q)=4.$$

XXXII.

1. (1) When the roots of $x^3-px^2+qx-r=0$ are in arithmetical progression.

Let the three roots be denoted by $a-d$, a, $a+d$.

Then $a-d+a+a+d=p$, $a(a-d)+(a-d)(a+d)+a(a+d)=q$, $a(a+d)(a-d)=r$, or $3a=p$, $3a^2-d^2=q$, $a(a^2-d^2)=r$.

$$a=\frac{p}{3},\ \frac{p^2}{3}-d^2=q, \text{ and } d^2=\frac{p^2}{3}-q;\ \text{also } \frac{p}{3}\left\{\frac{p^2}{9}-d^2\right\}=r, \text{ and } d^2=\frac{p^2}{9}-\frac{3r}{p}.$$

$$\text{Hence equating the values of } d^2,\ \frac{p^2}{3}-q=\frac{p^2}{9}-\frac{3r}{p},$$

$$\text{and } 2p^3-9pq+27r=0, \text{ the relation required.}$$

(2) When the roots of $x^3-px^2+qx-r=0$ are in harmonical progression.

For x write $\frac{1}{y}$, then the given equation becomes $y^3-\frac{qy^2}{r}+\frac{py}{r}-\frac{1}{r}=0$, which is an equation whose roots are in arithmetical progression, and the relation between the roots will be found to be $2q^3-9pqr+27r^2=0$.

(3) When the roots of $x^3-px^2+qx-r=0$ are in geometrical progression.

Let a, ab, ab^2 denote the roots of the equation,

$$\text{then } a+ab+ab^2=p,\ a^2b+a^2b^2+a^2b^3=q,\ a^3b^3=r,$$

$$\therefore \frac{a^2b(1+b+b^2)}{a(1+b+b^2)}=\frac{q}{p}, \text{ or } ab=\frac{q}{p}, \text{ and } a^3b^3=\frac{q^3}{p^3}, \text{ but } a^3b^3=r,$$

$$\therefore \frac{q^3}{p^3}=r, \text{ or } q^3-p^3r=0 \text{ is the required relation.}$$

It is also obvious that $r^{\frac{1}{3}}=ab$, the middle root.

Also the relations which subsist between p, q, r, s, in $x^4-px^3+qx^2-rx+s=0$, an equation of the fourth degree, may be found by the same process when the roots are in arithmetical, harmonical, and geometrical progression.

The solution of cubic equations whose roots are in harmonical progression may be effected in two ways, (1) by transforming the given equation into one whose roots shall be in arithmetical progression, or by assuming three quantities a, $\frac{2ac}{a+c}$, c, in harmonical progression to denote the roots.

4. If $a_1=a$, $a_2=a+b$, $a_3=a+2b$, $a_4=a+3b$, the given equation assumes by substitution the form

$$x^4+(4a+6b)x^3+(6a^2+18ab+11b^2)x^2+(4a^3+18a^2b+22ab^2+6b^3)x$$
$$+(a^4+6a^3b+11a^2b^2+6ab^3)=d.$$

The left side of this equation will become a complete square by the addition of b^4. Let b^4 be added to each side, and let the square root of each side be extracted.

$$\text{then } x^2+(2a+3b)x+(a^2+3ab+b^2)=\pm\sqrt{(d+b^4)},$$

from these quadratics the four roots can be found.

This method may be illustrated by the solution of the equations

$$(x+4)(x+6)(x+8)(x+10)=48, \text{ and } (x+3)(x+7)(x+11)(x+15)=644.$$

See example 2, under Exercise VII., p. 22 of Section VI.

5. Since the roots of $x^3-px^2+qx-r=0$ are in harmonical progression.

Let $\dfrac{1}{a-b}$, $\dfrac{1}{a}$, $\dfrac{1}{a+b}$ denote the three roots,

then $\dfrac{q}{r}$=the sum of the reciprocals of the roots$=a-b+a+a+b=3a$,

$$\therefore \frac{1}{a} \text{ the middle root } =\frac{3r}{q}. \text{ Let } x=\frac{3r}{q}, \text{ or } qx-3r=0$$

$$\text{then } \frac{x^3-px^2+qx-r}{qx-3r}=0,$$

or $q^2x^2-(pq-3r)qx+(q^3-3pqr+9r^2)=0$ is the equation which contains the roots $\dfrac{1}{a-b}$, $\dfrac{1}{a+b}$.

9. Since the roots of the given equation are in harmonical progression,

for x substitute $\dfrac{1}{y}$, then the equation becomes

$y^4+\dfrac{cy^3}{d}+\dfrac{by^2}{d}+\dfrac{a}{d}=0$, whose roots are in arithmetical progression.

Let $x+3y$, $x+y$, $x-y$, $x-3y$ denote the roots of this equation,

$$\text{then } 4x=\frac{c}{d},\ 6x^2-10y^2=\frac{b}{d},\ 4x^3-20xy^2=0,\ x^4-10x^2y^2+9y^4=\frac{a}{d},$$

from which by elimination may be found $c^2=4bd$ and $c^4+400ad=0$.

XXXIII.

7. Let a denote the possible root of the equation $x^3-qx+r=0$, then $x=a$, and $x-a=0$; $\therefore \dfrac{x^3-qx+r}{x-a}=0$, or $x^2+ax+a^2-q=0$, the quadratic equation which contains the two impossible roots, and taking $a=y+z$, the required expression will be found.

8. The equation $(x^3-2r)^3-27q^3x^3=0$ is an equation of nine dimensions, and includes the three equations

$$x^3-3qx+2r=0,\ x^3-3q\omega x+2r=0, \text{ and } x^3-3q\omega^2x+2r=0,$$

in which 1, ω, ω^2 denote the three cube roots of unity.

The first of these equations, $x^3-3qx+2r=0$, is only another form of $x^3-qx+r=0$, obtained by writing $3q$ for q and $2r$ for r, when $y=\dfrac{q}{z}$ and $z=\dfrac{q}{y}$, the two equations will give values of y and z, from which

$$x=y+z=\{r+(r^2-q^3)^{\frac{1}{2}}\}^{\frac{1}{3}}+\{r-(r^2-q^3)^{\frac{1}{2}}\}^{\frac{1}{3}}.$$

XXXIV.

1. See Art. 16, p. 20.

2. Let $\alpha^2,\beta^2,\gamma^2$ denote the three values of e^2 or y in $y^3+2qy^2+(q^2-4s)y-r^2=0$,

the reducing cubic of Descartes' solution of the biquadratic $x^4+qx^2+rx+s=0$, then $\alpha^2\beta^2\gamma^2=r^2$, $\alpha\beta\gamma=r$, and $-2q=\alpha^2+\beta^2+\gamma^2$.

$x^2+ex+f=x^2+ex+\frac{1}{2}\left(q+e^2-\frac{r}{e}\right)$. Let $e=\alpha$ one of the three values of e.

$$=x^2+\alpha x+\tfrac{1}{4}(\alpha^2-\beta^2-\gamma^2-2\beta\gamma)=0,$$

$$\therefore\ x^2+\alpha x+\frac{\alpha^2}{4}=\tfrac{1}{4}(\beta+\gamma)^2,\ \therefore\ x+\frac{\alpha}{2}=\pm\tfrac{1}{2}(\beta+\gamma),$$

and $x=\frac{1}{2}(-\alpha+\beta+\gamma)$ and $\frac{1}{2}(-\alpha-\beta-\gamma)$, two of the required roots.

In the same manner

$$x^2-ex+g=x^2-ex+\frac{1}{2}\left(q+e^2+\frac{r}{e}\right)$$

$$=x^2-\alpha x+\tfrac{1}{4}(\alpha^2-\beta^2-\gamma^2+2\beta\gamma)=0,$$

$$\text{and } x^2-\alpha x+\frac{\alpha^2}{4}=\tfrac{1}{4}(\beta-\gamma)^2,\ \therefore\ x-\frac{\alpha}{2}=\pm\tfrac{1}{2}(\beta-\gamma),$$

$\therefore\ x=\frac{1}{2}(\alpha+\beta-\gamma)$ and $\frac{1}{2}(\alpha-\beta+\gamma)$ are the other two roots of the biquadratic in terms of the roots of the reducing cubic.

3. See Art. 16, p. 20. 7. See Art. 17, p. 22. 8. See Art. 17, p. 22.

12. See Art. 16, p. 20, and Art. 18, p. 23.

XXXV.

2. Since α, β are two roots of the equation, $x^2-(\alpha+\beta)x+\alpha\beta=0$, which is one of the quadratic equations of which the biquadratic is composed.

Let γ, δ denote the roots of the other quadratic.

Since $\alpha+\beta+\gamma+\delta=0$, $\alpha+\beta=-(\gamma+\delta)$,

and $x^2+(\gamma+\delta)x+\gamma\delta=0$, or $x^2+(\alpha+\beta)+\gamma\delta=0$ is the quadratic containing the other two roots,

$$\therefore\ x=-\frac{\alpha+\beta}{2}\pm\left\{\left(\frac{\alpha+\beta}{2}\right)^2-\gamma\delta\right\}^{\frac{1}{2}}=-\frac{\alpha+\beta}{2}\pm\left\{\left(\frac{\alpha-\beta}{2}\right)^2+\alpha\beta-\gamma\delta\right\}^{\frac{1}{2}}.$$

$$\text{But } q=+\alpha\beta\gamma+\alpha\beta\delta-\alpha\gamma\delta-\beta\gamma\delta=+(\gamma+\delta)\alpha\beta-(\alpha+\beta)\gamma\delta=(\alpha+\beta)(\alpha\beta-\gamma\delta),$$

$$\therefore\ \frac{q}{\alpha+\beta}=\alpha\beta-\gamma\delta,\ \therefore\ x=-\frac{\alpha+\beta}{2}\pm\left\{\left(\frac{\alpha-\beta}{2}\right)^2+\frac{q}{\alpha+\beta}\right\}^{\frac{1}{2}}.$$

3. The equations are $p^2q-p(r-s)-q^2=0$, and $p^2s+q(r-s)=0$; and if $r-s$ be eliminated, the resulting equation is $p^3s+p^2q^2-q^3=0$.

4. First suppose the four roots real; since the last term of the equation is negative and equal to unity, the roots are of the form a, a^{-1}, b, $-b^{-1}$; and it may be shewn that $a+a^{-1}=\frac{1}{2}(p+q)$, and $b-b^{-1}=\frac{1}{2}(p-q)$.

5. First $x^4=-4a^3x+a^4$, add $2x^2a^2$ to each side of this equation, and $x^4+2x^2a^2=2x^2a^2-4a^3x+a^4$, complete the square of the left side,

$$\therefore\ x^4+2x^2a^2+a^4=2x^2a^2-4a^3x+2a^4=2a^2(x^2-2ax+a^2).$$

Extract the square root of each side, $\therefore\ x^2+a^2=\pm a(x-a)\sqrt{2}$, then $x^2+a^2=+a(x-a)\sqrt{2}$ and $x^2+a^2=-a(x-a)\sqrt{2}$ are the two quadratic equations from which two values of x will be fonnd to be possible and two impossible.

6. It is obvious from inspection that a, b, c respectively, satisfies the equation; and, therefore, three of the roots are known, that is $x=a$, $x=b$, $x=c$, and $x-a=0$, $x-b=0$, $x-c=0$, $\therefore\ (x-a)(x-b)(x-c)=0$.

Next the given equation can be put into the following form:

$$x^4=\frac{a^4\{x^2-(b+c)x+bc\}}{(a-b)(a-c)}+\frac{b^4\{x^2-(a+c)x+ac\}}{(b-c)(b-a)}+\frac{c^4\{x^2-(a+b)x+ab\}}{(c-a)(c-b)}$$

$$=\left\{\frac{a^4}{(a-b)(a-c)}+\frac{b^4}{(b-c)(b-a)}+\frac{c^4}{(c-a)(c-b)}\right\}x^2$$

$$-\left\{\frac{a^4(b+c)}{(a-b)(a-c)}+\frac{b^4(a+c)}{(b-c)(b-a)}+\frac{c^4(a+b)}{(c-a)(c-b)}\right\}x$$

$$+\left\{\frac{a^4bc}{(a-b)(a-c)}+\frac{b^4ac}{(b-c)(b-a)}+\frac{c^4ab}{(c-a)(c-b)}\right.$$

$$=\{(c+a)(c+b)+a^2+b^2\}x^2-(a+b)(b+c)(c+a)x+abc(a+b+c),$$

$$\therefore\ x^4-\{(c+a)(c+b)+a^2+b^2\}x^2+(a+b)(b+c)(c+a)x-abc(a+b+c)=0 \quad (1),$$

but $(x-a)(x-b)(x-c)$, or $x^3-(a+b+c)x^2+(ab+ac+bc)x-abc=0$ (2).

By dividing the equation (1) by the equation (2)

there results the equation $x+a+b+c=0$,

and consequently $x=-(a+b+c)$ is the fourth root of the equation.

7. Divide each term of the equation by x^3, and arrange the terms in pairs.

Assume $x+\frac{s}{x}=z$, and by the necessary substitutions, $z^3-pz^2+(q-3s)z+2ps-r=0$ is the cubic equation.

8. The equation can be put into the form $(x^9-1)-px(x^7-1)+qx^2(x^5-1)-rx^3(x^3-1)+sx^4(x-1)=0$, which is obviously divisible by $x-1$, $\therefore x=1$, one of the roots; depressing the equation it becomes

$$x^8-(p-1)x^7+(q-p+1)x^6-(r-q+p-1)x^5+(s-r+q-p+1)x^4-(r-q+p-1)x^3$$
$$+(q-p+1)x^2-(p-1)x+1=0.$$

Dividing by x^4 and arranging the terms,

$$\left(x^4+\frac{1}{x^4}\right)-(p-1)\left(x^3+\frac{1}{x^3}\right)+(q-p+1)\left(x^2+\frac{1}{x^2}\right)-(r-q+p-1)\left(x+\frac{1}{x}\right)$$
$$+(s-r+q-p+1)=0.$$

Let $x+\frac{1}{x}=z$, then $x^2+\frac{1}{x^2}=z^2-2$, $x^3+\frac{1}{x^3}=z^3-3z$, $x^4+\frac{1}{x^4}=z^4-4z^2+2$.

Making these substitutions and reducing the terms

$$z^4-(p-1)z^3+(q-p-3)z^2-(r+q+2p-2)z+(s-r-q+p+1)=0$$

is an equation of four dimensions.

9. The equation is $(x^9+1)+x(x^7+1)-9x^2(x^5+1)+3x^3(x^3+1)-8x^4(x+1)=0$.

Obviously -1 is a root of this equation, and $x+1=0$. Depressing the equation it becomes $x^8-9x^6+12x^5-20x^4+12x^3-9x^2+1=0$.

Divide by x^4, arrange the terms, and let $x+\frac{1}{x}=z$,

$$\text{then } z^4-13z^2+12z+22-22=0, \text{ or } z(z^3-13z+12)=0,$$
$$\therefore z=0, \text{ or } x+\frac{1}{x}=0, \text{ and } x=\pm\sqrt{-1},$$
$$\text{and } z^3-13z+12=0,$$
$$\text{or } z(z^2-1)-12(z-1)=0, \text{ or } (z-1)(z^2+z-12)=0;$$
$$\therefore z-1=0, \text{ or } x+\frac{1}{x}+1=0; \therefore x=\frac{1\pm\sqrt{-3}}{2},$$
$$\text{and } z^2+z-12=0,\ z^2+z+\frac{1}{4}=\frac{49}{4},\ z+\frac{1}{2}=\frac{\pm 7}{2};\ \therefore z=3 \text{ and } -4;$$
$$\therefore x+\frac{1}{x}=3, \text{ and } x=\frac{3\pm\sqrt{5}}{2},$$
$$x+\frac{1}{x}=-4, \text{ and } x=-2\pm\sqrt{3}.$$

The nine roots are $-2\pm\sqrt{3}$, $\frac{1}{2}(3\pm\sqrt{5})$, $\frac{1}{2}(1\pm\sqrt{-3})$, $\pm\sqrt{-1}$, -1.

10. Since $l^2+m^2+n^2=1$, the given equation may be put under the form

$$(l^2+m^2+n^2)v^4-\{(b^2+c^2)l^2+(c^2+a^2)m^2+(a^2+b^2)n^2\}v^2$$
$$+b^2c^2l^2+c^2a^2m^2+a^2b^2n^2=0,$$

and resolved into

$$(v^2-b^2)(v^2-c^2)l^2+(v^2-a^2)(v^2-c^2)m^2+(v^2-a^2)(v^2-b^2)=0,$$

and then dividing by $(v^2-b^2)(v^2-b^2)(v^2-c^2)$, the form required is obtained.

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